735.

NOTE ON THE THEORY OF APSIDAL SURFACES.

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I OBTAIN in the present Note a system of formulæ which lead very simply to the known theorem, that the apsidals of reciprocal surfaces are reciprocal; or, what is the same thing, that the reciprocal of the apsidal of a given surface is the apsidal of its reciprocal; the surfaces are referred to the same axes, and by the reciprocal is meant the reciprocal surface in regard to a sphere radius unity, having for its centre a determinate point, say the origin; and it is this same point which is used in the construction of the apsidal surfaces. The apsidal of a given surface is constructed as follows; considering the section by any plane through the fixed point, and in this section the apsidal radii from the fixed point (that is, the radii which meet the curve at right angles), then drawing a line through the fixed point at right angles to the plane, and on this line measuring off from the fixed point distances equal to the apsidal radii respectively, the locus of the extremities of these distances is the apsidal surface. We have the surface, its reciprocal, the apsidal of the surface, the apsidal of the reciprocal; and I take

$$(x,\ y,\ z),\ (x',\ y',\ z'),\ (X,\ Y,\ Z),\ (X',\ Y',\ Z')$$

for the coordinates of corresponding points on the four surfaces respectively.

The condition of reciprocity gives $xx' + yy' + zz' - 1 = 0$, and (the equations being $U = 0$, $U' = 0$) x', y', z' proportional to d_xU, d_yU, d_zU, and x, y, z proportional to $d_{x'}U'$, $d_{y'}U'$, $d_{z'}U'$; or, what is the same thing, we must have

$$x'dx + y'dy + z'dz = 0 \text{ and } xdx' + ydy' + zdz' = 0;$$

one of these is implied in the other, as appears at once by differentiating the equation $xx' + yy' + zz' - 1 = 0$.

The other two surfaces will therefore be reciprocal if only we have the like relations between the coordinates (X, Y, Z) and (X', Y', Z'); that is, if

$$XX' + YY' + ZZ' - 1 = 0,$$
$$X'dX + Y'dY + Z'dZ = 0,$$
$$XdX' + YdY' + ZdZ' = 0.$$

To find the apsidal surface, we consider an arbitrary section $x \cos\alpha + y \cos\beta + z \cos\gamma = 0$ of the surface $U = 0$, and seek to determine the apsidal radii thereof, that is, the maximum or minimum values of $R^2 = x^2 + y^2 + z^2$ when x, y, z vary subject to these two conditions. Writing x', y', z' to denote functions proportional to d_xU, d_yU, d_zU, we thus have the set of equations

$$x + \lambda x' + \mu \cos\alpha = 0,$$
$$y + \lambda y' + \mu \cos\beta = 0,$$
$$z + \lambda z' + \mu \cos\gamma = 0,$$

where λ, μ are indeterminate coefficients; taking then X, Y, Z as the coordinates of the extremity of the line drawn at right angles to the plane, we have $R^2 = X^2 + Y^2 + Z^2$, and $\cos\alpha, \cos\beta, \cos\gamma = \frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$; substituting these values in the equation

$$x \cos\alpha + y \cos\beta + z \cos\gamma = 0,$$

we have $Xx + Yy + Zz = 0$, and substituting in the other equations, and instead of λ, μ introducing the new indeterminate coefficients ρ, σ, we obtain

$$X, Y, Z = \rho x + \sigma x', \rho y + \sigma y', \rho z + \sigma z'.$$

Hence these last equations, together with

$$R^2 = X^2 + Y^2 + Z^2 = x^2 + y^2 + z^2,$$

and

$$Xx + Yy + Zz = 1,$$

contain the solution of the problem. If for convenience we introduce R'^2 to denote $x'^2 + y'^2 + z'^2$, and imagine the absolute values of x', y', z' determined so that $xx' + yy' + zz' = 1$, then substituting for X, Y, Z their values in the equations $X^2 + Y^2 + Z^2 = R^2$ and $Xx + Yy + Zz = 1$, we find

$$R^2 = \rho^2 R^2 + 2\rho\sigma + \sigma^2 R'^2,\ 0 = \rho R^2 + \sigma,$$

and thence

$$\rho^2 = \frac{1}{R^2 R'^2 - 1},\ \sigma = -\rho R^2,$$

or, finally assuming

$$\rho = \frac{1}{\sqrt{(R^2 R'^2 - 1)}},$$

we have

$$X, Y, Z = x - R^2 x', y - R^2 y', z - R^2 z',$$

each divided by

$$\sqrt{(R^2 R'^2 - 1)},$$

where I recall that x', y', z' are proportional to d_xU, d_yU, d_zU, and are such that $xx'+yy'+zz'=1$: they in fact denote

$$d_xU,\ d_yU,\ d_zU, \text{ each divided by } xd_xU+yd_yU+zd_zU;$$

and that R^2 and R'^2 denote $x^2+y^2+z^2$ and $x'^2+y'^2+z'^2$ respectively. The coordinates X, Y, Z of the point of the apsidal surface are thus determined as functions of x, y, z.

For the apsidal of the reciprocal surface, we have in like manner

$$X',\ Y',\ Z' = x'-R'^2x,\ y'-R'^2y,\ z'-R'^2z,$$

each divided by

$$-\sqrt{(R^2R'^2-1)},$$

and then the two sets of values give, not only

$$XX'+YY'+ZZ'=1,$$

as is obvious, but also

$$X'dX+Y'dY+Z'dZ=0, \text{ and } XdX'+YdY'+ZdZ'=0.$$

In fact, writing for a moment ρ, ρ' instead of R^2, R'^2, and $\sqrt{(R^2R'^2-1)}=\sqrt{(\rho\rho'-1)}$, $=\omega$, then

$$\begin{aligned}
&X'dX+Y'dY+Z'dZ\\
&\quad=\frac{x'-x\rho'}{\omega}\,d\,\frac{x-x'\rho}{\omega}+\&c.\\
&\quad=\frac{x'-x\rho'}{\omega}\left\{\frac{dx-\rho dx'-x'd\rho}{\omega}-\frac{(x-x'\rho)\,d\omega}{\omega^2}\right\}+\&c.\\
&\quad=\frac{1}{\omega^2}\{\quad x'dx+y'dy+z'dz\\
&\qquad\quad-\ \rho\ (x'dx'+y'dy'+z'dz')\\
&\qquad\quad-\ \ (x'^2+y'^2+z'^2)\,d\rho\\
&\qquad\quad-\ \rho'\,(xdx+ydy+zdz)\\
&\qquad\quad+\rho\rho'\,(xdx'+ydy'+zdz')\\
&\qquad\quad+\ \rho'\,(xx'+yy'+zz')\,d\rho\}\\
&\quad-\frac{d\omega}{\omega^3}\{\quad xx'+yy'+zz'\\
&\qquad\quad-\ \rho\ (x'^2+y'^2+z'^2)\\
&\qquad\quad-\ \rho'\,(x^2+y^2+z^2)\\
&\qquad\quad+\rho\rho'\,(xx'+yy'+zz')\},
\end{aligned}$$

or, since the terms in { } are

$$0-\rho\,.\,\tfrac{1}{2}d\rho'-\rho'd\rho-\rho'\,.\,\tfrac{1}{2}d\rho+0+\rho'd\rho,\ =-\tfrac{1}{2}(\rho d\rho'+\rho'd\rho),$$

and

$$1-\rho\rho'-\rho\rho'+\rho\rho',\ =1-\rho\rho',\ =-\omega^2,$$

this is

$$=\frac{1}{\omega^2}\{-\tfrac{1}{2}(\rho d\rho'+\rho'd\rho)+\omega\,d\omega\},\ =0,$$

in virtue of $\omega^2=\rho\rho'-1$. And similarly the other equation $XdX'+YdY'+ZdZ'=0$ might be directly verified.