472.

NOTE ON LAMBERT'S THEOREM FOR ELLIPTIC MOTION.

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CONSIDER any two positions, A, B, in an elliptic orbit, focus S, and semi-axis major $=a$; then if ρ, ρ', c denote the radius vectors SA, SB, and the chord AB respectively, and if $P, =\dfrac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$, be the periodic time, the time of passage from A to B is given by the formula

$$\text{Time } AB = \frac{P}{2\pi}(\chi - \chi' - \sin\chi + \sin\chi')$$

where

$$2a\cos\chi = 2a - \rho - \rho' - c,\quad 2a\cos\chi' = 2a - \rho - \rho' + c.$$

To fix the ideas we may consider the time of passage as being in every case positive; and, for Time AB, the motion from A as being towards the apocentre; Time BA will, of course, in like manner denote that the motion from B is towards the apocentre; and we thus have according to the positions of A, B, either Time $AB =$ Time BA; or else Time $AB +$ Time $BA = P$.

This being so (see the *Theoria Motus*, p. 120), χ will be always a positive arc between 0 and 360°; χ' a positive or negative arc between 0 and $\pm 180°$; and moreover χ' will be positive or negative according as the described focal angle is $< 180°$ or $> 180°$; whence, $\cos\chi'$ being known, the arc χ' is determined without ambiguity.

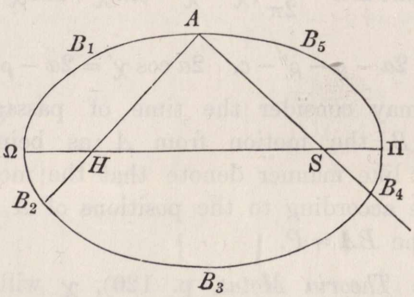
But as noticed in the place referred to, there is when only ρ, ρ', c, a, are known, a real ambiguity as regards the arc χ; viz. χ may be either the arc $>180°$ or the arc $<180°$, having for its cosine the given value of $\cos\chi$. For, given the points S, A, B, and the semi-axis major a, there exist two elliptic orbits determined by these data; and the two values of χ correspond to the times of passage between A and B, in these two orbits respectively. If, however, the actual orbit be given, there is no longer any real ambiguity; and it must be possible to decide between the two values of χ: the criterion is, in fact, a very simple one, viz. drawing a chord from A through the other focus H of the ellipse, this either separates, or it does not separate, B from the force-focus S; and I say that in the expression of Time AB, in the former case (viz. when chord A is a separator) we have $\chi < 180°$; in the latter case (viz. when chord A is not a separator) we have $\chi > 180°$.

It of course follows that, in the case of transition, when the line AB passes through H, we must have $\chi = 180°$: this is at once seen to be so; for $\chi = 180°$ gives the condition $4a = \rho + \rho' + c$; but if σ, σ', are the distances of AB from the focus H, then $2a = \rho + \sigma$, $2a = \rho' + \sigma'$, and the condition becomes $\sigma + \sigma' = c$; that is AB must pass through H.

As a verification of the new criterion, I consider the point A as having a fixed position on the orbit, but the point B as having successively different positions; and writing down the two formulæ

$$\text{Time } AB = \chi - \chi' - \sin\chi + \sin\chi',$$
$$\text{Time } BA = \omega - \omega' - \sin\omega + \sin\omega',$$

(where for simplicity the constant factor $P \div 2\pi$ is omitted) I proceed to compare these for different positions of the point B. We have, in every case, $\cos\omega = \cos\chi$, and $\cos\omega' = \cos\chi'$; whence (χ, ω being each positive and less than 360°) $\omega = \chi$ or else $\omega + \chi = 360°$, viz. the former equation subsists if ω, χ, are each less or each greater than 180°, the latter if the one is greater, the other less than 180°. And again (χ', ω' being each less than $\pm 180°$) we have $\omega' = \chi'$, or else $\omega' = -\chi'$, according as ω', χ' have the same or opposite signs.



Now in the figure, suppose that B occupies successively the different positions $B_1, B_2, \ldots B_5$, the criteria for χ, χ' (or ω, ω') give as follows,

	Ch. A.	Ch. B.	therefore			$\angle AS$	$\angle BSA$			
1	sep.	not	$\chi < 180°$	$\omega > 180°$	or $\omega + \chi = 2\pi$	$< 180°$	$> 180°$	$\chi' = +$	$\omega' = -$	or $\omega' = -\chi'$,
2	sep.	sep.	$<$	$<$	„ $\omega = \chi$	$<$	$<$	$+$	$+$	„ $\omega' = \chi'$,
3	not	not	$>$	$>$	„ $\omega = \chi$	$>$	$>$	$+$	$+$	„ $\omega' = \chi'$,
4	not	not	$>$	$>$	„ $\omega = \chi$	$>$	$>$	$-$	$-$	„ $\omega' = \chi'$,
5	not	sep.	$>$	$<$	„ $\omega + \chi = 2\pi$	$>$	$<$	$-$	$-$	„ $\omega' = -\chi'$.

Hence substituting for ω, ω' their values in terms of χ, χ', we have

$$\begin{aligned}
\text{Time } AB_{1\ldots5} &= \chi - \chi' - \sin\chi + \sin\chi', \\
\text{,, } BA_1 &= 2\pi - \chi + \chi' + \sin\chi - \sin\chi', \\
\text{,, } BA_2 &= \chi - \chi' - \sin\chi + \sin\chi', \\
\text{,, } BA_3 &= \chi - \chi' - \sin\chi + \sin\chi', \\
\text{,, } BA_4 &= \chi - \chi' - \sin\chi + \sin\chi', \\
\text{,, } BA_5 &= 2\pi - \chi + \chi' + \sin\chi - \sin\chi';
\end{aligned}$$

and thence (restoring the omitted factor $P \div 2\pi$)

$$\begin{aligned}
\text{Time } AB_1 + \text{Time } BA_1 &= P, \\
\text{,, } AB_2 - \text{ ,, } BA_2 &= 0, \\
\text{,, } AB_3 - \text{ ,, } BA_3 &= 0, \\
\text{,, } AB_4 - \text{ ,, } BA_4 &= 0, \\
\text{,, } AB_5 + \text{ ,, } BA_5 &= P,
\end{aligned}$$

which are the relations which in fact subsist between the times AB_1 and BA_1 &c.