474.

ON THE GEOMETRICAL THEORY OF SOLAR ECLIPSES.

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THE fundamental equation in a solar eclipse is, I think, most readily established as follows:

Take the centre of the Earth for origin, and consider a set of axes fixed in the Earth and moveable with it; viz., the axis of z directed towards the North Pole; those of x, y, in the plane of the Equator; the axis of x directed towards the point longitude 0°; that of y towards the point longitude 90° W. of Greenwich. Take a, b, c, for the coordinates of the Moon; k for its radius (assuming it to be spherical); a', b', c', for the coordinates of the Sun; k' for its radius (assuming it to be spherical); then, writing $\theta + \phi = 1$, the equation

$$\{\theta(x-a)+\phi(x-a')\}^2+\{\theta(y-b)+\phi(y-b')\}^2+\{\theta(z-c)+\phi(z-c')\}^2=(\theta k \pm \phi k')^2$$

is the equation of the surface of the Sun or Moon, according as $\theta, \phi = 1, 0$ or $=0, 1$: and for any values whatever of θ, ϕ, it is that of a variable sphere, such that the whole series of spheres have a common tangent cone. Writing the equation in the form

$$\begin{aligned}&\theta^2\{(x-a)^2+(y-b)^2+(z-c)^2-k^2\}\\ &+2\theta\phi\{(x-a)(x-a')+(y-b)(y-b')+(z-c)(z-c')\mp kk'\}\\ &+\ \phi^2\{(x-a')^2+(y-b')^2+(z-c')^2-k'^2\}=0,\end{aligned}$$

or, putting for shortness,

$$\begin{aligned}\rho &= a^2+b^2+c^2-k^2\\ \rho' &= a'^2+b'^2+c'^2-k'^2\\ \sigma &= aa'+bb'+cc'\mp kk'\\ P &= ax+by+cz\\ P' &= a'x+b'y+c'z,\end{aligned}$$

the equation is

$$\begin{aligned}&\theta^2\ (x^2+y^2+z^2-2P+\rho)\\ &+2\theta\phi\ (x^2+y^2+z^2-P-P'+\sigma)\\ +\ &\phi^2\,(x^2+y^2+z^2-2P'+\rho')=0,\end{aligned}$$

and the equation of the envelope consequently is

$$(x^2+y^2+z^2-2P+\rho)(x^2+y^2+z^2-2P'+\rho')-(x^2+y^2+z^2-P-P'+\sigma)^2=0,$$

that is

$$(x^2+y^2+z^2)(\rho+\rho'-2\sigma)-(P-P')^2-2(\rho'-\sigma)P-2(\rho-\sigma)P'+\rho\rho'-\sigma^2=0,$$

which is the equation of the cone in question.

Observe that one sphere of the series is a *point*, viz., taking first the upper signs if we have $\theta k+\phi k'=0$, that is

$$\theta=\frac{k'}{k'-k},\quad \phi=\frac{-k}{k'-k},$$

then the sphere in question is the point the coordinates whereof are

$$x=\frac{k'a-ka'}{k'-k},\quad y=\frac{k'b-kb'}{k'-k},\quad z=\frac{k'c-kc'}{k'-k},$$

which point is the vertex of the cone: it hence appears that, taking the upper signs, the cone is the *umbral* cone, having its vertex on this side of the Moon; and similarly taking the lower signs, then if we have $\theta k-\phi k'=0$, that is

$$\theta=\frac{k'}{k'+k},\quad \phi=\frac{k}{k'+k},$$

then the variable sphere will be the point the coordinates of which are

$$\frac{k'a+ka'}{k'+k},\quad \frac{k'b+kb'}{k'+k},\quad \frac{k'c+kc'}{k'+k},$$

which point is the vertex of the cone; viz. the cone is here the penumbral cone having its vertex between the Sun and Moon.

Taking as unity the Earth's equatorial radius, if p, p' are the parallaxes, κ, κ' the angular semi-diameters of the Moon and Sun respectively, then the distances are $\frac{1}{\sin p}$, $\frac{1}{\sin p'}$ and the radii are $\frac{\sin\kappa}{\sin p}$, $\frac{\sin\kappa'}{\sin p'}$ respectively; hence, if h, h' are the hour-angles west from Greenwich, Δ, Δ' the N.P.D.'s of the Moon and Sun respectively, we have

$$a=\frac{1}{\sin p}\sin\Delta\cos h,\qquad a'=\frac{1}{\sin p'}\sin\Delta'\cos h',$$

$$b=\frac{1}{\sin p}\sin\Delta\sin h,\qquad b'=\frac{1}{\sin p'}\sin\Delta'\sin h',$$

$$c=\frac{1}{\sin p}\cos\Delta\qquad,\qquad c'=\frac{1}{\sin p'}\cos\Delta',$$

$$k=\frac{\sin\kappa}{\sin p}\qquad,\qquad k'=\frac{\sin\kappa'}{\sin p'};$$

and thence

$$\rho = \frac{1}{\sin^2 p}(1 - \sin^2 \kappa'),$$

$$\rho' = \frac{1}{\sin^2 p'}(1 - \sin^2 \kappa'),$$

$$\sigma = \frac{1}{\sin p \sin p'}[\cos \Delta \cos \Delta' + \sin \Delta \sin \Delta' \cos (h' - h) \mp \sin \kappa \sin \kappa'],$$

$$P = \frac{1}{\sin p}\{\sin \Delta \ (x \cos h + y \sin h) + z \cos \Delta\},$$

$$P' = \frac{1}{\sin p'}\{\sin \Delta' (x \cos h' + y \sin h') + z \cos \Delta'\}.$$

Moreover, if the right ascensions of the Moon and Sun are α, α' respectively, and if the R.A. of the meridian of Greenwich (or sidereal time in angular measure) be $= \Sigma$, then we have

$$h = \Sigma - \alpha, \quad h' = \Sigma - \alpha'.$$

It is to be observed that $h - h'$, Δ, Δ' are slowly varying quantities, viz., their variation depends upon the variation of the celestial positions of the Sun and Moon; but h and h' depend on the diurnal motion, thus varying about 15° per hour; to put in evidence the rate of variation of the several angles h, h', Δ, Δ' during the continuance of the eclipse, instead of the foregoing values of h, h', I write

$$h' = \left\{E + \left(1 + \frac{E_1 - E}{24}\right)t\right\} 15^\circ,$$

where t is the Greenwich mean time, E, E_1 are the values (reckoned in parts of an hour) of the Equation of Time at the preceding and following mean noons respectively, taken positively or negatively, so that E, E_1 are the mean times of the two successive apparent noons respectively; whence also

$$h = \left\{E + \left(1 + \frac{E_1 - E}{24}\right)t\right\} 15^\circ - \alpha + \alpha';$$

and moreover

$$\begin{aligned} \alpha &= A + m\ (t - T), \\ \alpha' &= A + m' (t - T), \\ \Delta &= D + n\ \ (t - T), \\ \Delta' &= D' + n'\ (t - T), \end{aligned}$$

if T be the time of conjunction, A, A, D, D' the values at that instant of the R.A.'s and N.P.D.'s; m, m' and n, n' the horary motions in R.A. and N.P.D. respectively.

It appears to me not impossible but that the foregoing form of equation,

$$(x^2+y^2+z^2)(\rho+\rho'-2\sigma)-(P-P')^2-2(\rho'-\sigma)P-2(\rho-\sigma)P'+\rho\rho'-\sigma^2=0,$$

for the umbral or penumbral cone might present some advantage in reference to the calculation of the phenomena of an eclipse over the Earth generally: but in order to obtain in the most simple manner the equation of the same cone referred to a set of principal axes, I proceed as follows:

Writing

$$\begin{aligned} \mathrm{a} &= bc'-b'c, & \mathrm{f} &= a-a', \\ \mathrm{b} &= ca'-c'a, & \mathrm{g} &= b-b', \\ \mathrm{c} &= ab'-a'b, & \mathrm{h} &= c-c', \end{aligned}$$

(and therefore $\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0$).

Then, if

$$X=\frac{(\mathrm{bh}-\mathrm{cg})\,x+(\mathrm{cf}-\mathrm{ah})\,y+(\mathrm{ag}-\mathrm{bf})\,z}{\sqrt{\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2}\,\sqrt{\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2}},$$

$$Y=\frac{\mathrm{a}x+\mathrm{b}y+\mathrm{c}z}{\sqrt{\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2}},$$

$$Z=\frac{\mathrm{f}x+\mathrm{g}y+\mathrm{h}z}{\sqrt{\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2}},$$

X, Y, Z, will be coordinates referring to a new set of rectangular axes; viz., the origin is, as before, at the centre of the Earth, the axis of Z is parallel to the line joining the centres of the Sun and Moon; the axis of X cuts at right angles the last-mentioned line; and the axis of Y is perpendicular to the plane of the other two axes; or, what is the same thing, to the plane through the centres of the Earth, Sun, and Moon.

The coordinates of the vertex of the cone are therefore $X_\circ$, $Y_\circ$, $Z_\circ$, where these denote what the foregoing values of X, Y, Z, become on substituting therein for x, y, z, the values

$$\frac{k'a\mp ka'}{k'\mp k},\quad \frac{k'b\mp kb'}{k'\mp k},\quad \frac{k'c\mp kc'}{k'\mp k},$$

and the equation of the cone therefore is

$$(X-X_\circ)^2+(Y-Y_\circ)^2=\tan^2\lambda\,(Z-Z_\circ)^2,$$

where

$$\sin\lambda=\frac{k'\mp k}{G},$$

if for a moment G denotes the distance between the centres of the Sun and Moon. We have therefore

$$\tan\lambda = \frac{k' \mp k}{\sqrt{G^2 - (k' \mp k)^2}},$$

or since

$$G^2 = (a'-a)^2 + (b'-b)^2 + (c'-c)^2,$$

this is in fact

$$\tan\lambda = \frac{k' \mp k}{\sqrt{\rho + \rho' - 2\sigma}},$$

where ρ, ρ', σ signify as before; and thus $X_\circ$, $Y_\circ$, $Z_\circ$, $\tan\lambda$ are all of them given functions of a, b, c, k, a', b', c', k', and consequently of the before-mentioned astronomical data of the problem. The form is substantially the same as Bessel's equation (3), *Ast. Nach.* No. 321 (1837), (but the direction of the axes of X, Y is not identical with those of his x, y); and it is therefore unnecessary to consider here the application of it to the calculation of the eclipse for a given point on the Earth.