717.

ON THE TRIPLE THETA-FUNCTIONS.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 48—50.]

As a specimen of mathematical notation, viz. of the notation which appears to me the easiest to *read* and also to *print*, I give the definition and demonstration of the fundamental properties of the triple theta-functions.

Definition.

$$\vartheta(U,\ V,\ W) = \Sigma \exp. \Theta,$$

where

$$\Theta = (A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)^2 + 2(U,\ V,\ W)(l,\ m,\ n),$$

Σ denoting the sum in regard to all positive and negative integer values from $-\infty$ to $+\infty$ (zero included) of l, m, n respectively.

$\vartheta(U,\ V,\ W)$ is considered as a function of the arguments $(U,\ V,\ W)$, and it depends also on the parameters $(A,\ B,\ C,\ F,\ G,\ H)$.

First Property. $\vartheta(U,\ V,\ W) = 0$, for

$$U = \tfrac{1}{2}\{x\pi i + (A,\ H,\ G)(\alpha,\ \beta,\ \gamma)\},$$
$$V = \tfrac{1}{2}\{y\pi i + (H,\ B,\ F)(\alpha,\ \beta,\ \gamma)\},$$
$$W = \tfrac{1}{2}\{z\pi i + (G,\ F,\ C)(\alpha,\ \beta,\ \gamma)\},$$

x, y, z, α, β, γ being any positive or negative integer numbers, such that $\alpha x + \beta y + \gamma z$ = odd number.

Demonstration. It is only necessary to show that to each term of ϑ there corresponds a second term, such that the indices of the two exponentials differ by an odd multiple of πi.

Taking l, m, n as the integers which belong to the one term, those belonging to the other term are

$$-(l+\alpha),\quad -(m+\beta),\quad -(n+\gamma),$$

(where observe that one at least of the numbers α, β, γ being odd, this system of values is not in any case identical with l, m, n). The two exponents then are

$$\Theta, =(A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)^2+2(U,\ V,\ W)(l,\ m,\ n),$$

and

$$\Theta', =(A,\ B,\ C,\ F,\ G,\ H)(l+\alpha,\ m+\beta,\ n+\gamma)^2-2(U,\ V,\ W)(l+\alpha,\ m+\beta,\ n+\gamma);$$

viz. the value of Θ' is

$$\begin{aligned} = \quad & (A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)^2+(A,\ B,\ C,\ F,\ G,\ H)(\alpha,\ \beta,\ \gamma)^2 \\ & +2(A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)(\alpha,\ \beta,\ \gamma) \\ & -2(U,\ V,\ W)(l+\alpha,\ m+\beta,\ n+\gamma), \end{aligned}$$

and we then have

$$\begin{aligned} \Theta'-\Theta = & \ 2(A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)(\alpha,\ \beta,\ \gamma) \\ & +(A,\ B,\ C,\ F,\ G,\ H)(\alpha,\ \beta,\ \gamma)^2 \\ & -2(U,\ V,\ W)(2l+\alpha,\ 2m+\beta,\ 2n+\gamma). \end{aligned}$$

Substituting herein for U, V, W their values, the last term is

$$\begin{aligned} = & -\{(2l+\alpha)x+(2m+\beta)y+(2n+\gamma)z\} \\ & -2(A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)(\alpha,\ \beta,\ \gamma) \\ & -\ (A,\ B,\ C,\ F,\ G,\ H)(\alpha,\ \beta,\ \gamma)^2, \end{aligned}$$

and thence

$$\Theta'-\Theta=-\{(2l+\alpha)x+(2m+\beta)y+(2n+\gamma)z\}\,\pi i,$$

which proves the theorem.

As to the notation, remark that, after (A, B, C, F, G, H) has been once written out in full, we may instead of

$$(A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)^2, \text{ \&c., write } (A,\ \ldots)(l,\ m,\ n)^2, \text{ \&c.,}$$

and that we may use the like abbreviations

$(A, \ldots)(l, m, n)$, to denote $(A, H, G)(l, m, n)$ respectively,
$(H, \ldots)(l, m, n)$, „ $(H, B, F)(l, m, n)$ „ ,
$(G, \ldots)(l, m, n)$, „ $(G, F, C)(l, m, n)$ „ .

These are not only abbreviations, but they make the formulæ actually clearer, as bringing them into a smaller compass; and I accordingly use them in the demonstration which follows.

Second Property. If U_1, V_1, W_1 denote

$$U + x\pi i + (A, H, G)(\alpha, \beta, \gamma),$$
$$V + y\pi i + (H, B, F)(\alpha, \beta, \gamma),$$
$$W + z\pi i + (G, F, C)(\alpha, \beta, \gamma),$$

respectively, where $x, y, z, \alpha, \beta, \gamma$ are any positive or negative integers (zero values admissible), then

$$\vartheta(U_1, V_1, W_1) = \exp.\{-(A, B, C, F, G, H)(\alpha, \beta, \gamma)^2\}.\exp.\{-2(\alpha U + \beta V + \gamma W)\}.\vartheta(U, V, W),$$

or say

$$= \exp.\{-(A, \ldots)(\alpha, \beta, \gamma)^2\}.\exp.\{-2(\alpha U + \beta V + \gamma W)\}.\vartheta(U, V, W).$$

Demonstration. Writing $\vartheta(U_1, V_1, W_1) = \Sigma.\exp.\Theta_1$, then in the expression of Θ_1 we may in place of l, m, n write $l-\alpha, m-\beta, n-\gamma$; we thus obtain

$$\begin{aligned}\Theta_1 = (A, \ldots)(l-\alpha, m-\beta, n-\gamma)^2 + \{&(l-\alpha)[U + x\pi i + (A, \ldots)(\alpha, \beta, \gamma)]\\ &+ (m-\beta)[V + y\pi i + (H, \ldots)(\alpha, \beta, \gamma)]\\ &+ (n-\gamma)[W + z\pi i + (G, \ldots)(\alpha, \beta, \gamma)]\},\end{aligned}$$

which is

$$\begin{aligned}= \; &(A, \ldots)(l, m, n)^2\\ &+ 2(lU + mV + nW) + 2(lx + my + nz)\pi i + 2(A, \ldots)(l, m, n)(\alpha, \beta, \gamma)\\ &\qquad - 2(A, \ldots)(l, m, n)(\alpha, \beta, \gamma)\\ &- 2(\alpha U + \beta V + \gamma W) - 2(\alpha x + \beta y + \gamma z)\pi i - 2(A, \ldots)(\alpha, \beta, \gamma)^2\\ &\qquad + (A, \ldots)(\alpha, \beta, \gamma)^2,\end{aligned}$$

which is

$$\begin{aligned}= \; &(A, \ldots)(l, m, n)^2 + 2(lU + mV + nW)\\ &-(A, \ldots)(\alpha, \beta, \gamma)^2 - 2(\alpha U + \beta V + \gamma W)\\ &+ 2[(l-\alpha)x + (m-\beta)y + (n-\gamma)z]\pi i.\end{aligned}$$

Hence, rejecting the last line, which (as an even multiple of πi) leaves the exponential unaltered, we see that $\vartheta(U_1, V_1, W_1)$ is $= \vartheta(U, V, W)$ multiplied by the factor

$$\exp.\{-(A, \ldots)(\alpha, \beta, \gamma)^2\}.\exp.\{-2(\alpha U + \beta V + \gamma W)\},$$

which is the theorem in question.

In many cases a formula, which belongs to an indefinite number s of letters, is most easily intelligible when written out for three letters, but it is sometimes convenient to speak of the s letters $l, m, \ldots, n$, or even the s letters $l, \ldots, n$, and to write out the formulæ accordingly.