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ON RICCATI'S EQUATION.

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THE following is, it appears to me, the proper form in which to present the solution of Riccati's equation.

The equation may be written

$$\frac{dy}{dx} + y^2 = x^{2q-2},$$

which is integrable by algebraic and exponential functions if $(2i+1)q = \pm 1$, i being zero, or a positive integer. To effect the integration, writing $y = \frac{1}{u}\frac{du}{dx}$, we have

$$\frac{d^2u}{dx^2} = x^{2q-2}u.$$

The peculiar advantage of this well-known transformation has not (so far as I am aware) been explicitly stated; it puts in evidence the form under which the sought-for function y contains the constant of integration. In fact if $u = P$, $u = Q$ be two particular solutions of the equation in u, then the general solution is $u = CP + DQ$; and denoting by P', Q' the derived functions, the value of y is

$$y = \frac{CP' + DQ'}{CP + DQ},$$

showing the form under which the constant of integration $C \div D$ is contained in y. To complete the solution, assume

$$u = ze^{\frac{1}{q}x^q};$$

we find

$$\frac{d^2z}{dx^2} + 2x^{q-1}\frac{dz}{dx} + (q-1)\,x^{q-2}\,z = 0:$$

considering first the particular integral of the form

$$z = A + Bx^q + Cx^{2q} + Dx^{3q} + \&\text{c.},$$

we find that the equation will be satisfied if

$$\begin{aligned}
(\ q-1)\,A + \ q\,(\ q-1)\,B &= 0,\\
(3q-1)\,B + 2q\,(2q-1)\,C &= 0,\\
(5q-1)\,C + 3q\,(3q-1)\,D &= 0,\\
(7q-1)\,D + 4q\,(4q-1)\,E &= 0,\\
\&\text{c.}&
\end{aligned}$$

If $A = 1$, this is

$$\begin{aligned}
A &= \ 1,\\
B &= -\frac{q-1}{q\,(q-1)},\\
C &= +\frac{(q-1)\,(3q-1)}{q\,(q-1)\,2q\,(2q-1)},\\
D &= -\frac{(q-1)\,(3q-1)\,(5q-1)}{q\,(q-1)\,2q\,(2q-1)\,3q\,(3q-1)},\\
&\&\text{c.},
\end{aligned}$$

where it is to be noticed that the series may be considered to stop so soon as there is in the numerator a factor $= 0$. For instance, if $5q-1=0$, then if the particular integral had been assumed to be $z = A + Bx^q + Cx^{2q}$, the only conditions to be satisfied by the coefficients are the first and second equations giving the foregoing values of A, B, C. It is immaterial that the analytical expressions of F and the subsequent coefficients contain in the denominators the evanescent factor $5q-1$; the coefficients after C do not ever come into consideration.

Thus if $(2i+1)\,q = +1$, the series terminates, and we have for u the finite particular solution

$$u = P = \left(1 - \frac{q-1}{q\,(q-1)}\,x^q + \frac{(q-1)\,(3q-1)}{q\,(q-1)\,2q\,(2q-1)}\,x^{2q} - \&\text{c.}\right) e^{\frac{1}{q}x^q}:$$

and it is easy to see that we may herein change the sign of x^q, thereby obtaining another finite particular solution,

$$u = Q = \left(1 + \frac{q-1}{q\,(q-1)}\,x^q + \frac{(q-1)\,(3q-1)}{q\,(q-1)\,2q\,(2q-1)}\,x^{2q} + \&\text{c.}\right) e^{-\frac{1}{q}x^q}.$$

Reverting to the equation in z, we have next a particular solution of the form

$$z = Ax + Bx^{q+1} + Cx^{2q+1} + Dx^{3q+1} + \&\text{c.},$$

giving between the coefficients the relation

$$\begin{aligned}
(q+1)\,A + (q+1)\,q\,B &= 0,\\
(3q+1)\,B + (2q+1)\,2q\,C &= 0,\\
(5q+1)\,C + (3q+1)\,3q\,D &= 0,\\
(7q+1)\,D + (4q+1)\,4q\,E &= 0,\\
\&\text{c.}
\end{aligned}$$

If $A = 1$, we have

$$\begin{aligned}
A &= 1,\\
B &= -\frac{(q+1)}{(q+1)\,q},\\
C &= +\frac{(q+1)(3q+1)}{(q+1)\,q\,(2q+1)\,2q},\\
D &= -\frac{(q+1)(3q+1)(5q+1)}{(q+1)\,q\,(2q+1)\,2q\,(3q+1)\,3q},\\
\&\text{c.},
\end{aligned}$$

where, as in the former case, the series is considered to terminate as soon as there is an evanescent factor in the numerator, without any regard to the subsequent coefficients which contain in the denominators the same evanescent factor. [In particular, $q = -1$, we have the solution $z = x$.]

Hence if we have $(2i+1)\,q = -1$, the series terminates, and we have for u the finite particular solution,

$$u = P = x\left(1 - \frac{q+1}{(q+1)\,q}\,x^q + \frac{(q+1)(3q+1)}{(q+1)\,q\,(2q+1)\,2q}\,x^{2q} - \&\text{c.}\right) e^{\frac{1}{q}x^q}$$

from which, changing the sign of x^q, we deduce the other finite particular solution,

$$u = Q = x\left(1 + \frac{q+1}{(q+1)\,q}\,x^q + \frac{(q+1)(3q+1)}{(q+1)\,q\,(2q+1)\,2q}\,x^{2q} + \&\text{c.}\right) e^{-\frac{1}{q}x^q}.$$

Hence, in the equation

$$\frac{dy}{dx} + y^2 = x^{2q-2},$$

where $q\,(2i+1) = \pm 1$, we have (writing $D = 1$)

$$y = \frac{CP' + Q'}{CP + Q},$$

where C is the constant of integration, P, Q are finite series as above, and P', Q' are the derived functions of P and Q. Writing successively $i=0$, $i=1$, $i=2$, &c., we may tabulate the solutions

$$\frac{dy}{dx}+y^2=1, \qquad P=e^x, \qquad Q=e^{-x},$$

$$\frac{dy}{dx}+y^2=x^{-4}, \qquad P=xe^{-\frac{1}{x}}, \qquad Q=xe^{\frac{1}{x}},$$

$$\frac{dy}{dx}+y^2=x^{-\frac{4}{3}}, \qquad P=(1-3x^{\frac{1}{3}})\,e^{3x^{\frac{1}{3}}}, \qquad Q=(1+3x^{\frac{1}{3}})\,e^{-3x^{\frac{1}{3}}},$$

$$\frac{dy}{dx}+y^2=x^{-\frac{8}{3}}, \qquad P=x\,(1+3x^{-\frac{1}{3}})\,e^{-3x^{-\frac{1}{3}}}, \qquad Q=x\,(1-3x^{-\frac{1}{3}})\,e^{3x^{-\frac{1}{3}}},$$

$$\frac{dy}{dx}+y^2=x^{-\frac{8}{5}}, \qquad P=(1-5x^{\frac{1}{5}}+\tfrac{25}{3}x^{\frac{2}{5}})\,e^{5x^{\frac{1}{5}}}, \qquad Q=(1+5x^{\frac{1}{5}}+\tfrac{25}{3}x^{\frac{2}{5}})\,e^{-5x^{\frac{1}{5}}},$$

&c.

It is hardly necessary to make the final step of calculating P' and Q' and substituting in y; but, as an example, take the above equation $\frac{dy}{dx}+y^2=x^{-\frac{4}{3}}$: we have

$$y=\frac{-3x^{-\frac{1}{3}}\,(Ce^{3x^{\frac{1}{3}}}+e^{-3x^{\frac{1}{3}}})}{C\,(1-3x^{\frac{1}{3}})\,e^{3x^{\frac{1}{3}}}+(1+3x^{\frac{1}{3}})\,e^{-3x^{\frac{1}{3}}}},$$

which is readily identified with the solution, p. 98 of Boole's *Differential Equations* (Cambridge, 1859).

Cambridge, September 29, 1868.