466.

SECOND NOTE ON THE LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXV. (1864—1865), pp. 203—207.]

THE elliptic values of

r, the radius vector,
v, the longitude,
y, the latitude,

are functions of

a, the mean distance,
e, the excentricity,
γ, the tangent of the inclination,
l, the mean longitude,
c, the mean anomaly,
g, the mean distance from node;

see my Note in the last *Monthly Notice*, p. 182, [465], where, for the present purpose, $\frac{a}{r}$ should be written instead of $\frac{1}{r}$; and it is there shown that the disturbed values, attending only to the coefficients independent of m, are obtained by affecting a, e, γ, c, g, l, with the inequalities

$$\begin{aligned}
\delta a &= 0 \\
\delta e &= -\tfrac{5}{8}\gamma^2 e \quad \cos \quad 2c-2g \\
\delta\gamma &= +\tfrac{5}{8}\gamma e^2 \quad\ \ ,, \quad 2c-2g \\
\delta c &= +\tfrac{5}{8}\gamma^2 \quad \sin \quad 2c-2g \\
\delta g &= +\tfrac{5}{8}e^2 \quad\ \ ,, \quad 2c-2g \\
\delta l &= -\tfrac{5}{16}\gamma^2 e^2 \quad ,, \quad 2c-2g,
\end{aligned}$$

or, what is the same thing, adding to the elliptic values the inequalities

$$
\begin{array}{lll}
\delta \dfrac{a}{r} = -\tfrac{5}{4}\gamma^2 e^2 & \cos & 2g \\
\quad -\tfrac{5}{8}\gamma^2 e & ,, & c-2g, \\
\delta v = -\tfrac{5}{16}\gamma^2 e^2 & \sin & 2c \\
\quad -\tfrac{25}{16}\gamma^2 e^2 & ,, & 2g \\
\quad +\tfrac{5}{4}\gamma^2 e & ,, & c-2g \\
\quad -\tfrac{5}{16}\gamma^2 e^2 & ,, & 2c-2g, \\
\delta y = -\tfrac{5}{8}\gamma e^3 + \tfrac{5}{8}\gamma^3 e & \sin & c-g \\
\quad +\tfrac{5}{8}\gamma e^2 & ,, & 2c-g \\
\quad +\tfrac{5}{8}\gamma e^3 & ,, & 3c-g \\
\quad +\tfrac{5}{8}\gamma^3 e & ,, & c-3g.
\end{array}
$$

I propose to show how these results may be obtained by the method of the variation of the elements. For this purpose, treating a, e, γ, c, g, l, as elements, the proper formulæ are obtained very readily from those given in my "Memoir on the Problem of Disturbed Elliptic Motion," *Mem. R. Ast. Soc.*, vol. XXVII. (1859), pp. 1—29, [212]; viz., writing c in place of g, the formulæ, p. 25, give the variations of a, e, c, ϖ, θ, ϕ; we have then

$$
\begin{aligned}
g &= c + \varpi \\
l &= c + \varpi + \theta \\
\gamma &= \tan\phi,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
dg &= dc + d\varpi \\
dl &= dc + d\varpi + d\theta \\
d\gamma &= (1+\gamma^2)\, d\phi,
\end{aligned}
$$

which give for the transformation of the differential coefficients of Ω,

$$
\begin{aligned}
\frac{d\Omega}{dc} &= \frac{d\Omega}{dc} + \frac{d\Omega}{dg} + \frac{d\Omega}{dl} \\
\frac{d\Omega}{d\varpi} &= \frac{d\Omega}{dg} + \frac{d\Omega}{dl} \\
\frac{d\Omega}{d\theta} &= \frac{d\Omega}{dl} \\
\frac{d\Omega}{d\phi} &= (1+\gamma^2)\frac{d\Omega}{d\gamma}
\end{aligned}
$$

and the formulæ finally become

$$
\frac{da}{dt} = \frac{2}{na}\frac{d\Omega}{dc} + \frac{2}{na}\frac{d\Omega}{dg} + \frac{2}{na}\frac{d\Omega}{dl},
$$

$$
\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{d\Omega}{dc} + \frac{1-e^2-\sqrt{1-e^2}}{na^2 e}\frac{d\Omega}{dg} + \frac{1-e^2-\sqrt{1-e^2}}{na^2 e}\frac{d\Omega}{dl},
$$

$$\frac{d\gamma}{dt} = \frac{1+\gamma^2}{na^2\sqrt{1-e^2}\,\gamma}\frac{d\Omega}{dg} + \frac{(1+\gamma^2)(1-\sqrt{1+\gamma^2})}{na^2\sqrt{1-e^2}\,\gamma}\frac{d\Omega}{dl},$$

$$\frac{dc}{dt} = -\frac{2}{na}\frac{d\Omega}{da} - \frac{1-e^2}{na^2e}\frac{d\Omega}{de},$$

$$\frac{dg}{dt} = -\frac{2}{na}\frac{d\Omega}{da} - \frac{1-e^2-\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{de} - \frac{1+\gamma^2}{na^2\sqrt{1-e^2}\,\gamma}\frac{d\Omega}{d\gamma},$$

$$\frac{dl}{dt} = -\frac{2}{na}\frac{d\Omega}{da} - \frac{1-e^2-\sqrt{1+e^2}}{na^2e}\frac{d\Omega}{de} - \frac{(1+\gamma^2)(1-\sqrt{1+\gamma^2})}{na^2\sqrt{1-e^2}\,\gamma}\frac{d\Omega}{d\gamma}.$$

The disturbing function contains the term

$$m^2n^2a^2\left(+\tfrac{15}{16}e^2\gamma^2\right)\ \cos\ 2c-2g.$$

If after the differentiations we write for greater simplicity $a=1$, $n=1$, we have

$$\frac{d\Omega}{da} = +\tfrac{15}{8}m^2e^2\gamma^2\ \cos\ 2c-2g,$$

$$\frac{d\Omega}{de} = +\tfrac{15}{8}m^2e\,\gamma^2\ \ ,,\ \ 2c-2g,$$

$$\frac{d\Omega}{d\gamma} = +\tfrac{15}{8}m^2e^2\gamma\ \ ,,\ \ 2c-2g,$$

$$\frac{d\Omega}{dc} = -\tfrac{15}{8}m^2e^2\gamma^2\ \sin\ 2c-2g,$$

$$\frac{d\Omega}{dg} = -\tfrac{15}{8}m^2e^2\gamma^2\ \ ,,\ \ 2c-2g,$$

$$\frac{d\Omega}{dl} = 0,$$

and the formulæ for the variations give

$$\frac{da}{dt} = 2\left(\frac{d\Omega}{dc}+\frac{d\Omega}{dg}\right) = 0$$

$$\frac{de}{dt} = \frac{1}{e}\frac{d\Omega}{dc} = -\tfrac{15}{8}m^2e\gamma^2\ \sin\ 2c-2g,$$

$$\frac{d\gamma}{dt} = \frac{1}{\gamma}\frac{d\Omega}{dg} = -\tfrac{15}{8}m^2e^2\gamma\ \ ,,\ \ 2c-2g,$$

$$\frac{dc}{dt} = -\frac{1}{e}\frac{d\Omega}{de} = -\tfrac{15}{8}m^2\gamma^2\ \cos\ 2c-2g,$$

$$\frac{dg}{dt} = -\frac{1}{\gamma}\frac{d\Omega}{d\gamma} = -\tfrac{15}{8}m^2e^2\ \ ,,\ \ 2c-2g,$$

$$\frac{dl}{dt} = -2\frac{d\Omega}{da} + \tfrac{1}{2}e\frac{d\Omega}{de} + \tfrac{1}{2}\gamma\frac{d\Omega}{d\gamma} = \left(-\tfrac{15}{4}+\tfrac{15}{16}+\tfrac{15}{16}=\right) - \tfrac{15}{8}m^2e^2\gamma^2\ \ ,,\ \ 2c-2g,$$

but this value of $\frac{dl}{dt}$ is, as will presently be seen, incomplete.

Writing $a+\delta a$, $e+\delta e$, &c., in place of a, e, &c., and observing that the divisor for the integration of the term in $2c-2g$ is $2(c-g)$, $=-3m^2$, the first five equations give respectively

$$\begin{aligned}\delta a &= 0,\\ \delta e &= -\tfrac{5}{8}\gamma^2 e \quad \cos \quad 2c-2g,\\ \delta\gamma &= +\tfrac{5}{8}\gamma e^2 \quad \text{,,} \quad 2c-2g,\\ \delta c &= +\tfrac{5}{8}\gamma^2 \quad \sin \quad 2c-2g,\\ \delta g &= +\tfrac{5}{8}c^2 \quad \text{,,} \quad 2c-2g.\end{aligned}$$

The constant term in Ω is

$$= m^2 n^2 a^2 \left(\tfrac{1}{4}+\tfrac{3}{8}e^2-\tfrac{3}{8}\gamma^2\right),$$

and this gives in

$$\frac{dl}{dt},\ =-2\ \frac{d\Omega}{da}+\tfrac{1}{2}e\frac{d\Omega}{de}+\tfrac{1}{2}\gamma\frac{d\Omega}{d\gamma},$$

a term

$$m^2\left(-1-\tfrac{3}{2}e^2+\tfrac{3}{2}\gamma^2 + \tfrac{3}{8}e^2-\tfrac{3}{8}\gamma^2\right),$$

which is

$$= m^2\left(-1-\tfrac{9}{8}e^2+\tfrac{9}{8}\gamma^2\right).$$

Substituting for e, γ, their correct values $e+\delta e$, $\gamma+\delta\gamma$, it appears that $\dfrac{dl}{dt}$ contains the term

$$m^2\left(-\tfrac{9}{4}e\delta e+\tfrac{9}{4}\gamma\delta\gamma\right),$$

which is

$$\begin{aligned}&= m^2\left(\tfrac{45}{32}+\tfrac{45}{32}=\right)\tfrac{45}{16}\quad e^2\gamma^2 \quad \cos \quad 2c-2g,\\ &= \tfrac{45}{16}m^2e^2\gamma^2 \quad \text{,,} \quad 2c-2g,\end{aligned}$$

and joining to this the before-mentioned term

$$= -\tfrac{15}{8}m^2e^2\gamma^2 \quad \text{,,} \quad 2c-2g,$$

we find

$$\frac{dl}{dt} = \left(\tfrac{45}{16}-\tfrac{15}{8}=\right)\tfrac{15}{16}m^2e^2\gamma^2 \quad \text{,,} \quad 2c-2g,$$

whence, writing as above $l+\delta l$ for l, and integrating, we have

$$\delta l = -\tfrac{5}{16}\quad e^2\gamma^2 \quad \sin \quad 2c-2g,$$

and it thus appears that the values of δa, δe, $\delta\gamma$, δc, δg, δl, agree with those obtained in my former Note.