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AN ILLUSTRATION OF THE THEORY OF THE ϑ-FUNCTIONS.

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IF X be a given quartic function of x, and if u, or for convenience a constant multiple αu, be the value of the integral $\int \frac{dx}{\sqrt{(X)}}$ taken from a given inferior limit to the superior limit x; then, conversely, x is expressible as a function of u, viz. it is expressible in terms of ϑ-functions of u, where ϑu, or say $\vartheta(u, \mathfrak{F})$ ($\mathfrak{F}$ a parameter upon which the function depends), is given by definition as the sum of a series of exponentials of u; and it is possible from the assumed equation $\alpha u = \int \frac{dx}{\sqrt{(X)}}$, and the definition of ϑu, to obtain by general theory the actual formulæ for the determination of x as such a function of u.

I propose here to obtain these formulæ, in the case where X is a product of real factors, in a less scientific manner, by connecting the function ϑu (as given by such definition) with Jacobi's function Θ, and by reducing the integral $\int \frac{dx}{\sqrt{(X)}}$ by a linear substitution to the form of an elliptic integral; the object being merely to obtain for the case in question the actual formulæ for the expression of x in terms of ϑ-functions of u.

The definition of ϑu or, when the parameter is expressed, $\vartheta(u, \mathfrak{F})$ is

$$\vartheta u = \Sigma (-)^s e^{-\mathfrak{F}s^2 + 2isu},$$

where s has all positive or negative integer values, zero included, from $-\infty$ to $+\infty$ (that is, from $-S$ to $+S$, $S = \infty$); the parameter $\mathfrak{F}$, or (if imaginary) its real part, must be positive.

Evidently ϑu is an even function: $\vartheta(-u) = \vartheta u$. Moreover, it is at once seen that we have

$$\vartheta(u+\pi) = \vartheta u,\ \vartheta(u+i\mathfrak{F}) = -e^{\mathfrak{F}-2iu}\,\vartheta u,$$

whence also

$$\vartheta(u + m\pi + ni\mathfrak{F}),$$

where m and n are any positive or negative integers, is the product of ϑu by an exponential factor, or say simply that it is a multiple of ϑu.

Writing $u = -\tfrac{1}{2}i\mathfrak{F}$, we have $\vartheta(-\tfrac{1}{2}i\mathfrak{F}) = \vartheta(\tfrac{1}{2}i\mathfrak{F})$, that is,

$$\vartheta(\tfrac{1}{2}i\mathfrak{F}) = 0,$$

and therefore also

$$\vartheta\{m\pi + (n+\tfrac{1}{2})i\mathfrak{F}\} = 0.$$

The above properties are general, but if $\mathfrak{F}$ be real, then k, K, K', q being as in Jacobi (consequently k being real, positive, and less than 1, and K and K' real and positive), and assuming $\mathfrak{F} = \dfrac{\pi K'}{K}$, or, what is the same thing,

$$q\,(=e^{-\frac{\pi K'}{K}}) = e^{-\mathfrak{F}},$$

the function ϑ is given in terms of Jacobi's Θ by the equation $\vartheta u = \Theta\left(\dfrac{2Ku}{\pi}\right)$; or, what is the same thing, $\Theta u = \vartheta\left(\dfrac{\pi u}{2K}\right)$.

We hence at once obtain expressions of the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ in terms of ϑ, viz. these are

$$\operatorname{sn} u = \frac{-i}{\sqrt{k}}\, e^{-\frac{\pi}{4K}(K'-2iu)}\,\vartheta\left(\frac{\pi u}{2K} + \tfrac{1}{2}i\mathfrak{F}\right) \div \vartheta\left(\frac{\pi u}{2K}\right),$$

$$\operatorname{cn} u = \sqrt{\left(\frac{k'}{k}\right)}\, e^{-\frac{\pi}{4K}(K'-2iu)}\,\vartheta\left(\frac{\pi u}{2K} + \tfrac{1}{2}\pi + \tfrac{1}{2}i\mathfrak{F}\right) \div \vartheta\left(\frac{\pi u}{2K}\right),$$

$$\operatorname{dn} u = \sqrt{k'}\,\vartheta\left(\frac{\pi u}{2K} + \tfrac{1}{2}\pi\right) \div \vartheta\left(\frac{\pi u}{2K}\right).$$

Consider now the integral

$$\int_a \frac{dx}{\sqrt{\{(-)\,x-a\,.\,x-b\,.\,x-c\,.\,x-d\}}},\ = \int_a \frac{dx}{\sqrt{(X)}} \text{ suppose,}$$

where a, b, c, d are taken to be real, and in the order of increasing magnitude, viz. it is assumed that $b-a$, $c-a$, $d-a$, $c-b$, $d-b$, $d-c$ are all positive; x considered as the variable under the integral sign is always real; when it is between a and b or between c and d, X is positive, and we assume that $\sqrt{(X)}$ denotes the positive value of the radical; but if x is between b and c, X is negative, and we assume that the sign of $\sqrt{(X)}$ is taken so that $\dfrac{1}{\sqrt{(X)}}$ is equal to a positive multiple of i, and this being so the integral is taken from the inferior limit a to the superior limit x, which is real.

Take x a linear function of y, such that for

$$x = a,\ b,\ c,\ \ d,$$

$$y = 0,\ 1,\ \frac{1}{k^2},\ \infty\text{, respectively,}$$

so that, x increasing continuously from a to d, y will increase continuously from 0 to ∞. We have

$$k^2 = \frac{b-a\,.\,d-c}{d-b\,.\,c-a},$$

$$y = \frac{b-d}{b-a}\,\frac{x-a}{x-d},$$

$$1-y = \frac{d-a}{b-a}\,\frac{x-b}{x-d},$$

$$1-k^2y = \frac{d-a}{c-a}\,\frac{x-c}{x-d};$$

and, thence,

$$\sqrt{(y\,.\,1-y\,.\,1-k^2y)} = \frac{d-a}{c-a}\sqrt{\left(\frac{d-b}{c-a}\right)}\cdot\frac{\sqrt{(X)}}{(x-d)^2},$$

where $\sqrt{\left(\dfrac{d-b}{c-a}\right)}$ is taken to be positive, and the sign of $\sqrt{(X)}$ is fixed as above. Then for y between 0 and 1 or $> \dfrac{1}{k^2}$, $y\,.\,1-y\,.\,1-k^2y$ will be positive, and $\sqrt{(y\,.\,1-y\,.\,1-k^2y)}$ will also be positive; but y being between 1 and $\dfrac{1}{k^2}$, $y\,.\,1-y\,.\,1-k^2y$ will be negative, and the sign of the radical is such that $\dfrac{1}{\sqrt{(y\,.\,1-y\,.\,1-k^2y)}}$ is a positive multiple of i.

We have moreover

$$dy = \frac{d-a}{b-a}(d-b)\frac{dx}{(x-d)^2};$$

and therefore

$$\frac{dy}{\sqrt{(y\,.\,1-y\,.\,1-k^2y)}} = \sqrt{(d-b\,.\,c-a)}\,\frac{dx}{\sqrt{(X)}},$$

where $\sqrt{(d-b\,.\,c-a)}$ is positive; or, say,

$$\int_0 \frac{dy}{\sqrt{(y\,.\,1-y\,.\,1-k^2y)}} = \sqrt{(d-b\,.\,c-a)}\int_a \frac{dx}{\sqrt{(X)}}.$$

Hence, writing $y = z^2 = \text{sn}^2 u$, we have

$$2u = \sqrt{(d-b \,.\, c-a)} \int_a \frac{dx}{\sqrt{(X)}},$$

and it is to be further noticed that to

$$x = a,\ b,\ c,\ d,$$

correspond

$$\text{sn}\, u = 0,\ 1,\ \frac{1}{k},\ \infty,$$

or we may say

$$u = 0,\ \ K,\ \ K + iK',\ \ 2K + iK'.$$

Writing for shortness

$$\frac{2}{\sqrt{(d-b \,.\, c-a)}} = \alpha,$$

we have

$$\alpha u = \int_a \frac{dx}{\sqrt{(X)}};$$

and moreover

$$\alpha K = \int_a^b \frac{dx}{\sqrt{(X)}},$$

$$\alpha(K + iK') = \int_a^c \frac{dx}{\sqrt{(X)}},$$

$$\alpha(2K + iK') = \int_a^d \frac{dx}{\sqrt{(X)}},$$

or if for a moment we write

$$\int_0^a \frac{dx}{\sqrt{(X)}} = A, \text{ \&c.},$$

then these equations are

$$\alpha K = B - A,$$
$$\alpha(K + iK') = C - A,$$
$$\alpha(2K + iK') = D - A.$$

Hence $B + C - 2A = D - A$, that is, $A - B - C + D = 0$, or $B - A = D - C$, that is,

$$\int_a^b \frac{dx}{\sqrt{(X)}} = \int_c^d \frac{dx}{\sqrt{(X)}},$$

where observe as before that $x = a$ to $x = b$, or $x = c$ to $x = d$, X is positive, and the radical $\sqrt{(X)}$ is taken to be positive.

We have also

$$\alpha K = B - A = \int_a^b \frac{dx}{\sqrt{(X)}},$$

$$\alpha i K' = C - B = \int_b^c \frac{dx}{\sqrt{(X)}},$$

where, as before, from b to c, X is negative, and the sign of the radical is such that $\frac{1}{\sqrt{(X)}}$ is a positive multiple of i; the last formula may be more conveniently written

$$\alpha K' = \int_b^c \frac{dx}{\sqrt{(-X)}},$$

where, from b to c, $-X$ is positive, and $\sqrt{(-X)}$ is also taken to be positive.

Collecting the results, we have

$$\int_a \frac{dx}{\sqrt{(X)}} = \alpha u, \quad \alpha = \frac{2}{\sqrt{(d-b\,.\,c-a)}}, \quad k^2 = \frac{b-a\,.\,d-c}{d-b\,.\,c-a},$$

and also

$$k'^2 = \frac{d-a\,.\,c-b}{d-b\,.\,c-a},$$

and then conversely

$$x = \frac{a(d-b) + d(b-a)\operatorname{sn}^2 u}{(d-b)+(b-a)\operatorname{sn}^2 u};$$

or, what is the same thing,

$$\operatorname{sn}^2 u = \frac{b-d\,.\,x-a}{b-a\,.\,x-d},$$

$$\operatorname{cn}^2 u = \frac{d-a\,.\,x-b}{b-a\,.\,x-d},$$

$$\operatorname{dn}^2 u = \frac{d-a\,.\,x-c}{c-a\,.\,x-d};$$

where, in place of the elliptic functions we are to substitute their ϑ-values; it will be recollected that $\mathfrak{F}$, the parameter of the ϑ-functions, has the value

$$\mathfrak{F}\left(=\frac{\pi K'}{K}\right) = \pi \int_b^c \frac{dx}{\sqrt{(-X)}} \div \int_a^b \frac{dx}{\sqrt{(X)}},$$

and, as before,

$$K = \frac{1}{\alpha}\int_a^b \frac{dx}{\sqrt{(X)}}.$$

Hence, finally, α, k, k', K, $\mathfrak{F}$ denoting given functions of a, b, c, d, if as above

$$\int_a \frac{dx}{\sqrt{(X)}} = \alpha u,$$

we have conversely

$$\frac{b-d\,.\,x-a}{b-a\,.\,x-d} = -\frac{1}{k}\, e^{-\frac{1}{2}\mathfrak{F}+\frac{i\pi u}{2K}}\, \vartheta^2\left(\frac{\pi u}{2K} + \tfrac{1}{2}i\mathfrak{F}\right) \div \vartheta^2 \frac{\pi u}{2K},$$

$$\frac{d-a\,.\,x-b}{b-a\,.\,x-d} = \frac{k'}{k}\, e^{-\frac{1}{2}\mathfrak{F}+\frac{i\pi u}{2K}}\, \vartheta^2\left(\frac{\pi u}{2K} + \tfrac{1}{2}\pi + \tfrac{1}{2}i\mathfrak{F}\right) \div \vartheta^2 \frac{\pi u}{2K},$$

$$\frac{d-a\,.\,x-c}{c-a\,.\,x-d} = k'\, \vartheta^2\left(\frac{\pi u}{2K} + \tfrac{1}{2}\pi\right) \div \vartheta^2 \frac{\pi u}{2K},$$

which are the formulæ in question.

The problem is to obtain them (and that in the more general case where a, b, c, d have any given imaginary values) directly from the assumed equation

$$\int_a \frac{dx}{\sqrt{(X)}} = \alpha u,$$

and from the foregoing definition of the function ϑ.

It may be recalled that the function ϑu is a doubly infinite product

$$\vartheta u = \Pi\Pi \left\{1 - \frac{u}{m\pi + (n+\frac{1}{2})\, i\mathfrak{F}}\right\};$$

m and n positive or negative integers from $-\infty$ to $+\infty$; I purposely omit all further explanations as to limits; or, what is the same thing,

$$\vartheta \frac{\pi u}{2K} = \Pi\Pi \left\{1 - \frac{u}{2mK + (2n+1)\, iK'}\right\};$$

and consequently that, disregarding constant and exponential factors, the foregoing expressions of

$$\frac{b-d\,.\,x-a}{b-a\,.\,x-d}, \quad \frac{d-a\,.\,x-b}{b-a\,.\,x-d}, \quad \frac{d-a\,.\,x-c}{c-a\,.\,x-d},$$

are the squares of the expressions $\frac{X}{W}$, $\frac{Y}{W}$, $\frac{Z}{W}$, where X, Y, Z, W are respectively of the form

$$u\Pi\Pi\left\{1 + \frac{u}{(m,\ n)}\right\}, \quad \Pi\Pi\left\{1 + \frac{u}{(\overline{m},\ n)}\right\},$$

$$\Pi\Pi\left\{1 + \frac{u}{(\overline{m},\ \overline{n})}\right\}, \quad \Pi\Pi\left\{1 + \frac{u}{(m,\ \overline{n})}\right\},$$

where $(m,\ n) = 2mK + 2niK'$, and the stroke over the m or the n denotes that the $2m$ or the $2n$ (as the case may be) is to be changed into $2m+1$ or $2n+1$. But this is a transformation which has apparently no application to the ϑ-functions of more than one variable.