485.

PROBLEMS AND SOLUTIONS.

[From the *Mathematical Questions with their Solutions from the Educational Times*, vols. V. to XII. (1866—1869).]

[Vol. V., January to July, 1866, p. 17.]

1791. (PROPOSED by Professor CAYLEY.)—Given a quartic curve $U=0$, to find three cubic curves $P=0$, $Q=0$, $R=0$, each meeting the quartic in the same six points 1, 2, 3, 4, 5, 6, and such that $P=0$, $R=0$ may besides meet the quartic in the same three points a, b, c, and that $Q=0$, $R=0$ may besides meet the quartic in the same three points α, β, γ.

[Vol. V. pp. 25, 26.]

Note on the Problems in regard to a Conic defined by five Conditions of Intersection.

I use the word "intersection" rather than "contact" because it extends to the case of a 1-pointic intersection, which cannot be termed a contact. The conditions referred to are that the conic shall have with a given curve, at a point given or not given, a 1-pointic intersection, a 2-pointic intersection (= ordinary contact), a 3-pointic intersection, &c., as the case may be. It may be noticed that when the point on the curve is a given point, the condition of a k-pointic intersection is really only the condition that the conic shall pass through k given points; though from the circumstance that these are consecutive points on a conic, the formulæ for a conic passing through k discrete points require material alteration; for instance, in the two questions to find the equation of a conic passing through five given points, and to find the equation of a conic having at a given point of a given curve 5-pointic intersection with the curve, the forms of the solutions are very different from each other.

The foregoing remark shows, however, that it is proper to detach the conditions which relate to intersections at given points; and consequently attending only to the conditions which relate to intersection at an unascertained point (of course the intersections referred to must be at least 2-pointic, for otherwise there is no condition at all) we may consider the conics which pass through four points and satisfy one condition; or which pass through three points and satisfy two conditions; or which pass through two points and satisfy three conditions; or which pass through one point and satisfy four conditions; or which satisfy five conditions. Considering in particular the last case, let 1 denote that the conic has 2-pointic intersection, 2 that it has 3-pointic intersection, ... 5 that it has 6-pointic intersection with a given curve at an unascertained point.

Then the problems are in the first instance

5 ; 4, 1 ; 3, 2 ; 3, 1, 1 ; 2, 2, 1 ; 2, 1, 1, 1 ; 1, 1, 1, 1, 1.

But the intersections may be intersections with the same given curve or with different given curves; and we have thus in all 27 problems, viz. these are as given in the following table, where the colons (:) separate those conditions which refer to different curves:

No. of Prob.	Conditions.	No. of Prob.	Conditions.	No. of Prob.	Conditions.
1	5	10	3, 1 : 1	19	3 : 1 : 1
2	4, 1	11	3 : 1, 1	20	2 : 2 : 1
3	3, 2	12	2, 2 : 1	21	2, 1 : 1 : 1
4	3, 1, 1	13	2, 1 : 2	22	2 : 1, 1 : 1
5	2, 2, 1	14	2, 1, 1 : 1	23	1, 1, 1 : 1 : 1
6	2, 1, 1, 1	15	2, 1 : 1, 1	24	1, 1 : 1, 1 : 1
7	1, 1, 1, 1, 1	16	2 : 1, 1, 1	25	2 : 1 : 1 : 1
8	4 : 1	17	1, 1, 1, 1 : 1	26	1, 1 : 1 : 1 : 1
9	3 : 2	18	1, 1, 1 : 1, 1	27	1 : 1 : 1 : 1 : 1

Thus Problem 1 is to find a conic having 6-pointic intersection with a given curve; Problem 2 a conic having 5-pointic intersection and also 2-pointic intersection with a given curve... Problem 7 is to find a conic having five 2-pointic intersections with (touching at five distinct points) a given curve....Problem 27 is to find a conic having 2-pointic intersection with (touching) each of five given curves. Or we may in each case take the problem to be merely to find the number of the conics which satisfy the required conditions. This number is known in Prob. 1, for the case of a curve of the order m without singularities, viz. the number is $=m(12m-27)$. It is also known in Problems 25 and 26 in the case where the first curve (that to which the symbol 2, or 1, 1 relates) is a curve without singularities; and it is known in Prob. 27, viz. if m, n, p, q, r be the orders and M, N, P, Q, R the classes of the five curves respectively, then the number is $=(M, m)(N, n)(P, p)(Q, q)(R, r)\{1, 2, 4, 4, 2, 1\}$, that is, $1\,MNPQR + 2\,\Sigma MNPQ\,r + \&c.$ The number is not, I believe, known in any other of the problems. In particular, (Prob. 7) we do not as yet know the number of the conics which touch a given curve at five points. It would be interesting to obtain this number; but (judging from the analogous question of finding the double tangents of a curve) the problem is probably a very difficult one.

[Vol. v. p. 37.]

1857. (Proposed by Professor CAYLEY.)—If for shortness we put

$$P = x^3 + y^3 + z^3,\quad Q = yz^2 + y^2z + zx^2 + z^2x + xy^2 + x^2y,\quad R = xyz,$$

$$P_0 = a^3 + b^3 + c^3,\quad Q_0 = bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b,\quad R_0 = abc;$$

then (α, β, γ) being arbitrary, show that the cubic curves $\begin{vmatrix} \alpha, & \beta, & \gamma \\ P, & Q, & R \\ P_0, & Q_0, & R_0 \end{vmatrix} = 0$ pass all of them through the same nine points, lying six of them upon a conic and three of them upon a line; and find the equations of the conic and line, and the coordinates of the nine points of intersection; find also the values of $(\alpha : \beta : \gamma)$ in order that the cubic curve may break up into the conic and line.

[Vol. v. p. 37.]

1730. (Proposed by Professor CAYLEY.)—Show that (I) the condition in order that the roots k_1, k_2, k_3 of the equation

$$\gamma k^3 + (-g - \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \tfrac{3}{2}\gamma)\,k^2 + (-g - \tfrac{3}{2}\alpha - \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma)\,k - \alpha = 0 \qquad \text{(A)}$$

may be connected by a relation of the form

$$k_3(k_1 - k_2) - (k_2 - k_3) = 0, \qquad (1)$$

and (II) the result of the elimination of a, b, c from the equations

$$a^2(b + c) = -2\alpha, \qquad (2)$$

$$b^2(c + a) = 2\beta, \qquad (3)$$

$$c^2(a + b) = -2\gamma, \qquad (4)$$

$$(b - c)(c - a)(a - b) = -4g, \qquad (5)$$

are each

$$4(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,g^3 + 4(-\Sigma\alpha^3\beta + 4\Sigma\alpha^2\beta^2 - 2\Sigma\alpha^2\beta\gamma)\,g^2$$
$$+ (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,g + 2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = 0. \qquad \text{(B)}$$

[Vol. v. pp. 38, 39.]

1834. (Proposed by Professor CAYLEY.)—1. It is required to find on a given cubic curve three points A, B, C, such that, writing $x=0$, $y=0$, $z=0$ for the equations of the lines BC, CA, AB respectively, the cubic curve may be transformable into itself by the inverse substitution $(\alpha x^{-1}, \beta y^{-1}, \gamma z^{-1})$ in place of x, y, z respectively, α, β, γ being disposable constants.

2. In the cubic curve $ax(y^2+z^2)+by(z^2+x^2)+cz(x^2+y^2)+2lxyz=0$ the inverse points (x, y, z) and (x^{-1}, y^{-1}, z^{-1}) are corresponding points (that is, the tangents at these two points meet on the curve).

Solution by the PROPOSER, S. ROBERTS, M.A., *and others.*

Since the points A, B, C are on the curve, the equation is of the form

$$fy^2z+\quad gz^2x+\quad hx^2y+\quad iyz^2+\quad jzx^2+\quad kxy^2+2lxyz=0;$$

hence this equation must be equivalent to

$$\frac{f\beta^2\gamma}{y^2z}+\frac{g\gamma^2\alpha}{z^2x}+\frac{h\alpha^2\beta}{x^2y}+\frac{i\beta\gamma^2}{yz^2}+\frac{j\gamma\alpha^2}{zx^2}+\frac{k\alpha\beta^2}{xy^2}+\frac{2l\alpha\beta\gamma}{xyz}=0,$$

or,

$$j\frac{\alpha}{\beta}y^2z+k\frac{\beta}{\gamma}z^2x+i\frac{\gamma}{\alpha}x^2y+h\frac{\alpha}{\gamma}yz^2+f\frac{\beta}{\alpha}zx^2+g\frac{\gamma}{\beta}xy^2+2lxyz=0,$$

which will be the case if

$$f=j\frac{\alpha}{\beta},\quad g=k\frac{\beta}{\gamma},\quad h=i\frac{\gamma}{\alpha},\quad i=h\frac{\alpha}{\gamma},\quad j=f\frac{\beta}{\alpha},\quad k=g\frac{\gamma}{\beta}.$$

This implies $fgh=ijk$; and if this condition be satisfied, then $\alpha:\beta:\gamma$ can be determined, viz. we have $\alpha:\beta:\gamma=if:ij:hf$, which satisfy the remaining equations, so that the only condition is $fgh=ijk$.

Writing in the equation of the curve $x=0$, we find $fy^2z+iyz^2=0$, that is, the line $x=0$ meets the curve in the points $(x=0,\ y=0)$, $(x=0,\ z=0)$, and $(x=0,\ fy+iz=0)$. We have thus on the curve the three points

$$(x=0,\ fy+iz=0),\quad (y=0,\ gz+jx=0),\quad (z=0,\ hx+ky=0),$$

and in virtue of the assumed relation $fgh=ijk$, these three points lie in a line. Hence the points A, B, C must be such that BC, CA, AB respectively meet the curve in points A', B', C', which three points lie in a line; that is, we have a quadrilateral whereof the six angles A, B, C, A', B', C' all lie on the curve. It is well known that the opposite angles A and A', B and B', C and C' must be *corresponding points*, that is, points the tangents at which meet on the curve. And conversely taking A, C any two points on the curve, A' a corresponding point to A (any one of the four corresponding points), then AC, $A'C$ will meet the curve in the corresponding points B', B; and AB, $A'B'$ will meet on the curve in a point C' corresponding to C, giving the inscribed quadrilateral (A, B, C, A', B', C'); the triangle ABC is therefore constructed.

It is to be remarked that the equation $fgh = ijk$ being satisfied, we may without any real loss of generality write $f = j$, $g = k$, $h = i$, and therefore $\alpha = \beta = \gamma$; hence changing the constants we have the theorem: the inverse points (x, y, z), (x^{-1}, y^{-1}, z^{-1}) are corresponding points on the curve

$$ax(y^2 + z^2) + by(z^2 + x^2) + cx(x^2 + y^2) + 2lxyz = 0.$$

[Vol. v. pp. 57, 58.]

Addition to the Note on the Problems in regard to a Conic defined by five Conditions of Intersection.

Since writing the Note in question, I have found that a solution of Problem 7 has been given by M. De Jonquières in the paper "Du Contact des Courbes Planes, &c.," *Nouvelles Annales de Mathématiques*, vol. III. (1864), pp. 218—222: viz. the number of conics which touch a curve of the order n in five distinct points is stated to be

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(n^5 + 15n^4 - 55n^3 - 495n^2 + 1584n + 15).$$

There are given also the following results; the number of conics which pass through two given points and touch a curve of the order n in three distinct points is

$$\frac{n(n-1)(n-2)}{2}(n^3 + 6n^2 - 19n - 12),$$

and the number of conics which pass through a given point and touch a curve of the order n in four distinct points is

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(n^4 + 10n^3 - 37n^2 - 118n + 282).$$

These formulæ are given without demonstration, and with an expression of doubt as regards their exactness—("elles sont exactes, je crois"); they apply, of course, to a curve of the order n without singularities; but assuming them to be accurate, the means exist for adapting them to the case of a curve with singularities.

[There is also a paper on the same subject in the *Annales* for January, 1866 (pp. 17—20), from the Editor's *Note* to which we have introduced a correction (+ 15 instead of − 35) in the formula given above.]

[Vol. v. pp. 58, 59.]

1876. (Proposed by R. BALL, M.A.)—If three of the roots of the equation $(a, b, c, d, e \between x, 1)^4 = 0$ be in arithmetical progression, show that

$$55296\, H^3J - 2304\, aH^2I^2 - 16632\, a^2HIJ + 625\, a^3I^3 - 9261\, a^3J^2 = 0,$$

where

$$H = ac - b^2, \quad I = ae - 4bd + 3c^2, \quad J = ace + 2bcd - ad^2 - b^2e - c^3.$$

Solution by PROFESSOR CAYLEY.

Write $(a, b, c, d, e \between x, 1)^4 = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$; then putting for a moment $\beta+\gamma+\delta = p$, $\beta\gamma+\beta\delta+\gamma\delta = q$, $\beta\gamma\delta = r$, and forming the equation

$$(\beta+\gamma-2\delta)(\beta+\delta-2\gamma)(\gamma+\delta-2\beta) = 0,$$

this is easily reduced to

$$-2p^3 + 9pq - 27r = 0.$$

But we have

$$a(x^3 - px^2 + qx - r)(x-\alpha) = (a, b, c, d, e \between x, 1)^4,$$

and hence

$$p = -\frac{4b}{a} - \alpha, \quad q = \frac{6c}{a} + \frac{4b}{a}\alpha + \alpha^2, \quad r = -\frac{4d}{a} - \frac{6c}{a}\alpha - \frac{4b}{a}\alpha^2 - \alpha^3.$$

Substituting these values of p, q, r, the foregoing equation becomes, after all reductions,

$$(20\,a^3,\ 20\,a^2b,\ -16\,ab^2 + 36\,a^2c,\ 128\,b^3 - 216\,abc + 108\,a^2d \between \alpha, 1)^3 = 0,$$

and from this and the equation $(a, b, c, d, e \between \alpha, 1)^4 = 0$, eliminating α, we should find the condition for three roots in arithmetical progression. But it appears from the theory of invariants that the result of the elimination may be obtained by writing $b = 0$, and expressing the result so obtained in terms of a, H, I, J. Hence, writing in the two equations $b = 0$, the first equation contains the factor $4a^2$, and throwing this out, the equations become

$$5a\alpha^4 + 27c\alpha + 27d = 0, \quad a\alpha^4 + 6c\alpha^2 + 4d\alpha + e = 0;$$

or multiplying the first by α and reducing by means of the second, the two equations become

$$5a\alpha^3 + 27c\alpha + 27d = 0, \quad 3c\alpha^2 - 7d\alpha + 5e = 0.$$

The result is of the degree 5 in the coefficients, but in order to avoid fractions in the final result it is proper to multiply it by a^4; it then becomes

$$625\,a^6e^3 - 4050\,a^5c^2e^2 + 6561\,a^4c^4e - 1890\,a^5ced^2 + 13122\,a^4c^3d^2 + 9261\,a^5d^4 = 0.$$

But writing as above $b = 0$, we have

$$a = a, \quad c = \frac{H}{a}, \quad e = \frac{I}{a} - \frac{3H^2}{a^3}, \quad d^2 = -\frac{J}{a} + \frac{HI}{a^2} - \frac{4H^3}{a^4};$$

and substituting these values, the result is found to contain the terms $\frac{IH^4}{a}$, $\frac{H^6}{a^3}$ with coefficients which vanish; viz. the coefficient of the first of these terms is

$$+16875+24300+\ 6561+\ 7560+18792-74088,\ =0\,;$$

and the coefficient of the second of the two terms is

$$-16875-36450-19683-75168+148176,\ =0.$$

The remaining terms give

$$\left.\begin{array}{lcl} +\ \ 625 & = + & 625\,a^3I^3 \\ -\ 5625-4050-1890+9261 & = - & 2304\,aH^2I^2 \\ +\ 1890-18522 & = - & 16632\,a^2HIJ \\ -\ 18792+74088 & = + & 55296\,H^3J \\ +\ 9261 & = + & 9261\,a^3J^2 \end{array}\right\}=0,$$

which is the required result; a more convenient form of writing it is

$$(55296\,J,\ \ -768\,I^2,\ \ -5544\,IJ,\ \ 625\,I^3+9261\,J^2 \between H,\ a)^3=0.$$

Remark. If I and J denote as above the two invariants of the form $U=(a, b, c, d, e\between x, 1)^4$, and if we now use H to denote the Hessian of the form, viz.

$$H=(ac-b^2,\ \ \tfrac{1}{2}(ad-bc),\ \ \tfrac{1}{6}(ae+2bd-3c^2),\ \ \tfrac{1}{2}(be-cd),\ \ ce-d^2\between x,\ 1)^4,$$

then it appears by the theory of invariants that the equation of the twelfth order

$$(55296\,J,\ \ -768\,I^2,\ \ -5544\,IJ,\ \ 625\,I^3+9261\,J^2\between H,\ U)^3=0,$$

is such that each of its roots forms with some three of the roots of the equation $U=0$ a harmonic progression; viz. if the three roots are β, γ, δ, then we have

$$\frac{2}{x-\gamma}=\frac{1}{x-\beta}+\frac{1}{x-\delta},\ \text{or}\ x=\frac{2\beta\delta-(\beta+\delta)\,\gamma}{\beta+\delta-2\gamma};$$

so that the roots of the equation of the twelfth order are the twelve values of the last-mentioned function of three roots.

[Vol. v. pp. 65, 66.]

On the Problems in regard to a Conic defined by five Conditions of Intersection.

There has been recently published in the *Comptes Rendus* (t. LXII. pp. 177—183, January, 1866) an extract of a memoir "Additions to the Theory of Conics," by M. H. G. Zeuthen (of Copenhagen). The extract gives the solutions of fourteen problems, with a brief indication of the method employed for obtaining them. Of these problems, four relate to intersections at given points, the remaining ten are included among the twenty-seven problems enumerated in my *Note* on this subject in the January Number of the *Educational Times* (*Reprint,* vol. v., p. 25); but two of these ten are the problems 25 and 26 which are in my *Note* stated to have been solved; there are, consequently, of the twenty-seven problems, in all twelve which are solved: viz. these are where it is to be observed that Zeuthen's solutions apply to the case

No. of Prob.	1, 8, 10, 12, 14, 17, 19, 21, 23, 25, 26, 27
Zeuthen's No.	—, 14, 13, 11, 8, 3, 12, 7, 2, 6, 1, —

of a curve of a given order with given numbers of double points and cusps. The problems 25 and 26 had been previously solved only in the case of a curve without singularities. As to Prob. 27, the solution mentioned in my former *Note* is in fact applicable to the general case. The solution for Prob. 1 may also be extended to this general case, viz. for a curve of the order m with δ double points and κ cusps the required number is $= m(12m-27)-24\delta-27\kappa$; or, if n be the class, then this number is $=12n-15m+9\kappa$; so that all the twelve problems are solved in the general case.

The results obtained by M. de Jonquieres, as stated in my *Note* in the March Number (*Reprint,* vol. v., p. 57), seem to be all of them erroneous. In fact, for the number of conics passing through two given points and touching a curve of the order m in three distinct points (which is a particular case of Prob. 23), Zeuthen's formula applied to a curve without singularities gives this

$$=\tfrac{1}{6}m(m-2)(m^4+5m^3-17m^2-49m+108)$$

instead of the value

$$\tfrac{1}{2}m(m-1)(m-2)(m^3+6m^2-19m-12)$$

which is

$$=\tfrac{1}{6}m(m-2)(m^4+5m^3-25m^2+7m+12);$$

and I have by my own investigation verified Zeuthen's Number. So for the number of conics through a given point and touching a curve of the order m in four distinct points (which is a particular case of Prob. 17), Zeuthen's formula applied to a curve without singularities gives this

$$=\tfrac{1}{24}m(m-2)(m-3)(m^5+9m^4-15m^3-225m^2+140m+1050)$$

instead of the value

$$\tfrac{1}{24}m(m-1)(m-2)(m-3)(m^4+10m^3-37m^2-118m+282)$$

which is

$$=\tfrac{1}{24}m(m-2)(m-3)(m^5+9m^4-47m^3-81m^2+400m-282),$$

and it may I think be inferred that the expression obtained for the number of conics which touch a given curve in five distinct points (Prob. 7), containing as it does the factor $(m-1)$, is also erroneous.

I have obtained for Prob. 2 a solution which I believe to be accurate; viz. the number of the conics (4, 1), (that is, the conics which have with a given curve a 5-pointic intersection and also a 2-pointic intersection, or ordinary contact), is

$$= 10n^2 + 10nm - 20m^2 - 130n + 140m + 10\kappa(m + n - 9) - 4[(n-3)\kappa + (m-3)\iota]$$

where ι (the number of inflexions) is $= 3n - 3m + \kappa$, but I prefer to retain the foregoing form, without effecting the substitution.

[Vol. v. pp. 88, 89.]

1890. (Proposed by Professor CAYLEY.)—Find the equation of a conic passing through three given points and having double contact with a given conic.

Solution by the PROPOSER.

Let the given points be the angles of the triangle ($x=0$, $y=0$, $z=0$), and let the equation of the given conic be $U = (a, b, c, f, g, h \between x, y, z)^2 = 0$; then the equation of the required conic is

$$U - (x\sqrt{a} + y\sqrt{b} + z\sqrt{c})^2 = 0,$$

for this is a conic having double contact with the conic $U=0$, and, since the terms in (x^2, y^2, z^2) each vanish, it is also a conic passing through the given points.

It is clear that there are four conics satisfying the conditions of the Problem, viz. putting for shortness

$$P = x\sqrt{a} + y\sqrt{b} + z\sqrt{c}, \qquad P_1 = -x\sqrt{a} + y\sqrt{b} + z\sqrt{c},$$

$$P_2 = x\sqrt{a} - y\sqrt{b} + z\sqrt{c}, \qquad P_3 = x\sqrt{a} + y\sqrt{b} - z\sqrt{c},$$

the four conics are

$$U - P^2 = 0, \quad U - P_1^2 = 0, \quad U - P_2^2 = 0, \quad U - P_3^2 = 0.$$

It may be remarked that the conics P, P_1 have a fourth intersection lying on the line $y\sqrt{b} + z\sqrt{c} = 0$, and the conics P_2, P_3 a fourth intersection lying on the line $y\sqrt{b} - z\sqrt{c}$; which two lines are harmonics in regard to the lines $y=0$, $z=0$.

Similarly the conics P_1, P_2 have a fourth intersection on the line $x\sqrt{a} + z\sqrt{c} = 0$, and the conics P, P_3 a fourth intersection on the line $x\sqrt{a} - z\sqrt{c} = 0$; which two lines are harmonics in regard to the lines $z=0$, $x=0$. And the conics P_1, P_3 have a fourth intersection on the line $x\sqrt{a} + y\sqrt{b} = 0$, and the conics P, P_2 a fourth intersection on the line $x\sqrt{a} - y\sqrt{b} = 0$; which two lines are harmonics in regard to the lines $x=0$, $y=0$. It may further be remarked that the equations of any two of the four conics may be taken to be

$$\alpha yz + \beta zx + \gamma xy = 0, \quad \alpha' yz + \beta' zx + \gamma' xy = 0.$$

The general equation of a conic having double contact with each of these conics then is

$$n^2z^2 - 2n(\gamma\alpha' + \gamma'\alpha)\,yz - 2n(\gamma\beta' + \gamma'\beta)\,zx - 4n\gamma\gamma'xy + [(\beta\gamma' - \beta'\gamma)\,x - (\gamma\alpha' - \gamma'\alpha)\,y]^2 = 0,$$

where n is arbitrary: and, having double contact with this conic, we have (besides the above-mentioned two conics) two new conics each passing through the angles of the triangle; viz. writing for greater convenience

$$n = \frac{(\beta\gamma' - \beta'\gamma)(\gamma\alpha' - \gamma'\alpha)}{K - \gamma\gamma'}, \quad \text{or } K = \gamma\gamma' + \frac{(\beta\gamma' - \beta'\gamma)(\gamma\alpha' - \gamma'\alpha)}{n},$$

then the equations of the two new conics are

$$\gamma'\alpha\,yz + \gamma\beta'\,zx + Kxy = 0, \quad \gamma\alpha'\,yz + \gamma'\beta\,zx + Kxy = 0.$$

In fact, writing the equation under the form

$$\begin{aligned}&[xz + (\beta\gamma' - \beta'\gamma)\,x + (\gamma\alpha' - \gamma'\alpha)\,y]^2\\ &- 4\ (\beta\gamma' - \beta'\gamma)(\gamma\alpha' - \gamma'\alpha)\,xy - 4n\gamma\gamma'xy\\ &- 2n(\beta\gamma' - \beta'\gamma)\,xz - 2n(\beta\gamma' + \beta'\gamma)\,xz\\ &- 2n(\gamma\alpha' - \gamma'\alpha)\,yz - 2n(\gamma\alpha' + \gamma'\alpha)\,yz = 0,\end{aligned}$$

we at once see that this is a conic having double contact with the conic $\gamma'\alpha yz + \gamma\beta'zx + Kxy = 0$, the equation of the chord of contact being $nz + (\beta\gamma' - \beta'\gamma)\,x + (\gamma\alpha' - \gamma'\alpha)\,y = 0$: and similarly it has double contact with the conic $\gamma\alpha'\,yz + \gamma'\beta\,zx + Kxy = 0$, the equation of the chord of contact being $nz - (\beta\gamma' - \beta'\gamma)\,x - (\gamma\alpha' - \gamma'\alpha)\,y = 0$.

[Vol. V. pp. 99, 100.]

1554. (Proposed by Professor CAYLEY.)—Show that, in the ellipse and its circles of maximum and minimum curvature respectively, the semi-ordinates through the focus of the ellipse are

For the circle of maximum curvature $y_1 = a(1-e)(1+2e)^{\frac{1}{2}}$,

for the ellipse $y_2 = a(1-e^2)$,

for the circle of minimum curvature $y_3 = \dfrac{a\{(1-e^2+e^4)^{\frac{1}{2}} - e^2\}}{(1-e^2)^{\frac{1}{2}}}$,

and that these values are in the order of increasing magnitude.

[Vol. VI., July to December, 1866, pp. 18, 19.]

1931. (Proposed by Professor CAYLEY.)—Find the stationary tangents (or tangents at the inflexions) of the nodal cubic

$$x(y-z)^2 + y(z-x)^2 + z(x-y)^2 = 0.$$

Solution by the PROPOSER.

The equation may be transformed into the form

$$(-8x+y+z)^{\frac{1}{3}} + (x-8y+z)^{\frac{1}{3}} + (x+y-8z)^{\frac{1}{3}} = 0,$$

and it thence follows immediately that the stationary tangents are the lines

$$-8x+y+z=0, \quad x-8y+z=0, \quad x+y-8z=0,$$

respectively, and that the three points of contact, or inflexions, are the intersections of these lines with the line $x+y+z=0$.

In fact, writing

$$X = kx+y+z, \quad Y = x+ky+z, \quad Z = x+y+kz,$$

we have identically

$$\begin{aligned}
&(X+Y+Z)^3 - 27XYZ \\
&= (k+2)^3(x+y+z)^3 - 27(kx+y+z)(x+ky+z)(x+y+kz), \\
&= \quad (x^3+y^3+z^3)\{(k+2)^3 - 27k\} \\
&\quad + 3(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)\{(k+2)^3 - 9(k^2+k+1)\} \\
&\quad + 3\,xyz\,\{2(k+2)^3 - 9(k^3+3k+2)\} \\
&= (k-1)^2(k+8)(x^3+y^3+z^3) + 3(k-1)^3(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y) - 3(k-1)^2(7k+2)\,xyz.
\end{aligned}$$

Hence, writing $k=-8$, we have

$$\begin{aligned}
(X+Y+Z)^3 - 27XYZ &= -2187\{yz^2+y^2z+zx^2+z^2x+xy^2+x^2y-6xyz\}, \\
&= -2187\{x(y-z)^2 + y(z-x)^2 + z(x-y)^2\}.
\end{aligned}$$

The equation of the given curve is therefore

$$(X+Y+Z)^3 - 27XYZ = 0, \quad \text{or } X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0,$$

where of course X, Y, Z have the values

$$X = -8x+y+z, \quad Y = x-8y+z, \quad Z = x+y-8z.$$

[Vol. VI. pp. 35—39.]

1990. (Proposed by Professor SYLVESTER.)—Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

Solution by PROFESSOR CAYLEY.

Some preliminary explanations are required in regard to this remarkable theorem.

1. I call to mind that a circular cubic (or cubic through the two circular points at infinity) has 16 foci, which lie 4 together on 4 different circles; and that the

property of 4 concyclic foci is that taking any three of them A, B, C, the distances of a point P of the curve from these three foci are connected by a linear relation $\lambda . AP + \mu . BP + \nu . CP = 0$, where $\lambda \pm \mu \pm \nu = 0$, or if as is more convenient the distances are considered as $\pm$, then where $\lambda + \mu + \nu = 0$. A circular cubic may be determined so as to satisfy 7 conditions; having a focus at a given point is 2 conditions; hence a circular cubic may be determined so as to pass through three given points, and to have as foci two given points.

2. A Cartesian, or bicircular cuspidal quartic (that is a quartic having a cusp at each of the circular points at infinity) has nine foci, but of these there are three which lie in a line with the centre of the Cartesian (or intersection of the cuspidal tangents), and which are preeminently the foci of the Cartesian. We may, therefore, say that the Cartesian has three foci, which foci lie in a line, the axis of the Cartesian. A Cartesian may be determined to satisfy 6 conditions; having a focus at a given point is 2 conditions; but having for foci three given points on a line is 5 conditions; and hence a Cartesian may be found having for foci three given points on a line, and passing through a given point; there are in fact two such Cartesians, intersecting at right angles at the given point.

3. The theorem at first sight appears impossible; for take any three points F, G, H in a line and any other point A; then, as just remarked, there are, having F, G, H for foci and passing through A, two Cartesians. And we may draw through F, G, H, and with A for focus, a circular cubic depending upon two arbitrary parameters; the position of a second focus of the circular cubic is (on account of the two arbitrary parameters) *primâ facie* indeterminate; and this is confirmed by the remark that the circular cubic can actually be so determined as to have for focus an arbitrary point B; and yet the theorem in effect asserts that the foci concyclic with A, of the circular cubic, lie on one or other of the two Cartesians.

4. To explain this, it is to be remarked that the arbitrary point B is a focus which is either concyclic with A or else not concyclic with A. In the latter case, although B is arbitrary, yet the foci concyclic with A may and in fact do lie on one of the Cartesians; the difficulty is in the former case if it arises; viz., if we can describe a cubic through the points F, G, H in a line, and with A and B as *concyclic* foci; that is, if we can find a third focus C, such that the distances from A, B, C of a point P on the curve are connected by a relation $\lambda . AP + \mu . BP + \nu . CP = 0$, where $\lambda + \mu + \nu = 0$. It may be shown that this is in a sense possible, but that the resulting cubic is not a proper circular cubic, but is the cubic made up of the line FGH taken twice, and of the line infinity. To show this, since the required cubic passes through the points F, G, H we have

$$\begin{array}{l} \lambda . AF + \mu . BF + \nu . CF = 0 \\ \lambda . AG + \mu . BG + \nu . CG = 0 \\ \lambda . AH + \mu . BH + \nu . CH = 0 \\ \lambda \quad + \mu \quad + \nu \quad = 0 \end{array} \text{ and thence } \left| \begin{array}{cccc} AF, & AG, & AH, & 1 \\ BF, & BG, & BH, & 1 \\ CF, & CG, & CH, & 1 \end{array} \right| = 0,$$

being two conditions for the determination of the position of the point C; these give CG, CH as linear functions of CF; the distances CF, CG, CH of the point C from the points F, G, H in the line FGH are connected by a quadratic equation, and hence substituting for CG, CH their values in terms of CF, we have a quadratic equation for CF; as the given conditions are satisfied when C coincides with A or with B, the roots of this equation are $CF = AF$ and $CF = BF$. But if $CF = AF$, then the linear relations give $CG = AG$ and $CH = AH$, that is, C is a point opposite to A in regard to the line FGH. And similarly if $CF = BF$, then C is a point opposite to B in regard to the line FGH. But C being opposite to A or B, the fourth concyclic focus D will be opposite to B or A; that is, the pairs A, B and C, D of concyclic foci lie symmetrically on opposite sides of the line FGH; this of course implies that the four points lie on a circle.

5. Taking $Y = 0$ as the equation of the line FGH, $x^2 + y^2 - 1 = 0$ as the equation of the circle through the four points A, B, C, D, then these lie on a proper cubic

$$(x^2 + y^2 + 1)\,x + lx^2 + ny^2 = 0$$

(not passing through the points F, G, H) and the four foci are given as the intersections with the circle $x^2 + y^2 - 1 = 0$ of the pair of lines

$$x^2 - 2nx - nl = 0.$$

But if we attempt to describe with the same four foci a cubic

$$(x^2 + y^2 + 1)\,y + l'x^2 + 2m'xy + n'y^2 = 0,$$

then the foci are given as the intersections with the circle $x^2 + y^2 - 1 = 0$ of the conic

$$y^2 + 2m'x - 2l'y + m'^2 - n'l' = 0.$$

In order that these may coincide with the points (A, B, C, D) we must have

$$(x^2 - 2nx - nl) + (y^2 + 2m'x - 2l'y + m'^2 - n'l') = x^2 + y^2 - 1\,;$$

that is

$$m' = n, \quad l' = 0, \quad -nl + n^2 - n'l' = -1.$$

The last equation is $n'l' = n^2 + 1 - nl$, which, assuming that nl is not equal to $n^2 + 1$, {in this case the cubic $(x^2 + y^2 + 1)\,x + lx^2 + my^2 = 0$ would reduce itself to the line and conic $(x + n)\left(x^2 + y^2 + \dfrac{x}{n}\right) = 0$}, since $l' = 0$, gives $n' = \infty$, and therefore the cubic

$$(x^2 + y^2 + 1)\,y + l'x^2 + 2m'xy + n'y^2 = 0,$$

reduces itself to $y^2 = 0$, that is, the cubic in question reduces itself to the line FGH twice repeated, and the line infinity.

6. The conclusion is that F, G, H being given points on a line, and A and B being any other given points, there is not any proper cubic passing through F, G, H and having A, B for concyclic foci: and the *primâ facie* objection to the truth of the theorem is thus removed.

7. Considering the points F, G, H on a line and the point A as given, it has been seen that there are *two* Cartesians through A with the foci F, G, H; and the theorem asserts that in the circular cubics through F, G, H with the focus A, the foci concyclic with A lie on one or other of the two Cartesians: there are consequently through F, G, H with the focus A two systems of circular cubics corresponding to the two Cartesians respectively, each system depending upon two arbitrary parameters. But if we attend only to one of the two Cartesians and to the corresponding system of cubics, then the Cartesian passes through the four foci of each cubic, and if (instead of taking as given the points F, G, H and the focus A) we take as given the four concyclic foci A, B, C, D of a cubic, the theorem asserts that we have through A, B, C, D a Cartesian depending on two arbitrary parameters (or having for its axis an arbitrary line), and such that the foci of the Cartesian are the points of intersection F, G, H of its axis with the cubic. And I proceed to the proof of the theorem in this form.

8. The equation of a circular cubic having four foci on the circle $x^2+y^2-1=0$ is

$$(x^2+y^2+1)(Px+Qy)+lx^2+2mxy+ny^2=0:$$

and this being so, the four foci are the intersections of the circle with the conic

$$(Qx-Py)^2+2(-nP+mQ)x+2(mP-lQ)y+m^2-nl=0.$$

9. The general equation of a Cartesian is

$$(x^2+y^2+2Ax+2By+C)^2+2Dx+2Ey+F=0,$$

and by assuming for A, B, C, D, E, F, the following values which contain the two arbitrary parameters α and θ, viz. by writing

$$2A=\theta Q,\ 2B=-\theta P,\ C=\alpha-1,\ D=-n\theta^2P+(m\theta^2-\alpha\theta)Q,$$
$$E=(m\theta^2+\alpha\theta)P-l\theta^2Q,\ F=-\alpha^2+\theta^2(m^2-nl),$$

we have the equation of a system (the selected one out of two systems) of Cartesians through the four foci; in fact, substituting the foregoing values, the equation of the Cartesian is

$$\{x^2+y^2+\theta(Qx-Py)+\alpha-1\}^2-2\alpha\theta(Qx-Py)$$
$$+2\theta^2(-nP+mQ)x+2\theta^2(mP-lQ)y-\alpha^2+\theta^2(m^2-nl)=0,$$

and writing herein $x^2+y^2-1=0$, the equation reduces itself to

$$\theta^2\{(Qx-Py)^2+2(-nP+mQ)x+2(mP-lQ)y+m^2-nl\}=0,$$

verifying that the Cartesian passes through the four foci.

The coordinates of the centre of the Cartesian are $x=-A$, $y=-B$, and the equation of its axis is $E(x+A)-D(y+B)=0$; we have therefore to show that the points of intersection of this line with the cubic are the foci of the Cartesian.

10. To find where the line in question meets the cubic

$$(x^2+y^2+1)(Px+Qy)+lx^2+2mxy+ny^2=0,$$

writing in this equation $x=-A+D\Omega,\quad y=-B+E\Omega,$

we have for the determination of Ω the equation

$$\{A^2+B^2+1-2(AD+BE)\,\Omega+(D^2+E^2)\,\Omega^2\}\times$$
$$\{-AP-BQ+(DP+EQ)\,\Omega\}+(l,\ m,\ n\between -A+D\Omega,\ -B+E\Omega)^2=0,$$

or observing that we have $AP+BQ=0$, this equation becomes

$$\begin{aligned}&(D^2+E^2)(DP+EQ)\,\Omega^3\\ &+\{-2(AD+BE)(DP+EQ)\quad+\ lD^2+2mDE+nE^2\}\,\Omega^2\\ &+\{\ (A^2+B^2+1)(DP+EQ)\quad -2lAD-2m(AE+BD)-2nBE\}\,\Omega\\ &+\{\qquad lA^2+2mAB+nB^2\}=0.\end{aligned}$$

11. Substituting for A, B, D, E their values in terms of $(P,\ Q,\ \alpha,\ \theta)$, we find

$$\begin{aligned}DP+\ EQ &= -\theta^2(nP^2-2mPQ+lQ^2),\\ lA^2+2mAB+nB^2 &= \tfrac{1}{4}\theta^2(nP^2-2mPQ+lQ^2),\\ lAD+m(AE+BD)+nBE &= -\tfrac{1}{2}\alpha\theta^2(nP^2-2mPQ+lQ^2),\\ lD^2+2mDE+nE^2 &= \big((nl-m^2)\,\theta^4+\alpha^2\theta^2\big)(nP^2-2mPQ+lQ^2),\end{aligned}$$

and substituting these values in the equation for Ω, the whole equation divides by $\theta^2(nP^2-2mPQ+lQ^2)$, and it then becomes

$$4(D^2+E^2)\,\Omega^3+4\{-2(AD+BE)-(nl-m^2)\,\theta^2-\alpha^2\}\,\Omega^2+4\{A^2+B^2+1-\alpha\}\,\Omega-1=0,$$

or, putting for shortness

$$\begin{aligned}C' &= C-A^2-B^2, &&= \alpha-1-A^2-B^2,\\ F' &= F-2(AD+BE), &&= -\alpha^2-\theta^2(nl-m^2)-2(AD+BE),\end{aligned}$$

the equation in Ω is

$$4(D^2+E^2)\,\Omega^3+4F'\Omega^2-4C'\Omega-1=0,$$

so that, Ω satisfying this equation, the intersections of the axis with the cubic are given by $x=-A+D\Omega$, $y=-B+E\Omega$.

12. The equation of the Cartesian, writing therein $x+A=u$ and $y+B=v$, and attending to the values of C' and F', is

$$(u^2+v^2+C')^2+2Du+2Ev+F'=0.$$

And to find the foci, writing in this equation $u+\rho$, $v+i\rho$ in place of u, v, we find

$$\{u^2+v^2+C'+2(u+vi)\,\rho\}^2+2(D+Ei)\,\rho+2Du+2Ev+F'=0,$$

that is

$$(u^2+v^2+C')^2+2Du+2Ev+F'+\{2(u+vi)(u^2+v^2+C')+D+Ei\}\,2\rho+4(u+vi)^2\rho^2=0.$$

Expressing that this equation in ρ has two equal roots, we find

$$4(u+vi)^2\{(u^2+v^2+C')^2+2Du+2Ev+F'\}-\{2(u+vi)(u^2+v^2+C')+D+Ei\}^2=0,$$

that is

$$4(2Du+2Ev+F')(u+vi)^2-4(u^2+v^2+C')(u+vi)(D+Ei)-(D-Ei)^2=0,$$

which equation is in fact the equation of the three tangents from one of the circular points at infinity. Writing it under the form $U+Vi=0$, the nine foci of the Cartesian are given as the intersections of the two cubics $U=0$, $V=0$. But of these nine points, three, the foci that we are concerned with, lie on the axis, or line $Eu-Dv=0$; in fact, we have

$$\begin{aligned} U=&\,4(u^2-v^2)(2Du+2Ev+F')\\ &-4(uD-vE)(u^2+v^2+C')\\ &-(D^2-E^2), \end{aligned} \qquad\qquad \begin{aligned} V=&\,8uv(2Du+2Ev+F')\\ &-4(uE+vD)(u^2+v^2+C')\\ &-2DE; \end{aligned}$$

and hence

$$2DEU-(D^2-E^2)V=(Eu-Dv)\{8(Du+Ev)(2Du+2Ev+F')-4(D^2+E^2)(u^2+v^2+C')\}=0,$$

which shows that the nine points lie three of them on the line $Eu-Dv=0$, and the remaining six on the conic

$$2(Du+Ev)(2Du+2Ev+F')-(D^2+E^2)(u^2+v^2+C')=0.$$

13. We have thus the three foci given as the intersections of the axis $Eu-Dv=0$, with the cubic

$$U=4(u^2-v^2)(2Du+2Ev+F')-4(uD-vE)(u^2+v^2+C')-(D^2-E^2)=0;$$

or, writing in this last equation $u=D\Omega$, $v=E\Omega$, that is $x=-A+D\Omega$, $y=-B+E\Omega$, we have

$$u^2-v^2=(D^2-E^2)\Omega^2,\quad uD-vE=(D^2-E^2)\Omega.$$

The whole equation divides by (D^2-E^2), and omitting this factor, it is

$$4\Omega^2\{2(D^2+E^2)\Omega+F'\}-4\Omega\{(D^2+E^2)\Omega^2+C'\}-1=0,$$

that is

$$4(D^2+E^2)\Omega^3+4F'\Omega^2-4C'\Omega-1=0,$$

the same equation as the equation in Ω before obtained; that is the intersections of the cubic with the axis are the three foci of the Cartesian.

[Vol. VI. pp. 57—59.]

1949. (Proposed by Professor CAYLEY.)—Find the conic of five-pointic intersection at any point of the cuspidal cubic $y^3=x^2z$.

Solution by the PROPOSER.

The equation $y^3 = x^2 z$, is satisfied by the values $x : y : z = 1 : \theta : \theta^3$; and conversely, to any given value of the parameter θ there corresponds a point on the cubic $y^3 = x^2 z$. Consider the five points corresponding to the values $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ respectively; the equation of the conic through these five points is

$$\begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} = 0,$$

where the remaining four lines of the determinant are obtained from the second line by writing therein $\theta_2, \theta_3, \theta_4, \theta_5$ successively in place of θ_1. Writing for shortness $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ to denote the product of the differences of the quantities $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, the equation contains the factor $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, and we may therefore write it in the simplified form

$$\frac{1}{\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} = 0.$$

Hence putting in this equation $\theta_1 = C_2 = \theta_3 = \theta_4 = \theta_5 = \phi$, we have the equation of the conic of five-pointic intersection at the point (ϕ). The result in its reduced form may be obtained directly without much difficulty, but it is obtained most easily as follows: let the function on the left hand of the foregoing equation be represented by

$$(a, b, c, f, g, h \between x, y, z)^2,$$

then writing $x : y : z = 1 : \theta : \theta^3$, we have

$$(a, b, c, f, g, h \between 1, \theta, \theta^3)^2$$

$$= \frac{1}{\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} 1, & \theta^2, & \theta^6, & \theta^4, & \theta^3, & \theta \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix},$$

$$= \frac{(\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3)(\theta - \theta_4)(\theta - \theta_5)}{\zeta^{\frac{1}{2}}(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} 1, & \theta^2, & \theta^6, & \theta^4, & \theta^3, & \theta \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix},$$

$$= (\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3)(\theta - \theta_4)(\theta - \theta_5)(\theta + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5);$$

for the determinant, which is a function of the order 16 in the quantities $(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ conjointly, divides by $\zeta^{\frac{1}{2}}(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, which is a function of the order 15; and as the quotient is a symmetrical function of $\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, it must be equal, save to a numerical factor which may be disregarded, to $\theta + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5$.

Hence if ϕ be the parameter of the given point, writing $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \phi$, we have

$$(a, b, c, f, g, h \between 1, \theta, \theta^3)^2 = (\theta - \phi)^5 (\theta + 5\phi)$$
$$= (1, 0, -15, +40, -45, +24, -5 \between \theta, \phi)^6,$$

where the left-hand side is

$$a + b\theta^2 + c\theta^6 + f\theta^4 + g\theta^3 + h\theta, = \{c, 0, f, g, b, h, a\} (\theta, 1)^6,$$

that is we have

$$c = 1, \quad f = -15\phi^2, \quad g = 40\phi^3, \quad b = -45\phi^4, \quad h = 24\phi^5, \quad a = -5\phi^6,$$

and the equation of the conic of five-pointic intersection therefore is

$$(-5\phi^6, -45\phi^4, 1, -15\phi^2, 40\phi^3, 24\phi^5 \between x, y, z)^2 = 0,$$

or, what is the same thing,

$$-5\phi^6 x^2 - 45\phi^4 y^2 + z^2 - 15\phi^2 yz + 40\phi^3 zx + 24\phi^5 xy = 0,$$

which is the required result.

Note. The condition in order that any six points $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ of the cubic $y^3 = x^2 z$ may lie on a conic, is

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 = 0.$$

[Vol. vi. p. 65.]

1872. (Proposed by Professor Cayley.)—Show that the surfaces

$$xyz = 1, \quad yz + zx + xy + x + y + z + 3 = 0,$$

intersect in two distinct cubic curves; and find the equations of the cubic cones which have their vertices at the origin and pass through these curves respectively.

[Vol. vi. pp. 67—69.]

1969. (Proposed by Professor Sylvester.)—In two given great circles of a sphere intersecting at O are taken respectively two points P and Q, the arc joining which is of given length: prove that S, H two fixed points, and M a fixed line, in a plane may be found such that, for all positions of the arc PQ, a point M in the fixed line may be found satisfying the equations

$$SM \pm HM = \sin OP, \quad SM \mp HM = \sin OQ.$$

Solution by PROFESSOR CAYLEY.

1. In the spherical triangle OPQ, whereof the sides OP, OQ, PQ are θ, ϕ, β and the angle O is $=\alpha$, the relation between these quantities is $\cos\alpha = \dfrac{\cos\beta - \cos\theta\cos\phi}{\sin\theta\sin\phi}$; hence treating α, β as constants, and θ, ϕ as variable angles connected by the foregoing equation, it is required to show that we can find two fixed points S, H and a fixed line, such that taking M a variable point in this line and writing $SM = r$, $HM = s$, the relation between r and s (or equation of the fixed line in terms of r, s as coordinates of a point thereof) is obtained by substituting in the foregoing equation for θ and ϕ the values given by the two equations

$$\sin\theta = (r+s), \quad \sin\phi = (r-s),$$

or as, for the sake of homogeneity, it will be more convenient to write these equations,

$$m\sin\theta = (r+s), \quad m\sin\phi = (r-s).$$

2. Suppose that the perpendicular distances of S, H from the fixed line are a and b, and that the distance between the feet of the two perpendiculars is $2c$, then if x denote the distance of the point M from the midway point between the feet of the two perpendiculars, we have

$$r = \sqrt{\{(c+x)^2 + a^2\}}, \quad s = \sqrt{\{(c-x)^2 + b^2\}},$$

and (a, b, c) being properly determined, the elimination of x from these equations should give between (r, s) a relation equivalent to that obtained by the elimination of (θ, ϕ) from the before-mentioned equations. Or, what is the same thing, the elimination of (r, s, x) from the equations

$$m\sin\theta = r+s, \quad m\sin\phi = r-s, \quad r = \sqrt{\{(c+x)^2 + a^2\}}, \quad s = \sqrt{\{(c-x)^2 + b^2\}}$$

should give between (θ, ϕ) the relation

$$\cos\alpha = \frac{\cos\beta - \cos\theta\cos\phi}{\sin\theta\sin\phi};$$

that is, the last-mentioned equation should be obtained by the elimination of x from the equations

$$m(\sin\theta + \sin\phi) = 2\sqrt{\{(c+x)^2 + a^2\}}, \quad m(\sin\theta - \sin\phi) = 2\sqrt{\{(c-x)^2 + b^2\}}.$$

3. The equation in (θ, ϕ) may be written

$$\cos\beta - \cos\alpha\sin\theta\sin\phi = \cos\theta\cos\phi,$$

or, squaring and reducing,

$$\sin^2\theta + \sin^2\phi = \sin^2\beta + 2\cos\alpha\cos\beta\sin\theta\sin\phi + \sin^2\alpha\sin^2\theta\sin^2\phi,$$

that is

$$\sin^2\theta + \sin^2\phi = \frac{1 - \cos^2\alpha - \cos^2\beta}{\sin^2\alpha} + \left(\sin\alpha\sin\theta\sin\phi + \frac{\cos\alpha\cos\beta}{\sin\alpha}\right)^2.$$

But from the two equations in x, we have

$$m^2(\sin^2\theta+\sin^2\phi)=4c^2+2a^2+2b^2+4x^2,\quad m^2\sin\theta\sin\phi=4cx+a^2-b^2,$$

whence

$$2x=\frac{b^2-a^2+m^2\sin\theta\sin\phi}{2c},$$

therefore

$$\sin^2\theta+\sin^2\phi=\frac{4c^2+2b^2+2a^2}{m^2}+\left(\frac{b^2-a^2+m^2\sin\theta\sin\phi}{2cm}\right)^2.$$

Hence, comparing the two results, we have

$$\frac{1-\cos^2\alpha-\cos^2\beta}{\sin^2\alpha}=\frac{4c^2+2b^2+2a^2}{m^2},\quad \frac{\cos\alpha\cos\beta}{\sin\alpha}=\frac{b^2-a^2}{2cm},\quad \sin\alpha=\frac{m}{2c};$$

or, as these may also be written,

$$\sin\alpha=\frac{m}{2c},\quad \cos^2\alpha+\cos^2\beta=\frac{-b^2-a^2}{2c^2},\quad 2\cos\alpha\cos\beta=\frac{b^2-a^2}{2c^2};$$

whence

$$(\cos\alpha+\cos\beta)^2=\frac{-a^2}{c^2},\quad (\cos\alpha-\cos\beta)^2=\frac{-b^2}{c^2},\quad \sin\alpha=\frac{m}{2c};$$

so that m being put equal to unity, or otherwise assumed at pleasure, a, b, c are given functions of α, β. Or conversely, if a, b, c are assumed at pleasure, then α, β, m are given functions of these quantities.

5. It is to be remarked that (α, β) being real, a and b will be imaginary, and consequently the points S, H of Professor Sylvester's theorem are imaginary[1]; if, however, we write $-a^2$, $-b^2$ in place of a^2, b^2 respectively, then the radicals $\sqrt{\{(c+x)^2-a^2\}}$, $\sqrt{\{(c-x)^2-b^2\}}$ have a real geometrical interpretation. The system of relations between $(\alpha, \beta, a, b, c, m)$ becomes

$$(\cos\alpha+\cos\beta)^2=\frac{a^2}{c^2},\quad (\cos\alpha-\cos\beta)^2=\frac{b^2}{c^2},\quad \sin\alpha=\frac{m}{2c};$$

and considering (a, b, c) as given, we may write

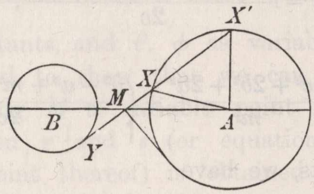
$$\cos\alpha=\frac{a+b}{2c},\quad \cos\beta=\frac{a-b}{2c},\quad m=\sqrt{\{4c^2-(a+b)^2\}},$$

viz. we have either this system or the similar system obtained by writing $-b$ in place of b.

6. Consider two circles with the radii a, b and having the distance of their centres $=2c$, and to fix the ideas assume that $2c>a+b$, that is, that the circles are

[1] Prof. Sylvester remarks that according as β is less or greater than α, we may find real values of θ, ϕ equal to one another in the one case and supplementary in the other. Hence we must in any case be able to make $r=0$ and $s=0$ indifferently, which shows *à priori* that the line being supposed real, each point S, H must be imaginary, but so that the squared distance of either from the line must be a *real negative quantity*, conformably to Prof. Cayley's important observation in the text. W. J. M.

exterior to each other. The foregoing equations signify that $90° - \alpha$, $90° - \beta$ are the inclinations to the line of centres of the inverse and the direct common tangents respectively, and that m is the length of the inverse common tangent. And the theorem is, that considering two circles as above, and taking M a variable point in



the line of centres, if r, s denote the tangential distances of M from the two circles respectively, and if m be the length of the inverse common tangent of the two circles, then the angles θ, ϕ determined by the equations

$$m \sin\theta = r + s, \quad m \sin\phi = r - s,$$

are connected by the relation

$$\cos\beta = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\alpha,$$

(α, β) being constant angles, determined as above.

7. It is to be remarked that, assuming

$$k = \frac{\sin\alpha}{\sin\beta} = \frac{\sqrt{\{4c^2 - (a+b)^2\}}}{\sqrt{\{4c^2 - (a-b)^2\}}},$$

that is, k = inverse common tangent ÷ direct common tangent, then we have

$$\cos\alpha = \sqrt{(1 - k^2\sin^2\beta)} = \Delta\beta,$$

or the equation in θ, ϕ becomes

$$\cos\beta = \cos\theta\cos\phi + \sin\theta\sin\phi\,\Delta\beta,$$

which is the algebraical equation connecting the amplitudes of the elliptic functions in the relation $F(\theta) + F(\phi) = F(\beta)$.

8. It is very noticeable that the above figure leads to another relation in elliptic functions, viz. it is the very figure employed for that purpose by Jacobi; in fact, considering therein YM as a variable tangent meeting the circle A in the two points X and X', then if 2ψ, $2\psi'$ denote the angles GAX, GAX' respectively, it is easy to see geometrically that we have $d\psi : d\psi' = YX : YX'$; where

$$(YX)^2 = (BX)^2 - b^2, \; = 4c^2 + a^2 + 4ac\cos 2\psi - b^2, \; = (2c+a)^2 - b^2 - 8ac\sin^2\psi,$$

and similarly $(YX')^2 = (2c+a)^2 - b^2 - 8ac\sin^2\psi'$, that is, writing $l^2 = \dfrac{8ac}{(2c+a)^2 - b^2}$, the differential equation is

$$\frac{d\psi}{\sqrt{(1 - l^2\sin^2\psi)}} - \frac{d\psi'}{\sqrt{(1 - l^2\sin^2\psi')}} = 0,$$

corresponding to an integral equation

$$F(\psi)-F(\psi')=F(\mu),$$

the modulus of the elliptic functions being

$$l, =\frac{\sqrt{8ac}}{\sqrt{\{(2c+a)^2-b^2\}}}.$$

In the problem above considered the modulus is

$$k, =\frac{\sqrt{\{4c^2-(a+b)^2\}}}{\sqrt{\{4c^2-(a-b)^2\}}},$$

and it is not very easy to see the connexion between the two results.

[Vol. VI. p. 81.]

Theorem: by PROFESSOR CAYLEY.

If (A, A'), (B, B') are four points (two real and the other two imaginary) related to each other as foci and antifoci (that is, if the lines AA', BB' intersect at right angles in a point O in such wise that $OA = OA' = i \,.\, OB = i \,.\, OB'$), then the product of the distances of any point P from the points A, A' is equal to the product of the distances of the same point P from the points B, B'.

In fact, the coordinates of A, A' may be taken to be $(\alpha, 0)$, $(-\alpha, 0)$, and those of B, B' to be $(0, \alpha i)$, $(0, -\alpha i)$; whence, if (x, y) are the coordinates of P, we have

$$(AP)^2=(x-\alpha)^2+y^2=(x-\alpha+iy)(x-\alpha-iy),$$

$$(A'P)^2=(x+\alpha)^2+y^2=(x+\alpha+iy)(x+\alpha-iy),$$

$$(BP)^2=x^2+(y-i\alpha)^2=(x+iy+\alpha)(x-iy-\alpha),$$

$$(B'P)^2=x^2+(y+i\alpha)^2=(x+iy-\alpha)(x-iy+\alpha),$$

from which the theorem is at once seen to be true.

An important application of the theorem consists in the means which it affords of passing from the foci (A, B, C, D) of a bicircular quartic, to the antifoci (A, B) and (C, D); viz. if these are (A', B', C', D'), then the equation $l\sqrt{(A)}+m\sqrt{(B)}+n\sqrt{(C)}=0$ must be transformable into $l'\sqrt{(A')}+m'\sqrt{(B')}+n'\sqrt{(C')}=0$. Writing these respectively under the forms

$$l^2A+m^2B-n^2C+2lm\sqrt{(AB)}=0, \quad l'^2A'+m'^2B'-n'^2C'+2l'm'\sqrt{(A'B')}=0,$$

the two radicals $\sqrt{(AB)}$, $\sqrt{(A'B')}$ are identical; and the remaining terms in the two equations respectively are rational functions, which when the ratios $l' : m' : n'$ are properly determined will be to each other in the ratio $lm : l'm'$; the two equations being thus identical.

[Vol. VI. p. 99.]

1970. (Proposed by Professor CAYLEY.)—Find the conditions in order that the conics

$$U=(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2=0,\quad U'=(a',\ b',\ c',\ f',\ g',\ h' \between x,\ y,\ z)^2=0,$$

may have double contact.

Solution by the PROPOSER.

The coefficients of the two conics must be so related that for a properly determined value of θ we shall have identically $U-\theta U'=(\lambda x+\mu y+\nu z)^2$; but when this is so, the inverse coefficients of the quadric function $U-\theta U'$ are each $=0$; that is, writing

$$(A,\ B,\ C,\ F,\ G,\ H)=(bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch)$$

$$(A',\ B',\ C',\ F',\ G',\ H')=(b'c'-f'^2,\ldots\quad g'h'-a'f',\ldots)$$

$$(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H})=(bc'+b'c-2ff',\ldots\quad gh'+g'h-af'-a'f,\ldots),$$

then we have the six equations $A-\theta\mathfrak{A}+\theta^2A'=0$, &c.

Or, eliminating θ, the required conditions are

$$\left\|\begin{matrix} A, & B, & C, & F, & G, & H \\ A', & B', & C', & F', & G', & H' \\ \mathfrak{A}, & \mathfrak{B}, & \mathfrak{C}, & \mathfrak{F}, & \mathfrak{G}, & \mathfrak{H} \end{matrix}\right\|=0,$$

equivalent to three relations between the two sets of coefficients.

[Vol. VII., January to July, 1867, pp. 17—19.]

2110. (Proposed by Professor CAYLEY.)—Prove that the locus of the foci of the parabolas which pass through three given points is a unicursal quintic curve passing through the two circular points at infinity.

Solution by the PROPOSER.

More generally it may be shown that for the conics which pass through three given points and touch a given line, the locus of the intersection of the tangents drawn from two fixed points Q, Q' on this line to each conic of the series is a unicursal quintic passing through the two points Q and Q'.

Taking the three given points to be the angles of the triangle ($x=0$, $y=0$, $z=0$), and the points Q, Q' to be the points (α, β, γ) and (α', β', γ') respectively, the equation of a conic through the three points is

$$fyz+gzx+hxy=0,$$

which conic will touch the line through the points $(\alpha, \beta, \gamma)(\alpha', \beta', \gamma')$, if

$$\sqrt{\{f(\beta\gamma'-\beta'\gamma)\}}+\sqrt{\{g(\gamma\alpha'-\gamma'\alpha)\}}+\sqrt{\{h(\alpha\beta'-\alpha'\beta)\}}=0.$$

The equation of the pair of tangents from (α, β, γ) to the conic is

$$(f^2, g^2, h^2, -gh, -hf, -bg)(\gamma y-\beta z, \alpha z-\gamma x, \beta x-\alpha y)^2=0,$$

that is

$$\begin{aligned} &x^2(g\gamma+h\beta)^2+y^2(h\alpha+f\gamma)^2+z^2(f\beta+g\alpha)^2\\ &+2yz\{2gh\alpha^2-(h\alpha+f\gamma)(f\beta+g\alpha)\}\\ &+2zx\{2hg\beta^2-(f\beta+g\alpha)(g\gamma+h\beta)\}\\ &+2xy\{2fg\gamma^2-(g\gamma+h\beta)(h\alpha+f\gamma)\}=0,\end{aligned}$$

but one of the tangents through (α, β, γ) being

$$x(\beta\gamma'-\beta'\gamma)+y(\gamma\alpha'-\gamma'\alpha)+z(\alpha\beta'-\alpha'\beta)=0,$$

it follows that the other tangent is

$$x\frac{(g\gamma+h\beta)^2}{\beta\gamma'-\beta'\gamma}+y\frac{(h\alpha+f\gamma)^2}{\gamma\alpha'-\gamma'\alpha}+z\frac{(f\beta+g\alpha)^2}{\alpha\beta'-\alpha'\beta}=0.$$

Hence, writing for shortness

$$\begin{aligned} A&=g\gamma+h\beta, & B&=h\alpha+f\gamma, & C&=f\beta+g\alpha,\\ A'&=g\gamma'+h\beta', & B'&=h\alpha'+f\gamma', & C'&=f\beta'+g\alpha',\end{aligned}$$

the equations of the tangents from Q, Q' respectively are

$$A^2\frac{x}{\beta\gamma'-\beta'\gamma}+B^2\frac{y}{\gamma\alpha'-\gamma'\alpha}+C^2\frac{z}{\alpha\beta'-\alpha'\beta}=0,$$

$$A'^2\frac{x}{\beta\gamma'-\beta'\gamma}+B'^2\frac{y}{\gamma\alpha'-\gamma'\alpha}+C'^2\frac{z}{\alpha\beta'-\alpha'\beta}=0,$$

and for the coordinates of the intersection of these tangents, we have

$$\frac{x}{\beta\gamma'-\beta'\gamma}:\frac{y}{\gamma\alpha'-\gamma'\alpha}:\frac{z}{\alpha\beta'-\alpha'\beta}=B^2C'^2-B'^2C^2:C^2A'^2-C'^2A^2:A^2B'^2-A'^2B^2.$$

$$\begin{aligned} BC'-B'C&=\qquad\qquad f\{-f(\beta\gamma'-\beta'\gamma)+g(\gamma\alpha'-\gamma'\alpha)+h(\alpha\beta'-\alpha'\beta)\}\\ BC'+B'C&=2gh\alpha\alpha'+f\{\ f(\beta\gamma'+\beta'\gamma)+g(\gamma\alpha'+\gamma'\alpha)+h(\alpha\beta'+\alpha'\beta)\}.\end{aligned}$$

To satisfy the equation

$$\sqrt{\{f(\beta\gamma'-\beta'\gamma)\}}+\sqrt{\{g(\gamma\alpha'-\gamma'\alpha)\}}+\sqrt{\{h(\alpha\beta'-\alpha'\beta)\}},$$

write

$$f=\frac{a^2}{\beta\gamma'-\beta'\gamma},\quad g=\frac{b^2}{\gamma\alpha'-\gamma'\alpha},\quad h=\frac{c^2}{\alpha\beta'-\alpha'\beta},$$

and therefore $a+b+c=0$; we then have

$$-f(\beta\gamma'-\beta'\gamma)+g(\gamma\alpha'-\gamma'\alpha)+h(\alpha\beta'-\alpha'\beta),=-a^2+b^2+c^2,=-2bc;$$

and thence

$$f\{-f(\beta\gamma'-\beta'\gamma)+g(\gamma\alpha'-\gamma'\alpha)+h(\alpha\beta'-\alpha'\beta)\}=\frac{a^2}{\beta\gamma'-\beta'\gamma}(-2bc),$$

and the equations become

$$x : y : z = a(BC'+B'C) : b(CA'+C'A) : c(AB'+A'B),$$

where $BC'+B'C$, $CA'+C'A$, $AB'+A'B$, substituting therein for f, g, h the values $\frac{a^2}{\beta\gamma'-\beta'\gamma}$, $\frac{b^2}{\gamma\alpha'-\gamma'\alpha}$, $\frac{c^2}{\alpha\beta'-\alpha'\beta}$, are respectively functions of the fourth degree in a, b, c; hence (a, b, c) being connected by the relation $a+b+c=0$, x, y, z are proportional to quintic functions of (a, b, c), or what is the same thing, writing $a, b, c = 1, \theta, -1-\theta$, then x, y, z are proportional to quintic functions of θ, that is, the locus is a unicursal quintic curve.

That the curve passes through the points $(\alpha', \beta', \gamma')$ and (α, β, γ) appears by considering the conics $fyz+gzx+hxy=0$, which pass through these points respectively.

For the first of these conics we have $f : g : h = \alpha(\beta\gamma'-\beta'\gamma) : \beta(\gamma\alpha'-\gamma'\alpha) : \alpha(\beta\gamma'-\beta'\gamma)$; the equation

$$A^2\frac{x}{\beta\gamma'-\beta'\gamma}+B^2\frac{y}{\gamma\alpha'-\gamma'\alpha}+C^2\frac{z}{\alpha\beta'-\alpha'\beta}=0,$$

reduces itself to $x(\beta\gamma'-\beta'\gamma)+y(\gamma\alpha'-\gamma'\alpha)+z(\alpha\beta'-\alpha'\beta)=0$, and as the other equation

$$A'^2\frac{x}{\beta\gamma'-\beta'\gamma}+B'^2\frac{y}{\gamma\alpha'-\gamma'\alpha}+C'^2\frac{z}{\gamma\alpha'-\gamma'\alpha}=0,$$

is that of a line through $(\alpha', \beta', \gamma')$ the two lines meet of course in the point $(\alpha', \beta', \gamma')$. And the like for the conic

$$f : g : h = \alpha'(\beta\gamma'-\beta'\gamma) : \beta'(\gamma\alpha'-\gamma'\alpha) : \gamma'(\alpha\beta'-\alpha'\beta).$$

If the triangle is equilateral, and (x, y, z) are respectively proportional to the perpendicular distances from the three sides, then we have for the circular points at infinity

$$(\alpha, \beta, \gamma)=(1, \omega, \omega^2), \quad (\alpha', \beta', \gamma')=(1, \omega^2, \omega),$$

where ω is an imaginary cube root of unity. These values give

$$\beta\gamma'-\beta'\gamma=\gamma\alpha'-\gamma'\alpha=\alpha\beta'-\alpha'\beta=\omega^2-\omega$$
$$\alpha\alpha'=\beta\beta'=\gamma\gamma'=1, \quad \beta\gamma'+\beta'\gamma=\gamma\alpha'+\gamma'\alpha=\alpha\beta'+\alpha'\beta=-1;$$

and the expressions for (x, y, z) take the form

$$\begin{aligned} x : y : z = {} & a\{2b^2c^2-a^2(a^2+b^2+c^2)\} \\ & : b\{2c^2a^2-b^2(a^2+b^2+c^2)\} \\ & : c\{2a^2b^2-c^2(a^2+b^2+c^2)\}, \end{aligned}$$

or, what is the same thing, reducing by means of the relation $a+b+c=0$,

$$x : y : z = a(a^4-2a^2bc-2b^2c^2) : b(b^4-2b^2ca-2c^2a^2) : c(c^4-2c^2ab-2a^2b^2),$$

and the equation of the curve is obtained by eliminating (a, b, c) from these equations and the before mentioned equation $a+b+c=0$.

N.B. The above is a particular case of the following general theorem of M. Chasles: If the conics of a system (μ, ν) all of them touch the line QQ', the locus of the intersection of the tangents through Q, Q' to each conic of the series is a curve of the order $\frac{1}{2}\mu+\nu$, having a $(\frac{1}{2}\mu)$-tuple point at the points Q, Q' respectively.

[Vol. VII. pp. 26, 27.]

2250. (Proposed by Professor CAYLEY.)—From the focal equation $x^2+y^2=(lx+n)^2$ of a conic, deduce the remaining three focal equations.

Solution by the PROPOSER.

We are to find α, β, L, M, N such that the equation

$$(x-\alpha)^2+(y-\beta)^2=(Lx+My+N)^2$$

may be identical with the given equation. It is at once seen that we must have $M=0$ or else $L=0$; the first supposition gives two solutions, one of which is the given equation itself, the other is

$$\left(x-\frac{2ln}{1-l^2}\right)^2+y^2=\left\{lx-n\,\frac{1+l^2}{1-l^2}\right\}^2.$$

The second supposition, $L=0$, gives two solutions, which only differ by the sign of $i\ (=\sqrt{-1})$, viz. these are

$$\left(x-\frac{ln}{1-l^2}\right)^2+\left(y\mp\frac{lni}{1-l^2}\right)^2=\frac{-(ly\pm ni)^2}{1-l^2}.$$

There is, of course, no difficulty in *verifying* the identity of each of the three forms with the given form $x^2+y^2=(lx+n)^2$.

[Vol. VII. pp. 33, 34.]

1991. (Proposed by Professor CAYLEY.)—Given a point and three lines; it is required to draw through the point a plane meeting the three lines in three points equidistant from the given point.

Solution by the PROPOSER.

Let O be the given point, $OA'=a$, $OB'=b$, $OC'=c$ the perpendiculars let fall from O on the given lines respectively. Take θ an arbitrary line, and from the points

A', B', C' measure off on the three lines respectively the distances $A'A = \pm\sqrt{(\theta^2 - a^2)}$, $B'B = \pm\sqrt{(\theta^2 - b^2)}$, $C'C = \pm\sqrt{(\theta^2 - c^2)}$, or, considering each radical as containing implicitly the sign $\pm$, what is the same thing, the distances $A'A = \sqrt{(\theta^2 - a^2)}$, $B'B = \sqrt{(\theta^2 - b^2)}$, $C'C = \sqrt{(\theta^2 - c^2)}$, then we have $OA = OB = OC (= \theta)$; and consequently the problem is to determine θ in such wise that the plane ABC may pass through the given point O: for we shall then have through O a plane meeting the three given lines in the points A, B, C equidistant from O.

The coordinates of A, B, C are linear functions of the radicals $\sqrt{(\theta^2 - a^2)}$, $\sqrt{(\theta^2 - b^2)}$, $\sqrt{(\theta^2 - c^2)}$ respectively. Taking O as origin, the condition in order that the plane ABC may pass through O is

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0,$$

and substituting for the coordinates their values in terms of θ, this is an equation linear in each of the three radicals, or say, an equation of the form

$$(\sqrt{(\theta^2 - a^2)},\ 1)(\sqrt{(\theta^2 - b^2)},\ 1)(\sqrt{(\theta^2 - c^2)},\ 1) = 0.$$

But we may represent any one of the three radicals, say $\sqrt{(\theta^2 - c^2)}$ by a single letter s; and this being so, we have $\sqrt{(\theta^2 - a^2)} = \sqrt{(s^2 + c^2 - a^2)} = \sqrt{P}$ suppose, and $\sqrt{(\theta^2 - b^2)} = \sqrt{(s^2 + c^2 - b^2)} = \sqrt{Q}$ suppose; and it is to be observed that there is no loss of generality in assuming that the distance $C'c = s$ is measured off from C' in a determinate sense, for as s passes from $-\infty$ to $+\infty$, we thus obtain for c every position whatever on the line in question; whereas the other two distances $A'A$, $B'B$, represented by the radicals $\sqrt{P}$ and $\sqrt{Q}$ respectively, remain each of them with the double sense $\pm$. The equation in s is of the form

$$(s,\ 1)(\sqrt{P},\ 1)(\sqrt{Q},\ 1) = 0,$$

or, what is the same thing, it is of the form

$$\alpha\sqrt{(PQ)} + \beta\sqrt{P} + \gamma\sqrt{Q} + \delta = 0,$$

where $(\alpha, \beta, \gamma, \delta)$ are respectively linear functions of s.

Proceeding to rationalise the equation, we have first

$$\alpha^2 PQ + 2\alpha\delta\sqrt{(PQ)} + \delta^2 = \beta^2 P + \gamma^2 Q + 2\beta\gamma\sqrt{(PQ)},$$

and then finally

$$(\alpha^2 PQ - \beta^2 P - \gamma^2 Q + \delta^2)^2 = 4(\beta\gamma - \alpha\delta)^2 PQ,$$

which, observing that P, Q are each of them of the second order in s, is an equation of the twelfth order in s; that is, the number of solutions is $= 12$.

The solution of the problem is greatly simplified when $a = b = c$, that is, when the three given lines are tangents to a sphere having its centre at the given point. We have in this case $\sqrt{P} = \pm s$, $\sqrt{Q} = \pm s$, or the equation in s is

$$(s,\ 1)(\pm s,\ 1)(\pm s,\ 1) = 0;$$

that is, the equation of the twelfth order breaks up into four equations each of the third order. The geometrical theory may also be further developed. In fact, assuming on each of the three lines respectively a certain sense as positive (and thus isolating a set of three solutions) the construction is, on the three lines, from the points A', B', C' respectively, measure off the distances $A'A = B'B = C'C = s$. Then the points A, B, C form on the three lines respectively three homographic series; that is, the lines BC, CA, AB are respectively generating lines of three hyperboloids, viz. hyperboloids which pass respectively through the second and third lines, the third and first lines, and the first and second lines. Taking the given point O as the centre of projection, and projecting the whole figure on any plane whatever, the projections of the lines BC are the tangents of a conic which is the projection of the visible contour of the hyperboloid generated by the lines BC; and the like for the lines CA and AB. Hence in the projection, or plane figure, we have a triangle whereof the sides A', B', C' are the projections of the three given lines respectively; inscribed in this triangle we have a variable triangle ABC, such that the side

BC envelopes a conic, say (A), which touches B' and C',

CA envelopes a conic, say (B), which touches C' and A',

AB envelopes a conic, say (C), which touches A' and B'.

The conics (A) (B) (C) have three common tangents, say L, M, N; the conics

(B) and (C) having besides the common tangent A',

(C) and (A) having besides the common tangent B',

(A) and (B) having besides the common tangent C',

so that the common tangents of the conics (B) and (C), (C) and (A), (A) and (B) are the lines A', B', C' each once, and the lines L, M, N each three times. In the entire series of triangles ABC there are three triangles which degenerate into the lines L, M, N respectively, these being in fact the projections of the triangles ABC of the solid figure which lie in a plane with O. Or, what is the same thing, the planes of the required triangles ABC of the solid figure are the planes through O and the three lines L, M, and N, respectively.

[Vol. VII. pp. 34—36.]

1993. (Proposed by T. COTTERILL, M.A.)—If P is a point on a circle, in which A and B are fixed points on a diameter at equal distances from its centre, the curve envelope of lines cutting harmonically the two circles whose centres are A and B and radii AP, BP respectively, is independent of the position of P on the circle.

Solution by PROFESSOR CAYLEY.

1. More generally, the problem may be thus stated: If two conics touch at I, J the lines OI, OJ respectively; if P be a variable point on the first conic, and OAB

a fixed line through O meeting the second conic in the points A and B; then considering the conic which passes through P and touches at I, J the lines AI, AJ respectively, and also the conic which passes through P and touches at I, J the lines BI, BJ respectively; the envelope of the lines which cut harmonically the last-mentioned two conics is a conic independent of the position of P.

2. Taking $x=0$, $y=0$, $z=0$ for the equations of the lines OI, JI, and OJ respectively, the equations of the two given conics are

$$xz-y^2=0, \quad kxz-y^2=0;$$

hence the coordinates of P may be taken to be

$$x:y:z=1:\theta:\theta^2,$$

and the coordinates of the points A and B may be taken to be

$$x:y:z=1:k\alpha:k\alpha^2, \text{ and } x:y:z=1:-k\alpha:k\alpha^2.$$

The equations of the lines AI, AJ are

$$k\alpha x-y=0, \quad z-\alpha y=0;$$

hence the equation of the conic touching these lines at the points I, J respectively, and also passing through the point P, is

$$\frac{(k\alpha x-y)(z-\alpha y)}{(k\alpha-\theta)(\theta-\alpha)}=\frac{y^2}{\theta},$$

and similarly the equations of the lines BI, BJ being

$$k\alpha x+y=0, \quad z+\alpha y=0,$$

the equation of the conic touching these lines at the points I, J respectively, and also passing through the point P, is

$$\frac{(k\alpha x+y)(z+\alpha y)}{(k\alpha+\theta)(\theta+\alpha)}=\frac{y^2}{\theta},$$

or multiplying out and reducing, if the equations of the two conics are represented by

$$(a, b, c, f, g, h \between x, y, z)^2=0, \quad (a', b', c', f', g', h' \between x, y, z)^2=0,$$

respectively, then the values of the coefficients are

$$\begin{aligned} a&=0, & a'&=0,\\ b&=2(k\alpha+\theta^2-k\alpha\theta), & b'&=2(-k\alpha^2-\theta^2-k\alpha\theta),\\ c&=0, & c'&=0,\\ f&=-\theta, & f'&=\theta,\\ g&=\theta k\alpha, & g'&=\theta k\alpha,\\ h&=-\theta k\alpha^2, & h'&=\theta k\alpha^2. \end{aligned}$$

Now the tangential equation of the envelope of the line which cuts harmonically the last-mentioned two conics, is

$$(bc' + b'c - 2ff', \ldots gh' + g'h - af' - a'f, \ldots \between \xi, \eta, \zeta)^2;$$

or substituting for a &c. a' &c., their values, it is found that the coefficients of this equation have all of them the common factor $2\theta^2$, and that omitting this factor the equation is independent of θ, viz. the tangential equation of the envelope in question is

$$(1, -k^2\alpha^2, k^2\alpha^4, 0, k(2k-1)\alpha^2, 0 \between \xi, \eta, \zeta)^2 = 0,$$

which proves the theorem.

3. *In particular*, if $k = 1$, that is if the points A, B lie on the conic $xz - y^2 = 0$, then the tangential equation of the envelope is

$$(1, -\alpha^2, \alpha^4, 0, \alpha^2, 0 \between \xi, \eta, \zeta)^2 = 0,$$

that is

$$\xi^2 - \alpha^2\eta^2 + \alpha^4\zeta^2 + 2\alpha^2\xi\zeta = 0;$$

or, what is the same thing, the equation is

$$(\xi - \alpha\eta + \alpha^2\zeta)(\xi + \alpha\eta + \alpha^2\zeta) = 0,$$

and thus the envelope breaks up into the two points

$$\xi - \alpha\eta + \alpha^2\zeta = 0, \quad \xi + \alpha\eta + \alpha^2\zeta = 0;$$

that is, the points $(1, -\alpha, \alpha^2)$ and $(1, \alpha, \alpha^2)$, which are the points A and B respectively. That is, in the problem in its original form, if the points A and B are the extremities of a diameter of a given circle, then the two constructed circles are a pair of orthotomic circles with the centres A and B respectively; and the theorem is the very obvious one, that any line through the centre of either circle cuts the two circles harmonically.

[Vol. VII. pp. 52, 53.]

2270. (Proposed by Professor CAYLEY.)—To reduce the equation of a bicircular quartic into the form $SS' - k^2L = 0$, where $S = 0$, $S' = 0$ are the equations of two circles, $L = 0$ the equation of a line. (See Salmon's *Higher Plane Curves*, p. 128.)

Solution by the PROPOSER.

The equation of a bicircular quartic may be taken to be

$$(x^2 + y^2)^2 + (u_1 + u_0)(x^2 + y^2) + v_2 + v_1 + v_0 = 0,$$

where, and in what follows, the subscript numbers denote the degrees in the coordinates (x, y) of the several functions to which they are attached.

Introducing an arbitrary constant θ_0, and putting the equation under the form

$$(x^2+y^2)^2+(u_1+u_0-\theta_0)(x^2+y^2)+\theta_0(x^2+y^2)+v_2+v_1+v_0=0,$$

this may be identified with

$$(x^2+y^2+p_1+p_0)(x^2+y^2+q_1+q_0)+r_1+r_0=0\ ;$$

viz. the conditions in order to this identity are

$$p_1+p_0+q_1+q_0=u_1+u_0-\theta_0,$$
$$(p_1+p_0)(q_1+q_0)+r_1+r_0=\theta_0(x^2+y^2)+v_2+v_1+v_0,$$

that is

$$p_1+q_1=u_1,\quad p_0+q_0=u_0-\theta_0,$$
$$p_1q_1=\theta_0(x^2+y^2)+v_2,\quad p_1q_0+p_0q_1+r_1=v_1,\quad p_0q_0+r_0=v_0.$$

Hence

$$(p_1-q_1)^2=u_1{}^2-4v_2-4\theta_0(x^2+y^2),$$

where the right-hand side is a quadric function $(x,\ y)^2$, which, when the discriminant thereof is put $=0$, (that is, when θ_0 is determined as the root of a quadric equation,) is a perfect square, p_1-q_1 is then a known linear function, and p_1+q_1 being equal to the linear function u_1, we have p_1 and q_1 as linear functions of $(x,\ y)$. We may take for the constants p_0 and q_0 any values satisfying the equation $p_0+q_0=u_0-\theta_0$; and we then have

$$r_1=v_1-p_1q_0-p_0q_1,\quad r_0=v_0-p_0q_0,$$

which completes the determination; the form

$$(x^2+y^2+p_1+p_0)(x^2+y^2+q_1+q_0)+r_1+r_0=0$$

is of course the same as the proposed form $SS'-k^3L=0$.

COR. A somewhat more convenient form is $UU'-k^2V=0$, where $U=0$, $U'=0$ are the equations of two evanescent circles (pairs of imaginary lines), $V=0$ the equation of a circle; in fact the original form $SS'-k^3L=0$ may be written $(S-\alpha)(S'-\alpha')+(\alpha S'+\alpha' S-\alpha\alpha'-k^3L)=0$, which, when α, α' are so determined that $S-\alpha=0$, $S'-\alpha'=0$ may be evanescent circles, is of the required form $UU'-k^2V=0$. The equation $UU'=0$ is that of the two pairs of tangents to the curve at the circular points at infinity respectively; in fact, writing $U=pq$, $U'=p'q'$, each of the lines $p=0$, $q=0$, $p'=0$, $q'=0$ meets the circle $V=0$ in one or other of the circular points at infinity, and therefore only in a single point not at infinity; hence each of these lines meets the curve $UU'-k^2V=0$ three times in one of the circular points at infinity, that is, the line in question is a tangent to one of the two branches through the circular point at infinity.

[Vol. VII. pp. 87, 88.]

2309. (Proposed by Professor CAYLEY.)—Show that for n things

1 – (no. of partitions into 2 parts) + 1 . 2 (no. of partitions into 3 parts) . . .
$\pm\,1\,.\,2\,.\,3\,.\,.\,(n-1)$ (no. of partitions into n parts) $=0$.

For instance, $n=4$; partitions of (a, b, c, d) into two parts are (a, bcd), (b, cda), (c, dab), (d, abc), (ab, cd), (ac, db), (ad, bc); no. is $=7$. Partitions into three parts are (ab, c, d), (ac, b, d), (ad, b, c), (bc, a, d), (b, d, ac), (cd, a, b); no. is $=6$. Partition into 4 parts is (a, b, c, d); no. is $=1$. And we have

$$1-1.7+2.6-6.1=13-13=0.$$

Solution by the PROPOSER.

Write $n=a\alpha+b\beta+c\gamma+\ldots$, where $a, \beta, \gamma\ldots$ are positive integers all of them different, and $\alpha, \beta, \gamma\ldots$ are positive integers; and consider the partitions wherein we have a parts each of α things, b parts each of β things, &c. Writing as usual $\Pi(n)=1.2.3\ldots n$, the number of partitions of the form in question is

$$=\frac{\Pi n}{\Pi a.\Pi b\ldots(\Pi\alpha)^a(\Pi\beta)^b\ldots};$$

whence, putting for shortness $\alpha+\beta+\ldots=p$, the theorem may be written

$$\Sigma(-)^{p-1}\frac{\Pi(p-1)\,\Pi n}{\Pi a.\Pi b\ldots(\Pi\alpha)^a(\Pi\beta)^b\ldots}=0,$$

the summation extending to all the partitions $n=a\alpha+b\beta+\ldots$, as explained above.

Now if the n quantities $x, y, z,\ldots$ are the nth roots of unity, we have $x+y+z\ldots=0$, and therefore also $(x+y+z\ldots)^n=0$, and the general term of the left-hand is

$$\frac{\Pi n}{(\Pi\alpha)^a(\Pi\beta)^b\ldots}[\alpha^a\beta^b\ldots],$$

where $[\alpha^a\beta^b\ldots]$ denotes the symmetrical function $\Sigma x^\alpha y^\alpha\ldots(a \text{ factors})\, u^\beta v^\beta\ldots(b \text{ factors})\ldots$ of the roots $x, y, z, u, v\ldots$ of the equation $\theta^n-1=0$; where, as above, $n=a\alpha+b\beta+\ldots$. Now by a formula not, I believe, generally known, but which is given on p. 175 of the translation of Hirsch's Algebra (Hirsch's *Collection of Examples &c. on the Literal Calculus and Algebra*, translated by the Rev. J. A. Ross, London, 1827), the value of the sum in question is $=(-)^{p-1}\dfrac{\Pi(p-1)}{\Pi a.\Pi b\ldots}n$, where $p=a+b+\ldots$, (the sign $\pm$, given in the formula as quoted, is at once seen to be $(-)^{p-1}$); whence, substituting and omitting the factor n, we have

$$\Sigma(-)^{p-1}\frac{\Pi(p-1)\,\Pi n}{\Pi a.\Pi b\ldots(\Pi\alpha)^a(\Pi\beta)^b\ldots}=0,$$

which is the required theorem.

OBSERVATION. In Cauchy's *Exercices d'Analyse &c.*, t. III., p. 173, is given a formula relating to the same mode of partition of the number n, viz. this is

$$\Sigma\frac{\Pi n}{\Pi a.\Pi b\ldots\alpha^a\beta^b\ldots}=\Pi n.$$

I have somewhere made the remark that, on the left-hand side, the terms which belong to the odd and the even values of $a+b+\ldots(=p)$ are equal, and that we have therefore

$$\Sigma(-)^{p-1}\frac{\Pi n}{\Pi a\,.\,\Pi b\ldots\alpha^a\beta^b\ldots}=0,$$

which is a theorem having a curious analogy with that demonstrated above.

[Vol. VII. pp. 99—102.]

2286. (Proposed by W. H. LAVERTY.)—If we have $(n-2)$ sets of n quantities each, $(\alpha_1, \alpha_2\ldots\alpha_n)$, $(\beta_1, \beta_2\ldots\beta_n)$, ... $(\lambda_1, \lambda_2\ldots\lambda_n)$, connected with the n quantities $(r_1, r_2\ldots r_n)$ by $\frac{1}{2}n(n-1)$ equations of which the type form is

$$(\alpha_k-\alpha_l)^2+(\beta_k-\beta_l)^2+\ldots(\lambda_k-\lambda_l)^2=r_k^2+r_l^2;$$

then show that

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\ldots+\frac{1}{r_n^2}=0 \text{ and } \frac{P_1}{r_1^2}+\frac{P_2}{r_2^2}+\ldots\frac{P_n}{r_n^2}=0,$$

where P is any one of the quantities $\alpha, \beta, \gamma\ldots\lambda$.

Solution by PROFESSOR CAYLEY.

Consider the case $n=4$; we have between $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $(\beta_1, \beta_2, \beta_3, \beta_4)$, ... (r_1, r_2, r_3, r_4) six equations, such as the equation

$$(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2=r_1^2+r_2^2; \qquad (12)$$

and it is in effect required to show that these equations give

$$\frac{1}{r_1^2}:\frac{1}{r_2^2}:\frac{1}{r_3^2}:\frac{1}{r_4^2}=(234):-(341):(412):-(123),$$

where

$$(123)=\begin{vmatrix}\alpha_1, & \beta_1, & 1\\ \alpha_2, & \beta_2, & 1\\ \alpha_3, & \beta_3, & 1\end{vmatrix}, \text{ \&c.},$$

viz. considering (α_1, β_1), (α_2, β_2), (α_3, β_3), (α_4, β_4) as the rectangular coordinates of four points in a plane, then (123) is the area (taken with a proper sign) of the triangle formed by the points 1, 2, 3; and the like for (234) &c.

Combining the equations as follows,

$$(12)+(34)-(13)-(24),$$

the r's disappear, and we have an equation

$$(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)+(\beta_1-\beta_4)(\beta_2-\beta_3)=0,$$

which shows that the lines 14 and 23 intersect at right angles; similarly the lines 12 and 34, and also the lines 13 and 24, intersect at right angles; or starting from the given points 1, 2, 3, the point 4 is the intersection of the perpendiculars let fall from the angles 1, 2, 3 of the triangle 123 on the opposite sides respectively.

Again combining the equations as follows,

$$(12)+(13)-(23),$$

we obtain

$$r_1^2=(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)+(\beta_1-\beta_2)(\beta_1-\beta_3).$$

The entire system of equations will remain unaltered if we pass from the original axes to any other system of rectangular axes; hence taking the axes of x in the sense from 1 to 2 along the line 12, $\beta_1-\beta_2$ becomes $=0$, and we have

$$\alpha_2-\alpha_1=12,\quad \alpha_3-\alpha_1=1\,(12,\ 34);$$

viz. $\alpha_2-\alpha_1$ is the distance 12 of the points 1 and 2, $\alpha_3-\alpha_1$ is the distance 1 (12, 34) of the point 1 from the point (12, 34) which is the intersection of the lines 12 and 34; we have therefore

$$r_1^2=12\,.\,1\,(12,\ 34).$$

But similarly

$$r_2^2=21\,.\,2\,(12,\ 34),\ =12\,.\,(12,\ 34)\,2,$$

(since $21=-12$ and $2\,(12,\ 34)=-(12,\ 34)\,2$). And we have therefore

$$r_1^2 : r_2^2=1\,(12,\ 34) : (12,\ 34)\,2,\ \text{ or }\ \frac{1}{r_1^2} : \frac{1}{r_2^2}=(12,\ 34)\,2 : 1\,(12,\ 34).$$

Write

$$\lambda=\frac{(12,\ 34)\,2}{12},\quad \mu=\frac{1\,(12,\ 34)}{12}$$

where 1 (12, 34) and (12, 34) 2 are as above the distances from 1 to (12, 34) and from (12, 34) to 2; and, in the denominators, 12 is the distance from 1 to 2; we have $\lambda+\mu=1$; the coordinates of (12, 34) are $\lambda\alpha_1+\mu\alpha_2$, $\lambda\beta_1+\mu\beta_2$, and the values of λ, μ are obtained by writing $\lambda\alpha_1+\mu\alpha_2$, $\lambda\beta_1+\mu\beta_2$, $\lambda+\mu$ for x, y, 1 in the equations

$$\begin{vmatrix} x, & y, & 1 \\ \alpha_3, & \beta_3, & 1 \\ \alpha_4, & \beta_4, & 1 \end{vmatrix}=0$$

of the line 34. Making this substitution, we find

$$\lambda\,(134)+\mu\,(234)=0,$$

where as above

$$(134)=\begin{vmatrix} \alpha_1, & \beta_1, & 1 \\ \alpha_3, & \beta_3, & 1 \\ \alpha_4, & \beta_4, & 1 \end{vmatrix},\ \text{\&c.},$$

we have therefore

$$\lambda : \mu = (234) : -(134) = (234) : -(341),$$

or, what is the same thing,

$$(12,\ 34)\,2 : 1\,(12,\ 34) = (234) : -(341);$$

and consequently

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} = (234) : -(341);$$

or completing the system by symmetry

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r_3^2} : \frac{1}{r_4^2} = (234) : -(341) : (412) : -(123),$$

which is the required result.

In the case $n=5$, we have between

$$(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5),\ (\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5),\ (\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5),\ (r_1,\ r_2,\ r_3,\ r_4,\ r_5)$$

ten equations such as the equation

$$(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+(\gamma_1-\gamma_2)^2 = r_1^2+r_2^2. \tag{12}$$

We obtain as before the equation

$$(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)+(\beta_1-\beta_4)(\beta_2-\beta_3)+(\gamma_1-\gamma_4)(\gamma_2-\gamma_3)=0,$$

which, considering $(\alpha_1,\ \beta_1,\ \gamma_1)$ &c. as the rectangular coordinates of five points 1, 2, 3, 4, 5 in space, signifies that the line 14 is at right angles to the line 23; the five points are therefore such that the line joining any two of them is at right angles to the line joining any other two of them, whence also the line joining any two is at right angles to the plane through the remaining three points. (The points 1, 2, 3, 4 form a tetrahedron such that that 12 and 34, also 13 and 42, also 14 and 23 are at right angles to each other, two of these conditions imply the third; and this being so, if a further condition be satisfied, the perpendiculars from 1, 2, 3, and 4 on the opposite faces respectively, will meet in a point 5, and we shall have the system of points 1, 2, 3, 4, 5 related as above.)

We further obtain as before

$$r_1^2 = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)+(\beta_1-\beta_2)(\beta_1-\beta_3)+(\gamma_1-\gamma_2)(\gamma_1-\gamma_3),$$

or taking the axis of x in the sense from 1 to 2 along the line 12, we have $\beta_1-\beta_2=0$, $\gamma_1-\gamma_2=0$, and the equation becomes

$$r_1^2 = 12\,.\,1\,(12,\ 345),$$

and similarly

$$r_2^2 = 12\,.\,(12,\ 345)\,2;$$

whence

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} = (12,\ 345)\,2 : (12,\ 345)\,1.$$

Writing them $\lambda = \frac{(12.345)\,2}{12}$, $\mu = \frac{1\,(12,\ 345)}{12}$ (and therefore $\lambda + \mu = 1$) we find (λ, μ) by substituting $\lambda\alpha_1 + \mu\alpha_2$, $\lambda\beta_1 + \mu\beta_2$, $\lambda\gamma_1 + \mu\gamma_2$, $\lambda + \mu$ for x, y, z, 1 in the equation

$$\begin{vmatrix} x, & y, & z, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \\ \alpha_5, & \beta_5, & \gamma_5, & 1 \end{vmatrix} = 0$$

of the plane 345; we have thus

$$\lambda\,(1345) + \mu\,(2345) = 0,$$

that is

$$\lambda : \mu = (2345) : -(1345) = (2345) : (3451),$$

whence

$$(12,\ 345)\,2 : 1\,(12,\ 345) = (2345) : (3451),$$

that is

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} = (2345) : (3451),$$

or completing by symmetry

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r_3^2} : \frac{1}{r_4^2} : \frac{1}{r_5^2} = (2345) : (3451) : (4512) : (5123) : (1234),$$

which is the theorem for the case $n = 5$. The general case depends, it is clear, upon similar reasoning in a $(n-2)$dimensional geometry; leading to the conception in this geometry of a figure of $(n-1)$ points such that the line joining any two of them is at right angles to the line joining any other two of them.

[Vol. VII. p. 106.]

2331. (Proposed by Professor CAYLEY.)—Show that it is possible to find (X, Y, Z) linear functions of the trilinear coordinates (x, y, z) such that the equations $xX = yY = zZ$ may determine four given points.

[Vol. VIII., July to December, 1867, p. 26.]

2321. (Proposed by Professor CAYLEY.)—Given a conic, to find four points such that all the conics through the four points may have their centres in the given conic.

[Vol. VIII. p. 36.]

2371. (Proposed by Professor CAYLEY.)—(4). If P, Q be two points taken at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle?

[Vol. VIII. pp. 51, 52.]

Note on Question 1990. *By* PROFESSOR CAYLEY.

The theorem of paragraph 4 (*Reprint*, vol. VI. p. 88), (ascribed by Professor Sylvester to Mr Crofton), that "if a circle and a straight line be cut by any transversal in three points, these will be the foci of a system of Cartesian ovals having double contact with one another at two fixed points," may be enunciated under a more complete form, as follows:

If in a given circle the chords PP_1, BC meet in A, then each of the two Cartesians, foci A, B, C, which pass through P, will also pass through P_1; and moreover, if α, α' be the diametrals of the chord PP_1 (that is, the extremities of the diameter at right angles to PP_1) then the tangents at P, P_1 to one of the Cartesians will be αP, αP_1 respectively, and to the other of them $\alpha' P$, $\alpha' P_1$ respectively, these tangents being thus independent of the position of the chord BC; and thence also thus;

Given the points A, B, C *in lineâ*, and the point P;

through P, B, C draw a circle (A) and let PA meet this in P_1,
„ P, C, A „ (B) „ PB „ P_2,
„ P, A, B „ (C) „ PC „ P_3,

then each of the Cartesians, foci A, B, C, which pass through P will also pass through P_1, P_2, P_3; and if

α, α' are the diametrals of PP_1 in circle (A),
β, β' „ PP_2 „ (B),
γ, γ' „ PP_3 „ (C),

then (the points of the several pairs being properly selected) the points (α, β, γ) and the points $(\alpha', \beta', \gamma')$ will each lie in a line through P, viz. the lines $P\alpha\beta\gamma$ and $P\alpha'\beta'\gamma'$ will be the tangents at P to the two Cartesians respectively.

The two Cartesians meet in the points P, P_1, P_2, P_3, and in the symmetrically situated points in regard to the axis ABC; the theorem contains as part of itself the well-known property that the two Cartesians cut at right angles at each of their points of intersection; it gives moreover the construction of the following problem:—given the foci A, B, C, and one intersection P of a pair of triconfocal Cartesians, to find the remaining intersections, and the tangents at each of the intersections.

[Vol. VIII. pp. 70—72.]

1911. (Proposed by Professor CAYLEY.)—Given four points, and also the "conic of centres"—viz. the conic which is the locus of the centres of the several conics which pass through the four given points; then if a conic through the four given points has for its centre a given point on the conic of centres, it is required to find a construction for the asymptotes of this conic.

Solution by the PROPOSER.

1. Consider four given points, and in connection therewith a given line IJ; the locus of the poles of IJ, in regard to the several conics which pass through the four points, is a conic, the "conic of poles." Consider a particular conic Θ, through the four points; the pole of IJ in regard to the conic Θ is a point C on the conic of poles, and the tangents from C to the conic Θ meet the conic of poles in two points H, K; the chord of intersection HK passes through the point Π which is the pole of IJ in regard to the conic of poles. Moreover, the polars of a point C', in regard to the several conics through the four points, meet in a point Ω', the "common pole" of C', and in particular if C' be the point C on the conic of poles, then the common pole is a point Ω on the line IJ; this being so, the line HK passes (as already mentioned) through Π, and the lines HK and $\Pi\Omega$ are harmonics in regard to the conic of poles.

2. Assuming the foregoing properties, then, given the four points, the line IJ, the conic of poles, and the point C on this conic; we may construct Π the pole of IJ in regard to the conic of poles; and also Ω the common pole of C; the line HK is then given as a line passing through Π, and harmonic to $\Pi\Omega$ in regard to the conic of poles; this line meets the conic of poles in the points H, K; and then CH, CK are the tangents from C to a conic Θ which passes through the four points.

3. In particular if IJ be the line infinity, then the conic of poles is the conic of centres; Π is the centre of this conic; Ω is as before the common pole of C; HK is given as the diameter of the conic of centres, conjugate to $\Pi\Omega$; H, K are the extremities of this diameter; and then CH, CK are the asymptotes of the conic through the four points, which has the point C for its centre; and the asymptotes are therefore constructed as required. If the points H, K are imaginary, the asymptotes will be also imaginary; the conic Θ is in this case an ellipse.

4. It is hardly necessary to remark, in regard to the construction of the point Ω, that we have among the conics through the four points, three pairs of lines meeting in points P, Q, R respectively (it is clear that the conic of poles passes through these three points); the harmonics of CP, CQ, CR in regard to the three pairs of lines respectively meet in a point, which is the required point Ω. In the particular case where the point C is on the conic of centres, the three harmonics are parallel; it is therefore sufficient to construct *one* of them; and the line HK is then the diameter of the conic of poles, conjugate to the harmonic so constructed.

5. It remains to prove the properties assumed in (1). We may take $z=0$ for the equation of the line IJ, $x=0$, $y=0$ for the equations of the tangents to the conic Θ at its intersections with the line IJ, so that we have $(x=0,\ y=0)$ for the coordinates of the point C; the equation of the conic Θ will be of the form $z^2-xy=0$, and the four points may then be taken to be the intersections of the conic $z^2-xy=0$, and the arbitrary conic

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=0.$$

The equation of the conic of centres is found to be

$$x(ax+hy+gz)-y(hx+by+fz)=0,\ \text{or}\ ax^2-by^2+gzx-hxy=0\ ;$$

or, as it may also be written,

$$(2a,\ -2b,\ 0,\ -f,\ g,\ 0 \between x,\ y,\ z)^2 = 0;$$

and it is convenient to remark that the equation in line coordinates (or condition that this conic may be touched by the line $\xi x + \eta y + \zeta z = 0$) is

$$(-f^2,\ -g^2,\ -4ab,\ 2af,\ 2bg,\ -fg \between \xi,\ \eta,\ \zeta)^2 = 0.$$

The line $x=0$ meets the conic of poles in the point $x=0$, $by+fz=0$, and the line $y=0$ meets the same conic in the point $y=0$, $ax+gz=0$; hence the line HK, which is the line joining these two points, has for its equation

$$af\,x + bg\,y + fg\,z = 0;$$

and it only remains to be shown that this line passes through the point Π, and is the harmonic of the line $\Pi\Omega$ in regard to the conic of centres. The point Π is the pole of the line $z=0$ in regard to the conic of centres, its coordinates are at once found to be

$$x : y : z = bg : af : -2ab;$$

and we thence see that Π is a point on the line HK. The point Ω is given as the intersection of the polars of C in regard to the conics $z^2 - xy = 0$, and $(a, b, c, f, g, h \between x, y, z)^2 = 0$ respectively; that is, as the intersection of the lines $z=0$, and $gx+fy+cz=0$; its coordinates therefore are

$$x : y : z = -f : g : 0.$$

Hence the equation of the line $\Pi\Omega$ is

$$2abg\,x + 2abf\,y + (af^2 + bg^2)\,z = 0.$$

Now, in general, if we have a conic the line-equation whereof is $(A, B, C, F, G, H \between \xi, \eta, \zeta)^2 = 0$, then the condition in order that, in regard thereto, the lines $\lambda x + \mu y + \nu z = 0$ and $\lambda' x + \mu' y + \nu' z = 0$ may be harmonics, is

$$(A,\ B,\ C,\ F,\ G,\ H \between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu') = 0;$$

that is

$$A\lambda\lambda' + B\mu\mu' + C\nu\nu' + F(\mu\nu' + \mu'\nu) + G(\nu\lambda' + \nu'\lambda) + H(\lambda\mu' + \lambda'\mu) = 0.$$

Hence, in order that the two lines HK and $\Pi\Omega$ may be harmonics in regard to the conic of centres, we should have

$$(-f^2,\ -g^2,\ -4ab,\ 2af,\ 2bg,\ -fg \between af,\ bg,\ fg \between 2abg,\ 2abf,\ af^2 + bg^2) = 0.$$

But developing, and omitting the common factor $abfg$, which enters into all the terms, this equation is

$$-(2af^2) - (2bg^2) - 4(af^2 + bg^2) + \{4af^2 + 2(af^2 + bg^2)\} + \{4bg^2 + 2(af^2 + bg^2)\} - 2(af^2 + bg^2) = 0,$$

which is identically true; and the lines HK and $\Pi\Omega$ are therefore harmonics in regard to the conic of centres.

[Vol VIII. p. 74.]

2371. (Proposed by Professor CAYLEY.)—If P, Q be two points taken at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle?

[Vol. VIII. pp. 86, 87.]

2466. (Proposed by H. MURPHY.)—If four points A, B, C, D be either in the same plane or not, and if the three rectangles $AB.CD$, $AC.DB$, $AD.BC$ be taken; the sum of any two of them is greater than the third, except when the points lie on the circumference of a circle.

Solution by PROFESSOR CAYLEY.

Write for shortness $BC=f$, $CA=g$, $AB=h$; $AD=a$, $BD=b$, $CD=c$; then, Lemma, if r be the radius of the sphere circumscribed about the tetrahedron $ABCD$, we have

$$4r^2\left\{\begin{array}{l}-a^4f^2-b^4g^2-c^4h^2-f^2g^2h^2\\+(a^2f^2+b^2c^2)(g^2+h^2-f^2)\\+(b^2g^2+c^2a^2)(h^2+f^2-g^2)\\+(c^2h^2+a^2b^2)(f^2+g^2-h^2)\end{array}\right\}=2b^2c^2g^2h^2+2c^2a^2h^2f^2+2a^2b^2f^2g^2-a^4f^4-b^4g^4-c^4h^4,$$

where the left-hand side is $=576V^2r^2$, if V be the volume of the tetrahedron.

Suppose first that the points are not in the same plane, then the left-hand side ($=576V^2r^2$) is positive; therefore the right-hand side is also positive, or putting for shortness $af=\alpha$, $bg=\beta$, $ch=\gamma$, we have

$$2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4=+,\text{ that is, }4\beta^2\gamma^2-(\alpha^2-\beta^2-\gamma^2)^2=+,$$

and thence $\alpha<\beta+\gamma$; for if α were equal to or greater than $\beta+\gamma$, say $\alpha=\beta+\gamma+x$, the left-hand side would be $4\beta^2\gamma^2-\{2\beta\gamma+2(\beta+\gamma)x+x^2\}^2$, which vanishes if $x=0$, and is negative for x positive. Similarly $\beta<\gamma+\alpha$, $\gamma<\alpha+\beta$; and the theorem is thus proved for the case where the four points are not in a plane.

Starting from this general case, if we imagine the point D continually to approach and ultimately to coincide with the plane ABC, but so as not to be in the circle ABC, then the expression $2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4$, which does not vanish in the limit, is throughout equal to the positive quantity $576V^2r^2$ (in the limit V is $=0$ and $r=\infty$, but Vr is finite, and of course V^2r^2 is positive), that is, the expression in question is $=+$, and the theorem follows as before. Of course when the four points are in a circle, then the expression is $=0$, and consequently one of the quantities α, β, γ is equal to the sum of the other two.

The lemma is at once proved by means of my theorem for the relation between the distances of five points in space, {*Cambridge Mathematical Journal*, vol. II. (1841), p. 269, [1],} viz. if the point 1 is the centre of the circumscribed sphere, and the points 2, 3, 4, 5 are the points A, B, C, D respectively, then the relation in question, viz.

$$\begin{vmatrix} 0\ , & (12)^2, & (13)^2, & (14)^2, & (15)^2, & 1 \\ (21)^2, & 0\ , & (23)^2, & (24)^2, & (25)^2, & 1 \\ (31)^2, & (32)^2, & 0\ , & (34)^2, & (35)^2, & 1 \\ (41)^2, & (42)^2, & (43)^2, & 0\ , & (45)^2, & 1 \\ (51)^2, & (52)^2, & (53)^2, & (54)^2, & 0\ , & 1 \\ 1\ , & 1\ , & 1\ , & 1\ , & 1\ , & 0 \end{vmatrix} = 0$$

becomes

$$\begin{vmatrix} 0, & r^2, & r^2, & r^2, & r^2, & 1 \\ r^2, & 0, & h^2, & g^2, & a^2, & 1 \\ r^2, & h^2, & 0, & f^2, & b^2, & 1 \\ r^2, & g^2, & f^2, & 0, & c^2, & 1 \\ r^2, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

Multiplying the last line by $-r^2$ and adding it to the first line, this is

$$\begin{vmatrix} -r^2, & 0, & 0, & 0, & 0, & 1 \\ r^2, & 0, & h^2, & g^2, & a^2, & 1 \\ r^2, & h^2 & 0, & f^2, & b^2, & 1 \\ r^2, & g^2, & f^2, & 0, & c^2, & 1 \\ r^2, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

and then proceeding in the same way with the first and last columns the equation is

$$\begin{vmatrix} -2r^2, & 0, & 0, & 0, & 0, & 1 \\ 0\ , & 0, & h^2, & g^2, & a^2, & 1 \\ 0\ , & h^2, & 0, & f^2, & b^2, & 1 \\ 0\ , & g^2, & f^2, & 0, & c^2, & 1 \\ 0\ , & a^2, & b^2, & c^2, & 0, & 1 \\ 1\ , & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is in fact the equation of the Lemma. See my papers in the *Quarterly Journal of Mathematics*, vol. III. (1859), pp. 275—277, [286], and vol. V. (1861), pp. 381—384, [297].

COR.—It appears by the demonstration that for any four points not in the same plane, the expression

$$-a^4f^2-b^4g^2-c^4h^2-f^2g^2h^2$$
$$+(a^2f^2+b^2c^2)(g^2+h^2-f^2)+(b^2g^2+c^2a^2)(h^2+f^2-g^2)+(c^2h^2+a^2b^2)(f^2+g^2-h^2)$$

is always positive.

[Vol. VIII. pp. 105, 106.]

2472. (Proposed by Professor CAYLEY.)—Through four points on a circle to draw a conic such that an axis may pass through the centre of the circle.

Solution by the PROPOSER.

Let the equation of the conic be $(a, b, c, f, g, h \between x, y, 1)^2=0$, then if as usual the inverse coefficients are represented by (A, B, C, F, G, H), the equation of the two axes is

$$(a-b)(Cx-G)(Cy-F)+h[(Cx-G)^2-(Cy-F)^2]=0,$$

whence if an axis pass through the origin

$$(a-b)FG+h(G^2-F^2)=0.$$

Consider now the circle $x^2+y^2-1=0$ and on it the four points in which it is intersected by the conic $(a, b, c, f, g, h \between x, y, 1)^2=0$; then for any conics through the four points we have

$$(a, b, c, f, g, h \between x, y, 1)^2+\lambda(x^2+y^2-1)=0;$$

so that, taking this for the equation of the required conic, and representing it by

$$(a', b', c', f', g', h' \between x, y, 1)^2=0,$$

the values of the coefficients are

$$a'=a+\lambda,\quad b'=b+\lambda,\quad c'=c+\lambda,\quad f'=f,\quad g'=g,\quad h'=h,$$

and we thence have

$$F'=F-\lambda f,\quad G'=G-\lambda g,\quad a'-b'=a-b,\quad h'=h.$$

The required relation is

$$(a'-b')F'G'+h'(G'^2-F'^2)=0,$$

that is

$$(a-b)(F-\lambda f)(G-\lambda g)+h\{(G-\lambda g)^2-(F-\lambda f)^2\}=0,$$

a quadric equation in λ; and substituting for λ each of its two values, we have the two required conics

$$(a, b, c, f, g, h \between x, y, 1)^2+\lambda(x^2+y^2-1)=0,$$

for each of which an axis passes through the centre of the circle.

[Vol. IX., January to June, 1868, pp. 20, 21.]

Note on Question 2471. *By* PROFESSOR CAYLEY.

In the singularly beautiful solution which Mr Woolhouse has given of this question (see *Reprint*, vol. VIII. p. 100), it is important to note what is the analytical problem solved, and how the solution is obtained. Considering a plane area bounded by any closed convex curve, and in it three points P, P', P'', Mr Woolhouse investigates the average area of the triangle $PP'P''$, viz. this depends on the sextuple integral

$$\int \pm \{x'y'' - x''y' + x''y - xy'' + xy' - x'y\}\, dx\, dy\, dx'\, dy'\, dx''\, dy'',$$

where the sign $\pm$ has to be taken so that $\pm\{\ \}$ shall be positive, and where the integration in respect to each set of coordinates extends over the entire closed area; the difficulty is as to the mode of dealing with the discontinuous sign. It is remarked that the integral is

$$= 6 \int \pm \{x'y'' - x''y' + x''y - xy'' + xy' - x'y\}\, dx\, dy\, dx'\, dy'\, dx''\, dy'';$$

the variables in this last expression being restricted in such wise that x, x'', x' are in the order of increasing magnitude; the term $\pm\{\ \}$ is of the form $\pm(x'-x)(y''-\beta)$, where β is independent of y, and where (as is easily seen) if v'', u'' be the upper and lower ordinate corresponding to the abscissa x'', then β lies between the values u'' and v''. But $x'-x$ is positive, hence the sign $\pm$ must be so taken that $\pm(y''-\beta)$ shall be positive, that is, from $y''=u''$ to $y''=\beta$ the sign is $-$, and from $y''=\beta$ to $y''=v''$ the sign is $+$.

Hence for the integration in regard of y'' we have

$$\int \pm (y''-\beta)\, dy'' = \int_\beta^{v''} + (y''-\beta)\, dy'' + \int_{u'}^{\beta} - (y''-\beta)\, dy'', = \tfrac{1}{2}(v''-\beta)^2 + \tfrac{1}{2}(\beta - u'')^2;$$

and the discontinuous sign $\pm$ is thus got rid of. The remaining integrations are then effected in the order x'', y', y, x', x, the limits being for x'' from x to x', for y from u' to v', and for y from u to v (if the upper and lower ordinates corresponding to the abscissa x and x' are v, u and v', u' respectively) and finally for x' from x to the maximum abscissa, and for x from the minimum to the maximum abscissa. The final result involves only single definite integrals between the extreme values of x, the functions under the integral sign containing indefinite integrations from the same arbitrary inferior limit, say $x=0$; the form of the result (previous to its simplification by taking the axes to be principal axes through the centre of gravity of the area) is however somewhat complicated; and it would not be easy to show *a posteriori*, that the value is invariantive, that is, independent of the position of the axes: that this is so is of course apparent from the original form of the integral.

[Vol. IX. pp. 38, 39.]

2530. (Proposed by Professor CAYLEY.)—Trace the curve

$$\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{(x+iy)}} + \frac{1}{\sqrt{(x-iy)}} = 0,$$

where the coordinates x, y, z are the perpendicular distances of the current point P from the sides of an equilateral triangle, the coordinates being positive for a point within the triangle.

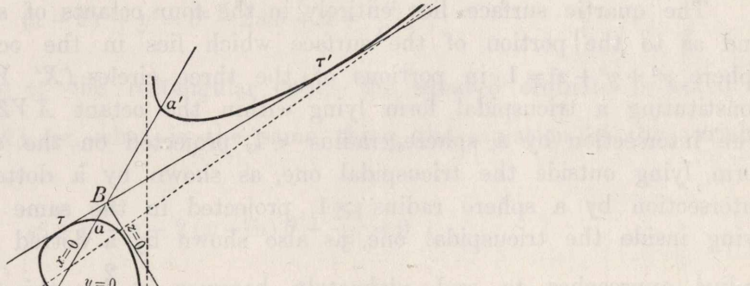
Solution by the PROPOSER.

The form of the equation shows that the curve is a tricuspidal quartic, having a real cusp at the point $(x=0, y=0)$, and two imaginary cusps at the points $(z=0, x+iy=0)$ and $(z=0, x-iy=0)$. The rationalised form of the equation is

$$(x^2+y^2)^2 - 4zx(x^2+y^2) - 4z^2y^2 = 0.$$

$x=0$ gives $y^2(y^2-4z^2)=0$, the point C twice, and two other real points α, α' on the line BC.

$y=0$ gives $x^3(x-4z)=0$, the point C three times, and a real point β on the line CA.



It is easy to find that there is a double tangent $z+2x=0$, viz. $z+2x=0$ gives $(3x^2-y^2)^2=0$, two points τ, τ' (each twice) on the line in question.

Laying down these points, it appears that the curve must have two real asymptotes, and that the form is as shown in the figure.

[Vol. IX. pp. 55, 56.]

2553. (Proposed by Professor CAYLEY.)—Show that the surface $y^2z^2+z^2x^2+x^2y^2-2xyz=0$ meets the sphere $x^2+y^2+z^2=1$ in four circles; and explain in a general manner the form of the curve of intersection of the surface by any other sphere having the same centre, and thence the form of the surface itself (being a particular case of Steiner's surface, and which by the homographic transformations $w^{-1}x$, $w^{-1}y$, $w^{-1}z$ for x, y, z gives $y^2z^2 + z^2x^2 + x^2y^2 - 2wxyz = 0$, the general equation of Steiner's surface).

Solution by the PROPOSER.

Take X, X', Y, Y', Z, Z' the intersections of the sphere $x^2 + y^2 + z^2 = 1$ by the three axes respectively; then we have $x^2 + y^2 + z^2 = 1$, $x + y + z = -1$, the equations of the circle through the points X', Y', Z'; and from these two equations we deduce $yz + zx + xy = 0$, and thence

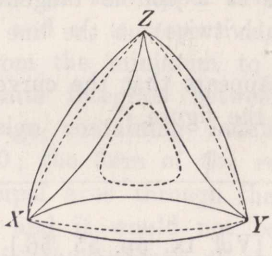
$$y^2z^2 + z^2x^2 + x^2y^2 + 2xyz\,(x + y + z) = 0,$$

that is

$$y^2z^2 + z^2x^2 + x^2y^2 - 2xyz = 0\,;$$

so that the circle lies on the quartic surface; and by changing successively the signs of each two of the three coordinates, we have three other circles lying on the sphere and also on the quartic surface; viz. we have in all four circles, the above-mentioned circle through (X', Y', Z'), and three other circles through (X', Y, Z), (X, Y', Z), (X, Y, Z') respectively, making together a curve of the order 8, the complete intersection of the quartic surface by the sphere.

The quartic surface lies entirely in the four octants of space xyz, $xy'z'$, $x'yz'$, $x'y'z$; and as to the portion of the surface which lies in the octant xyz, this meets the sphere $x^2 + y^2 + z^2 = 1$ in portions of the three circles (X', Y, Z) (X, Y', Z) (X, Y, Z') constituting a tricuspidal form lying within the octant XYZ as shown in the figure. The intersection by a sphere, radius < 1, projected on the octant XYZ, is a trinodal form, lying outside the tricuspidal one, as shown by a dotted line in the figure; the intersection by a sphere radius > 1, projected in the same way, is a trigonoid form lying inside the tricuspidal one, as also shown by a dotted line in the figure; as the radius approaches to and ultimately becomes $= \dfrac{2}{\sqrt{3}}$, this diminishes, and becomes ultimately a mere point, and when the radius is greater than this value the intersection is imaginary.



Imagine on the solid sphere, radius $= 1$, the four tricuspidal forms lying in alternate octants as above; cut away down to the centre the portions lying without these tricuspidal forms; and build up on the tricuspidal forms, until the greatest distance from the centre becomes $=\frac{2}{\sqrt{3}}$; we have a solid figure with four prominences situate as the summits of a tetrahedron, the bounding surface whereof is the surface in question: it is to be added that the axes are nodal lines on the surface, viz. the portions which lie within the solid figure are the intersections of two real sheets of the surface, the portions which lie without the solid figure are isolated, or acnodal, lines on the surface.

[Vol. IX. pp. 73, 74.]

2573. (Proposed by Professor CAYLEY.)—The envelope of a variable circle having for its diameter the double ordinate of a rectangular cubic is a Cartesian.

{DEFINITION. The expression "a rectangular cubic" is used to express a cubic with three real asymptotes, having a diameter at right angles to one of the asymptotes and at an angle of 45° to each of the other two asymptotes, viz. the equation of such a cubic is $xy^2 = x^3 + bx^2 + cx + d$.}

Solution by the PROPOSER.

The equation of the variable circle may be taken to be

$$(x-\theta)^2 + y^2 = \theta^2 - 2m\theta + \alpha + \frac{2A}{\theta},$$

viz. θ being the abscissa of the rectangular cubic, the squared ordinate is taken to be $=\frac{1}{\theta}(\theta^3 - 2m\theta^2 + \alpha\theta + 2A)$, or, what is the same thing, the equation of the variable circle is

$$x^2 + y^2 - \alpha - 2(x-m)\theta - \frac{2A}{\theta} = 0.$$

Hence, taking the derived equation in regard to θ, we have

$$x - m - \frac{A}{\theta^2} = 0,$$

and thence

$$x^2 + y^2 - \alpha = \frac{4A}{\theta};$$

therefore

$$(x^2 + y^2 - \alpha)^2 = \frac{16A^2}{\theta^2} = 16A(x-m);$$

that is, the equation of the envelope is

$$(x^2 + y^2 - \alpha)^2 - 16A(x-m) = 0,$$

which is a known form of the equation of a Cartesian.

[Vol. IX. pp. 82, 83.]

2493. (Proposed by Professor CAYLEY.)—1. Given the conic $U=0$ (but observe that the *function* U contains implicitly an arbitrary constant factor which is *not* given) and also the conic $U+1=0$, to construct the conic $U+l=0$, where l is a given constant.

2. Given the conics $U=0$, $U+1=0$, $V=0$, $V+1=0$, and the constants θ, k, to construct the conic $\theta U+\theta^{-1}V+2k=0$.

Solution by the PROPOSER.

1. The conics $U=0$, $U+1=0$, $U+l=0$ are obviously concentric similar and similarly situated conics, and if drawing a line in any direction from the centre, the radius-vectors for the three conics respectively are r, r', R, then it is easy to see that we have

$$R^2 = lr'^2 + (1-l)\,r^2.$$

There is no difficulty in constructing geometrically the radius R, and then the conic $U+l=0$ is given as the concentric similar and similarly situated conic passing through the extremity of this radius.

2. To construct the conic $\theta U+\theta^{-1}V+2k=0$. By what precedes, we may construct the two conics $\theta U+k=0$, $\theta^{-1}V+k=0$; the four points of intersection of these lie on the required conic $\theta U+\theta^{-1}V+2k=0$, and also on the conic $\theta U-\theta^{-1}V=0$; which last conic is consequently given as a conic passing through the four points in question, and also through the four points of intersection of the given conics $U=0$, $V=0$. But the conic $\theta U-\theta^{-1}V=0$ being constructed, the conic $\theta U+\theta^{-1}V=0$ can also be constructed; viz. the tangents of these two conics and of the conics $U=0$, $V=0$, at each of the four intersections $U=0$, $V=0$, form a harmonic pencil; and we have thus the conic $\theta U+\theta^{-1}V=0$ a conic passing through four given points, and having at each of these a given tangent. And then finally the required conic $\theta U+\theta^{-1}V+2k=0$ is given as a conic concentric similar and similarly situated with the conic $\theta U+\theta^{-1}V=0$, and passing through the four given points

$$\theta U+k=0, \quad \theta^{-1}V+k=0.$$

3. Treating k as an absolute constant but θ as a variable parameter, the envelope of the conic $\theta U+\theta^{-1}V+2k=0$ is the quartic curve $UV-k^2=0$. This is a curve used by Plücker (in the *Theorie der algebraischen Curven*) for the purpose of showing that the 28 double tangents of a quartic curve may be all of them real. In fact, if $U=0$, $V=0$ be ellipses intersecting in four real points; and if, moreover, the implicit constants be such that U is positive for points *without* the first ellipse, V positive for points *within* the second ellipse, then since $UV, =k^2$, is positive for all points of the curve in question, the curve must be wholly situate in the four closed spaces which lie outside the one and inside the other of the two ellipses; consisting therefore of four detached portions. And when k is sufficiently small, then the figure of each portion is that of a concavo-convex lens with its angles rounded off: viz. each such portion has a real double tangent of its own. Any two portions have obviously four real double tangents, and hence the total number of real double tangents is $4+6\times 4, =28$.

4. A construction has been given by Aronhold (*Berl. Monatsber.*, July, 1864) by which, taking any 7 given lines as double tangents of a quartic curve, the remaining 21 double tangents can be constructed, and which, when the seven given lines are real, leads to a system of 28 real double tangents; but wishing to construct the figure of the 28 real double tangents, it occurred to me that the easier manner might be to construct Plücker's curve $UV - k^2 = 0$, as the envelope of the conic $\theta U + \theta^{-1} V + 2k = 0$, and then to draw the tangents of this curve: the construction is, however, practically one of considerable difficulty, and I have not yet accomplished it.

[Vol. IX. p. 87.]

2451. (Proposed by Professor CAYLEY.)—If A, B, C, D are the intersections of a conic by a circle, then the antipoints of A, B, and the antipoints of C, D, lie on a confocal conic.

N.B. If AB, $A'B'$ intersect at right angles in a point O in such wise that $OA' = OB' = i \cdot OA = i \cdot OB$ {where $i = \sqrt{(-1)}$ as usual}, then A', B' are the antipoints of A, B, and conversely.

[Vol. IX. pp. 101—103.]

2590. (Proposed by Professor CAYLEY.)—It is required to verify Professor Kummer's theorem that "if a quartic surface is such that every section by a plane through a certain fixed point is a pair of conics, the surface is a pair of quadric surfaces (except only in the case where it is a quartic cone having its vertex at the fixed point)."

Solution by the PROPOSER.

The theorem may be more generally stated as follows; if a surface is such that every section through a certain fixed point (is or) includes a proper conic, then the surface (is or) includes a proper quadric surface. In order to the demonstration, I premise the following *Lemma:* If a surface is such that every section through a certain fixed line includes a conic, then the line meets each of these conics in the same two points.

In fact, if the line meet the surface in any n points, then it is clear that each of the conics will meet the line in some two of these n points; and as the plane of the section passes continuously from any one to any other position, the two points of intersection with the conic cannot pass abruptly from being some two to being some other two of the n points, that is, they are always the same two points.

Consider now a surface which is such that every section through a fixed point O includes a conic; and consider three lines xx', yy', zz' meeting in the point O. Let the conics in the planes yz, zx, xy be A, B, C respectively; then since the conics through the line xx' all pass through the same two points, and since B, C are two of these conics, B and C meet xx' in the same two points X, X'; similarly C and A meet yy' in the same two points Y, Y'; and A, B meet zz' in the same two points Z, Z'; that is, we have the conics A, B, C intersecting

B, C in the two points X, X',
C, A " " Y, Y',
A, B " " Z, Z';

hence taking on the conics A, B, C the points α, β, γ respectively, and drawing a quadric surface Σ through the nine points X, X', Y, Y', Z, Z', α, β, γ, this meets the conic A in the five points Y, Y', Z, Z', α; that is, it passes through the conic A, and similarly it passes through the conic B, and through the conic C.

Consider how any plane whatever through O intersecting the conics A, B, C in the points L and L', M and M', N and N' respectively; the section of the quadric surface Σ by the plane in question is a conic through the six points L, L', M, M', N, N'. But the section of the surface includes a conic through these same six points, and which is consequently the same conic; in fact, the section of the surface by the plane in question includes a conic, and since every section through the line LL' includes a conic through the same two points, and one of these conics is the conic A which passes through the points L and L', the conic in question passes through the points L and L'; and similarly it passes through the points M and M', and through the points N and N'. That is, for any plane whatever through O, the section of the surface includes the conic which is the section of the quadric surface Σ, and the surface thus includes as part of itself the quartic surface Σ.

The foregoing demonstration ceases, however, to be applicable if O is a point on the surface, and the conic included in the section through O is always a conic passing through the point O. In the case where O is a non-singular point of the surface (that is, where there is at O a unique tangent plane) a like demonstration applies. Take through O a section, and let this include the conic A; on A take any point O' and through OO' a section including the conic B; we have thus the conics A, B intersecting in the points O, O'. Take through O any plane meeting the conics A, B in the points X, Y respectively—the section by this plane includes a conic C passing through the points O, X, Y; and each of the conics A, B, C touches at O the same plane, viz. the tangent plane of the surface. Hence, taking on the conic A the point α, consecutive to O, and any other point α'; on the conic B the point β, consecutive to O, and any other point β'; and on the conic C a point γ'; we may, through the nine points O, α, β, O', α', β', X, Y, γ' describe a quadric surface Σ; this will touch at O the tangent plane of the surface, that is, it will touch the conic C, or (what is the same thing) pass through a point γ of this conic consecutive to O. Hence the quadric surface meets the conic A in the five points O, O', α, α', X, that is, it entirely contains the conic A; similarly it meets the conic B in five points O, O', B, B', Y, that is, it entirely contains the conic B; and it meets the conic C in the five points O, γ, X, Y, γ', that is, it entirely contains this conic. And it may then be shown as before that the surface will include the quadric surface Σ. But there still remains for consideration the case where O is a conical point on the surface, and I do not at present see how this is to be treated.

I remark that, taking three lines xx', yy', zz' which meet in a point O, then if a surface be such that every section through xx' includes a conic, every section through yy' includes a conic, and every section through zz' includes a conic; and if besides the two points, say X, X', on the conics through the line xx' are ordinary points on the surface, then the surface will include a quadric surface. In fact, if the surface has at each of the points X, X' an ordinary tangent plane, then the conics through xx', and (as conics of the series) the two conics B, C all of them touch the two tangent planes; hence, constructing as before the quadric surface Σ, this also touches the two tangent planes: and taking through xx' a plane meeting the conic A in the points L, L', the section of the surface includes a conic which touches the section of the quadric surface Σ at the points X, X', and besides meets it in the points L, L'; such conic coincides therefore with the section of the quadric surface Σ; that is, every section of the surface through the line xx' includes the conic which is the section of the quadric surface Σ; and the surface thus includes as part of itself the quadric surface Σ.

[Vol. x., July to December, 1868, pp. 17—19.]

2609. (Proposed by Professor CAYLEY.)—Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find, on the first a point A', on the second a point B', and on the third a point C', such that the intersections of the lines

$A'B'$ and AC, $A'C'$ and AB, lie on the first conic;

$B'C'$ and BA, $B'A'$ and BC, lie on the second conic;

$C'A'$ and CB, $C'B'$ and CA, lie on the third conic.

Solution by the PROPOSER.

Let the six intersections in question be called α, α'; β, β'; γ, γ', respectively; then BC intersects second conic in β', third conic in γ; CA intersects third conic in γ', first conic in α; AB intersects first conic in α', second conic in β; and we have

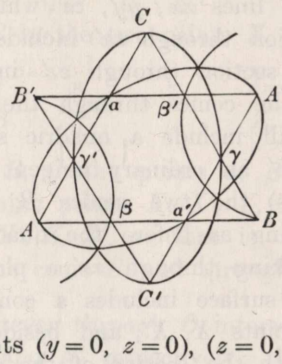
A' the intersection of $\alpha\beta'$, $\gamma\alpha'$,

B' the intersection of $\beta\gamma'$, $\alpha\beta'$,

C' the intersection of $\gamma\alpha'$, $\beta\gamma'$;

and it has to be shown that the points A', B', C' so determined lie—A' on the first conic, B' on the second conic, C' on the third conic.

Taking $x=0$, $y=0$, $z=0$ for the equations of the sides of the triangle ABC, the equations of the three conics may be taken to be $U=0$, $V=0$, $W=0$, where the functions U, V, W are such that identically $U+V+W=0$; and then observing that



the conics pass through the points $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$, respectively, we see that the equations may be taken to be

$$\begin{aligned}
&(\ 0,\ -b,\ c,\ f_1,\ g_1,\ h_1)(x,\ y,\ z)^2=0,\\
&(\ a,\ 0,\ -c,\ f_2,\ g_2,\ h_2)(x,\ y,\ z)^2=0,\\
&(-a,\ b,\ 0,\ f_3,\ g_3,\ h_3)(x,\ y,\ z)^2=0,
\end{aligned}$$

where

$$f_1+f_2+f_3=0,\quad g_1+g_2+g_3=0,\quad h_1+h_2+h_3=0.$$

The coordinates of the points α, β, γ, α', β', γ' are at once found to be

$$\begin{aligned}
&\alpha,\ (\ c,\ 0,\ -2g_1);\quad &&\alpha',\ (\ b,\ 2h_1,\ 0)\\
&\beta,\ (-2h_2,\ a,\ 0\ );\quad &&\beta',\ (\ 0,\ c,\ 2f_2)\\
&\gamma,\ (\ 0,\ -2f_3,\ b\ );\quad &&\gamma',\ (2g_3,\ 0,\ a\ );
\end{aligned}$$

and hence the equations of $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$ are

$$\begin{aligned}
&\beta\gamma';\quad && ax+2h_2y-2g_3z=0,\\
&\gamma\alpha';\quad && -2h_1x+by+2f_3z=0,\\
&\alpha\beta';\quad && 2g_1x-2f_2y+cz=0.
\end{aligned}$$

Hence for the point A', which is the intersection of $\gamma\alpha'$, $\alpha\beta'$, coordinates are

$$bc+4f_2f_3,\quad 4f_3g_1+2ch_1,\quad 4h_1f_2-2bg_1;$$

and A' will be on the first conic if only

$$(0,-b,\ c,\ f_1,\ g_1,\ h_1)(bc+4f_2f_3,\quad 4f_3g_1+2ch_1,\quad 4h_1f_2-2bg_1)^2=0,$$

viz. this equation is

$$\begin{aligned}
&-\ b\ (\ 16f_3^2g_1^2 + 16f_3g_1h_1c + 4h_1^2c^2)\\
&+\ c\ (\ 16h_1^2f_2^2 - 16f_2g_1h_1b + 4g_1^2b^2)\\
&+2f_1(\ 16g_1h_1f_2f_3 - 8g_1^2f_3b + 8h_1^2f_2c - 4g_1h_1bc)\\
&+2g_1(+16h_1f_2^2f_3 - 8g_1f_2f_3b + 4h_1f_2bc - 2g_1b^2c\ )\\
&+2h_1(+16g_1f_2f_3^2 + 8h_1f_2f_3c + 4g_1f_3bc + 2h_1bc^2\ )=0,
\end{aligned}$$

viz. this is easily found to be

$$8(2g_1f_3+ch_1)(2h_1f_2-bg_1)(f_1+f_2+f_3)=0,$$

which is satisfied in virtue of $f_1+f_2+f_3=0$; that is, A' is on the first conic; and similarly, in virtue of $g_1+g_2+g_3=0$, B' is on the second conic; and in virtue of $h_1+h_2+h_3=0$, C' is on the third conic. But the same thing appears at once by the remark that the equations of the three conics are

$$\begin{aligned}
-y(-2h_1x+\ \ by+2f_3z)+z(\ \ 2g_1x-2f_2y+\ \ cz)&=0,\\
-z(\ \ 2g_1x-2f_2y+\ \ cz)+x(\ \ ax+2h_2y-2g_3z)&=0,\\
-x(\ \ ax+2h_2y-2g_3z)+y(-2h_1x+\ \ by+2f_3z)&=0.
\end{aligned}$$

It may be added that, taking (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) as the coordinates of the four points of intersection of the three conics, the first conic is given by means of these four points and the fifth point $(y=0,\ z=0)$; and similarly for the other two conics; whence, denoting the determinants formed with any four columns out of the matrix

$$\left\|\begin{matrix}
x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1\\
x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2\\
x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3\\
x_4^2, & y_4^2, & z_4^2, & y_4z_4, & z_4x_4, & x_4y_4
\end{matrix}\right\|$$

by 1234, 1235, &c., we easily find the equations of the three conics, viz. these may be written

$$\begin{array}{rcccccc}
 & x^2\ , & y^2\ , & z^2\ , & yz\ , & zx\ , & xy\\
1456\,(& 0\ , & +3456, & -2456, & +2356, & +2364, & +2345)=0,\\
2356\,(& -3456, & 0\ , & +1456, & +3156, & +3164, & +3145)=0,\\
3456\,(& 2456, & -1456, & 0\ , & +1256, & +1264, & +1245)=0,
\end{array}$$

the exterior factors 1456, 2356, 3456 being introduced in order to bring the equations into the above-mentioned form, wherein the sum of the quadric functions is $=0$.

[Vol. x. pp. 88, 89.]

2743. (Proposed by M. JENKINS, M.A.)—Show that if p be a prime number and m and n any positive integers, the highest power of p contained in $\dfrac{\Pi\,(m+n)}{\Pi\,(m)\,\Pi\,(n)}$ may be obtained by expressing $m+n$ and either m or n in the scale of p; the number of times that it would be necessary to borrow in subtracting the latter number from the former being the index of the power of p required.

Solution by PROFESSOR CAYLEY.

1. In adding any two numbers, we carry a certain number of times; and it is easy to see that the sum of the digits of the two components, less the sum of the digits of the sum, is equal to nine times the number of carryings; moreover, that the number of carryings is equal to the number of borrowings, if either of the components be subtracted from the sum.

2. The same thing is true in any scale of notation, only, instead of nine, we have the radix of the scale, less unity: say the theorem is

$$S(m) + S(n) - S(m+n) = (p-1)x.$$

3. If p be a prime number, the number of times that the factor p occurs in $\Pi(m)$ is

$$E\left(\frac{m}{p}\right) + E\left(\frac{m}{p^2}\right) + E\left(\frac{m}{p^3}\right) + \&c.,$$

where $E\left(\frac{m}{p}\right)$ denotes the integer part of $\frac{m}{p}$, and similarly $E\left(\frac{m}{p^2}\right)$ &c. the integer part of $\frac{m}{p^2}$, &c.; the series is, of course, finite.

Hence the number of times that the factor p occurs in $\frac{\Pi(m+n)}{\Pi(m)\Pi(n)}$ is

$$N = E\left(\frac{m+n}{p}\right) + E\left(\frac{m+n}{p^2}\right) + \&c. - E\left(\frac{m}{p}\right) - E\left(\frac{m}{p^2}\right) - \&c. - E\left(\frac{n}{p}\right) - E\left(\frac{n}{p^2}\right) - \&c.$$

4. Hence, expressing m, n, $m+n$ in the scale to the radix p, suppose

$$m = a + bp + cp^2 + dp^3, \quad n = a' + b'p + c'p^2 + d'p^3, \quad m + n = \alpha + \beta p + \gamma p^2 + \delta p^3,$$

we have

$$E\left(\frac{m}{p}\right) + E\left(\frac{m}{p^2}\right) + \&c. = b + cp + dp^2 + c + dp + d = d(p^2+p+1) + c(p+1) + b;$$

and similarly for $\qquad E\left(\frac{n}{p}\right) + \&c.,\ E\left(\frac{m+n}{p}\right) + \&c. \ldots;$

whence

$$\begin{aligned}(p-1)N = {} & \delta\ (p^3-1) + \gamma\,(p^2-1) + \beta\,(p-1) \\ & - d\ (p^3-1) - c\,(p^2-1) - b\ (p-1) \\ & - d'(p^3-1) - c'(p^2-1) - b'(p-1) \\ = {} & \{m+n-S(m+n)\} - \{m - S(m)\} - \{n - S(n)\} \\ = {} & S(m) + S(n) - S(m+n),\ = (p-1)x,\end{aligned}$$

if x be the number of times of carrying for the sum $m+n$, or of borrowing for the difference $(m+n)-m$ or $(m+n)-n$; that is, $N=x$, the required theorem. I remark that although the foregoing expression of the number N is a very elegant and ingenious one, yet the original form of N, as given at the end of (3), is the natural and proper expression of the number of times that the factor p occurs in the binomial coefficient $\dfrac{\Pi(m+n)}{\Pi(m)\,\Pi(n)}$.

[Vol. x. p. 98.]

2756. (Proposed by J. GRIFFITHS, M.A.)—Show that an infinite number of triangles can be described such that each has the same circumscribing, nine-point, and self-conjugate circles as a given triangle.

Solution by PROFESSOR CAYLEY.

It is a known theorem that if two triads of points, say A, B, C and A', B', C', are self-conjugate in regard to a conic S, they lie in a conic Σ. Take the conic S and the points A, B, C as given; then Σ will be a conic passing through A, B, C; and if on this conic we take any point A', and then take B' to be either of the intersections of the conic Σ by the polar of A in regard to S, and finally construct C' as the pole of $A'B'$ in regard to S, then, by what precedes, C' will be on a conic through A, B, C, A', B', that is, on the conic Σ. Or given the conic S, the triangle ABC, and the conic Σ through A, B, C, we obtain an infinity of triangles $A'B'C'$, self-conjugate in regard to S and inscribed in Σ. That is, if S, Σ are circles, we have an infinity of triangles self-conjugate in regard to the circle S and inscribed in the circle Σ; and inasmuch as the nine-points circle can be constructed by means of the two circles S, Σ alone, the triangles have all of them the same nine-points circle.

[Vol. x. p. 108.]

2737. (Proposed by Professor CAYLEY.)—Find *in solido* the locus of a point P, such that from it two given points A, C, and two given points B, D, subtend equal angles.

[Vol. XI., January to June, 1869, pp. 33—38.]

2718. (Proposed by Professor CAYLEY.)—Find *in plano* the locus of a point P, such that from it two given points A, C, and two given points B, D, subtend equal angles.

2757. (Proposed by Professor CAYLEY.)—If

$$x_0^2+y_0^2=1, \quad \text{and} \quad \begin{vmatrix} x, & y, & 1 \\ x_0, & y_0, & 1 \\ x_1, & y_1, & 1 \end{vmatrix} = L;$$
$$x_1^2+y_1^2=1,$$

show that each of the equations

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xx_0+yy_0-1)^2}=\frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xx_1+yy_1-1)^2}, \qquad (1)$$

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xy_0-x_0y)^2-(x-x_0)^2-(y-y_0)^2}=\frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xy_1-x_1y)^2-(x-x_1)^2-(y-y_1)^2}, \qquad (2)$$

represents the right line $L=0$ and a cubic curve.

1819. (Proposed by C. TAYLOR, M.A.)—From two fixed points on a given conic pairs of tangents are drawn to a variable confocal conic, and with the fixed points as foci a conic is described passing through any one of the four points of intersection. Show that its tangent or normal at that point passes through a fixed point.

Solution of the above Problems by PROFESSOR CAYLEY.

1. It is easy to see that drawing through the points A, C a circle, and through B, D a circle, such that the radii of the two circles are proportional to the lengths AC, BD, then that the required locus is that of the intersections of the two variable circles.

Take $AC=2l$, MO perpendicular to it at its middle point M, and $=p$; a, b the coordinates of M, and λ the inclination of p to the axis of x; then

$$\text{coordinates of } O \text{ are} \quad a+p\cos\lambda, \quad b+p\sin\lambda,$$
$$\text{coordinates of } A, C \text{ are } a\pm l\sin\lambda, \quad b\mp l\cos\lambda,$$

and hence the equation of a circle, centre O and passing through A, C, is

$$(x-a-p\cos\lambda)^2+(y-b-p\sin\lambda)^2=l^2+p^2;$$

or, what is the same thing,

$$(x-a)^2+(y-b)^2-l^2=2p\,[(x-a)\cos\lambda+(y-b)\sin\lambda].$$

If $2m$, q, c, d, μ refer in like manner to the points B, D, then the equation of a circle, centre say Q, and passing through B, D, is

$$(x-c)^2+(y-d)^2-m^2=2q\,[(x-c)\cos\mu+(y-d)\sin\mu];$$

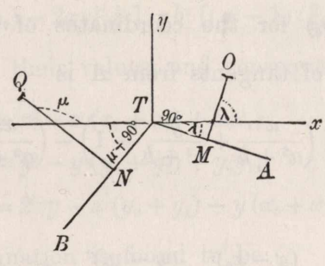
and the condition as to the radii is $l^2+p^2 : m^2+q^2 = l^2 : m^2$, that is, $p^2 : q^2 = l^2 : m^2$, or $p : q = \pm\, l : m$. And we thus have for the equation of the required locus

$$\frac{(x-a)^2+(y-b)^2-l^2}{(x-a)\cos\lambda+(y-b)\sin\lambda}=\pm\frac{l}{m}\,\frac{(x-c)^2+(y-d)^2-m^2}{(x-c)\cos\mu+(y-d)\sin\mu},$$

viz. the locus is composed of two cubics, which are at once seen to be *circular* cubics. One of these will however belong (at least for some positions of the four points) to the case of the subtended angles being *equal*, the other to that of the subtended angles being *supplementary*; and we may say that the required locus is a circular cubic.

2. If two of the points coincide, suppose C, D at T; then, taking T as the origin, we may write

$$a = l\,\sin\lambda,\quad b=-l\,\cos\lambda,$$
$$c=-m\sin\mu,\quad d=m\cos\mu,$$



and the equation becomes

$$\frac{x^2+y^2+2l\,(x\sin\lambda-y\cos\lambda)}{x\cos\lambda+y\sin\lambda}=\pm\frac{l}{m}\,\frac{x^2+y^2+2m\,(x\sin\mu-y\cos\mu)}{x\cos\mu+y\sin\mu},$$

viz. this is

$$(x^2+y^2)\,[m\,(x\cos\mu+y\sin\mu)\mp l\,(x\cos\lambda+y\sin\lambda)]-2lm\,\{(x\sin\lambda-y\cos\lambda)\,(x\cos\mu+y\sin\mu)$$
$$\pm\,(x\sin\mu-y\cos\mu)\,(x\cos\lambda+y\sin\lambda)\}=0.$$

Taking the lower signs, the term in $\{\ \}$ is $(x^2+y^2)\sin(\lambda-\mu)$, and the equation is

$$(x^2+y^2)\,\{m\,(x\cos\mu+y\sin\mu)+l\,(x\cos\lambda+y\sin\lambda)-2lm\sin(\lambda-\mu)\}=0,$$

viz. this is $x^2+y^2=0$, and a line which is readily seen to be the line AB; and in fact from any point whatever of this line the points A, T and the points B, T subtend *supplementary* angles.

Taking the upper signs, the equation is

$$(x^2+y^2)\,[m\,(x\cos\mu+y\sin\mu)-l\,(x\cos\lambda+y\sin\lambda)]$$
$$-2lm\,\{(x^2-y^2)\sin(\lambda+\mu)-xy\cos(\lambda+\mu)\}=0,$$

which is the locus for *equal* angles, a circular cubic as in the case of the four distinct points.

3. The question is connected with Question 1819, which is given above. In fact, taking A, B for the fixed points on the given conic, and P for the intersection of any two of the tangents, if in the conic (foci A, B) which passes through P, the tangent or normal at P passes through a fixed point T, then it is clear that at P the points A, T and B, T subtend equal angles, and consequently the locus of P should be a circular cubic as above. The theorem will therefore be proved if it be shown that the locus of P considered as the intersection of tangents from A, B to the variable confocal conic is in fact the foregoing circular cubic. I remark that the fixed point T is in fact the intersection of the tangents AT, BT to the given conic at the points A, B respectively.

4. Consider the points A, B, (which we may in the first instance take to be arbitrary points, but we shall afterwards suppose them to be situate on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,) and from each of them draw a pair of tangents to the confocal conic $\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}=1$. Take (x_0, y_0) for the coordinates of A, and (x_1, y_1) for those of B; then the equation of the pair of tangents from A is

$$\left(\frac{x_0^2}{a^2+h}+\frac{y_0^2}{b^2+h}-1\right)\left(\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}-1\right)-\left(\frac{xx_0}{a^2+h}+\frac{yy_0}{b^2+h}-1\right)^2=0,$$

or, what is the same thing,

$$\frac{(xy_0-x_0y)^2}{(a^2+h)(b^2+h)}-\frac{(x-x_0)^2}{a^2+h}-\frac{(y-y_0)^2}{b^2+h}=0,$$

that is

$$(xy_0-x_0y)^2-(b^2+h)(x-x_0)^2-(a^2+h)(y-y_0)^2=0,$$

or as this may also be written

$$(xy_0-x_0y)^2-b^2(x-x_0)^2-a^2(y-y_0)^2=h\,[(x-x_0)^2+(y-y_0)^2]\,;$$

and similarly for the tangents from B we have

$$(xy_1-x_1y)^2-b^2(x-x_1)^2-a^2(y-y_1)^2=h\,[(x-x_1)^2+(y-y_1)^2]\,;$$

in which equations the points (x_0, y_0), (x_1, y_1) are in fact any two points whatever.

5. Eliminating h, we have as the locus of the intersection of the tangents

$$\frac{(xy_0-x_0y)^2-b^2(x-x_0)^2-a^2(y-y_0)^2}{(x-x_0)^2+(y-y_0)^2}=\frac{(xy_1-x_1y)^2-b^2(x-x_1)^2-a^2(y-y_1)^2}{(x-x_1)^2+(y-y_1)^2},$$

which is apparently a quartic curve; but it is obvious *à priori* that the locus includes as part of itself the line AB which joins the two given points. In fact, there is in the series of confocal conics one conic which touches the line in question, and since for this conic one of the tangents from A and also one of the tangents from B is the line AB, we see that every point of the line AB belongs to the required locus. The locus is thus made up of the line in question and of the cubic curve.

6. To effect the reduction it will be convenient to write ax, by in the place of x, y, (ax_0, by_0, ax_1, by_1 in place of x_0, y_0, x_1, y_1,) and thus consider the equation under the form

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xy_0-x_0y)^2-(x-x_0)^2-(y-y_0)^2}=\frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xy_1-x_1y)^2-(x-x_1)^2-(y-y_1)^2};$$

it is to be shown that this equation represents the line $L=0$, and a cubic curve.

Writing for a moment $x_0=x+\xi_0$, $y_0=y+\eta_0$, and $x_1=x+\xi_1$, $y_1=y+\eta_1$, the equation becomes

$$\frac{a^2\xi_0{}^2+b^2\eta_0{}^2}{(x\eta_0-y\xi_0)^2-\xi_0{}^2-\eta_0{}^2}=\frac{a^2\xi_1{}^2+b^2\eta_1{}^2}{(x\eta_1-y\xi_1)^2-\xi_1{}^2-\eta_1{}^2},$$

and hence, multiplying out, the equation is at once seen to contain the factor $\xi_0\eta_1-\xi_1\eta_0$ (which is in fact the determinant just mentioned), and when divested of this factor the equation is

$$a^2[(x^2-1)(\xi_0\eta_1+\xi_1\eta_0)-2xy\xi_0\xi_1]=b^2[(y^2-1)(\xi_0\eta_1+\xi_1\eta_0)-2xy\eta_0\eta_1].$$

Writing herein for ξ_0, η_0, ξ_1, η_1 their values, and consequently

$$\xi_0\xi_1=x^2-x(x_0+x_1)+x_0x_1,$$

$$\eta_0\eta_1=y^2-y(y_0+y_1)+y_0y_1,$$

$$\xi_0\eta_1+\xi_1\eta_0=2xy-x(y_0+y_1)-y(x_0+x_1)+x_0y_1+x_1y_0,$$

and arranging the terms, the equation is found to be

$$(a^2x^2+b^2y^2)[-x(y_1+y_0)-y(x_1+x_0)]+(a^2x^2+b^2y^2)(x_0y_1+x_1y_0)-2xy[a^2(1+x_0x_1)-b^2(1+y_0y_1)]$$
$$+(a^2-b^2)[x(y_1+y_0)+y(x_1+x_0)-(x_0y_1+x_1y_0)]=0,$$

which is the required cubic curve.

7. Restoring the original coordinates, or writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{x_0}{a}$, &c. in place of x, y, x_0, &c., we have

$$(x^2+y^2)[-x(y_1+y_0)+y(x_1+x_0)]+(x^2-y^2)(x_0y_1+x_1y_0)-2xy(a^2-b^2+x_0x_1-y_0y_1)$$
$$+(a^2-b^2)[x(y_1+y_0)+y(x_1+x_0)-(x_0y_1+x_1y_0)]=0,$$

which is a circular cubic the locus of the intersections of the tangents from the arbitrary points (x_0, y_0), (x_1, y_1) to the series of confocal conics $\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}=1$; the origin of the coordinates is at the centre of the conics.

8. Supposing that the points (x_0, y_0), (x_1, y_1) are on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and that we have consequently $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$, the equations of the tangents at these points respectively are $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$, $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$; and hence, writing for shortness $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$, we find $x=-\frac{a^2\alpha}{\gamma}$, $y=-\frac{b^2\beta}{\gamma}$ as the coordinates of the point of intersection T, of the two tangents; and in order to transform to this point as origin, we must in place of x, y write $x-\frac{a^2\alpha}{\gamma}$, $y-\frac{b^2\beta}{\gamma}$ respectively. Or what is more convenient, we may in the equation at the end of (6), in which it is to be now assumed that $x_0^2+y_0^2=1$, $x_1^2+y_1^2=1$, write $x-\frac{\alpha}{\gamma}$, $y-\frac{\beta}{\gamma}$ for x, y, and then restore the original coordinates by writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{x_0}{a}$, &c., for x, y, x_0, &c., and $\frac{\alpha}{b}$, $\frac{\beta}{a}$, $\frac{\gamma}{ab}$ for α, β, γ, these quantities throughout signifying $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$. I however obtained the equation referred to the point T as origin by a different process, as follows:

9. Starting from the equation at the commencement of (5), I found that the points (x_0, y_0), (x_1, y_1) being on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation could be transformed into the form

$$\frac{\left(\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-1\right)^2}{(x-x_0)^2+(y-y_0)^2}=\frac{\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2}{(x-x_1)^2+(y-y_1)^2},$$

an equation which (not, as the original one, for all values of (x_0, y_0), (x_1, y_1), but) for values of (x_0, y_0), (x_1, y_1) such that $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$, breaks up into the line AB and a cubic curve.

10. To simplify the transformation, write as before ax, by, ax_0, &c., for x, y, x_0, &c. We have thus to consider the equation

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xx_0+yy_0-1)^2}=\frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xx_1+yy_1-1)^2},$$

where $x_0^2+y_0^2=1$, $x_1^2+y_1^2=1$, and which equation, I say, breaks up into the line $L=0$, and into a cubic.

Write for shortness $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$, so that the equation of the last-mentioned line is $\alpha x+\beta y+\gamma=0$. Then it may be verified that, in virtue of the relations between (x_0, y_0), (x_1, y_1), we have identically

$$(x-x_0)(xx_1+yy_1-1)+(x-x_1)(xx_0+yy_0-1)=(\alpha x+\beta y+\gamma)\frac{x_0+x_1}{\alpha\gamma}(\gamma x+\alpha),$$

$$(x-x_0)(xx_1+yy_1-1)-(x-x_1)(xx_0+yy_0-1)=\beta x^2-\alpha xy-\gamma y-\beta;$$

and, similarly,

$$(y-y_0)(xx_1+yy_1-1)+(y-y_1)(xx_0+yy_0-1)=(\alpha x+\beta y+\gamma)\frac{y_0+y_1}{\beta\gamma}(\gamma y+\beta),$$

$$(y-y_0)(xx_1+yy_1-1)-(y-y_1)(xx_0+yy_0-1)=\beta xy-\alpha y^2+\gamma x+\alpha.$$

11. The equation in question may be written $a^2P+b^2Q=0$, where

$$P=(x-x_0)^2(xx_1+yy_1-1)^2-(x-x_1)^2(xx_0+yy_0-1)^2,$$
$$Q=(y-y_0)^2(xx_1+yy_1-1)^2-(y-y_1)^2(xx_0+yy_0-1)^2,$$

values which are given by means of the formulæ just obtained; there is a common factor $\alpha x+\beta y+\gamma$ which is to be thrown out; and we have also, as is at once verified, $\dfrac{y_0+y_1}{\beta}=\dfrac{x_0+x_1}{\alpha}$, so that these equal factors may be thrown out. We thus obtain the cubic equation

$$a^2(\gamma x+\alpha)(\beta x^2-\alpha xy-\gamma y-\beta)+b^2(\gamma y+\beta)(\beta xy-\alpha y^2+\gamma x+\alpha)=0.$$

This is simplified by writing $x-\dfrac{\alpha}{\gamma}$ for x, $y-\dfrac{\beta}{\gamma}$ for y. It thus becomes

$$a^2x[(\gamma x-\alpha)(\beta x-\alpha y)-\gamma^2y]+b^2y[(\gamma y-\beta)(\beta x-\alpha y)+\gamma^2x]=0;$$

or, what is the same thing,

$$a^2x[\gamma x(\beta x-\alpha y)-\alpha\beta x+(\alpha^2-\gamma^2)y]+b^2y[\gamma y(\beta x-\alpha y)-(\beta^2-\gamma^2)x+\alpha\beta y]=0;$$

that is

$$\gamma(a^2x^2+b^2y^2)(\beta x-\alpha y)+a^2[-\alpha\beta x^2+(\alpha^2-\gamma^2)xy]+b^2[-(\beta^2-\gamma^2)xy+\alpha\beta y^2]=0.$$

12. Restoring $\dfrac{x}{a}$, $\dfrac{x_0}{a}$, $\dfrac{x_1}{a}$ for x, x_0, x_1, and $\dfrac{y}{a}$, $\dfrac{y_0}{a}$, $\dfrac{y_1}{a}$ for y, y_0, y_1; writing consequently $\dfrac{\alpha}{b}$, $\dfrac{\beta}{a}$, $\dfrac{\gamma}{ab}$ in place of α, β, γ, if α, β, γ are still used to denote $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$, the equation becomes

$$\gamma(x^2+y^2)[b^2\beta x-a^2\alpha y]+a^2[-b^2\alpha\beta x^2+(a^2\alpha^2-\gamma^2)xy]+b^2[-(b^2\beta^2-\gamma^2)xy+a^2\alpha\beta y^2]=0,$$

where now, as originally, $\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}=1$, $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}=1$; viz. this is the equation, referred to the point T as origin, of the locus of the point P considered as the intersection of tangents from A, B to the variable confocal conic; and it is consequently the equation which would be obtained as indicated in (8). The locus is thus a circular cubic; the equation is identical in form with that obtained (2) for the locus of the point at which A, T and B, T subtend equal angles, and the complete identification of the two equations may be effected without difficulty.

13. I may remark that M. Chasles has given (*Comptes Rendus*, tom. 58, February, 1864) the theorem that the locus of the intersections of the tangents drawn from a fixed conic to the conics of a system (μ, ν) is a curve of the order 3ν. The confocal

conics, *quà* conics touching four fixed lines, are a system (0, 1); hence, taking for the fixed conic the two points A, B, we have, as a very particular case, the foregoing theorem, that the locus of the intersections of the tangents drawn from two fixed points to a variable confocal conic is a cubic curve.

[Vol. XI. p. 49.]

Note on Question 2740. *By* PROFESSOR CAYLEY.

The envelope of the curve

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

(where A, B, C, D, E are any functions of the coordinates, and θ is a variable parameter,) is obtained in the particular case $E=0$ (Salmon's *Higher Plane Curves*, p. 116), and the same process is applicable in the general case where E is not $=0$. From the great variety of the problems which depend upon the determination of such an envelope, the result is an important one, *and ought to be familiarly known* to students of analytical geometry. We have only to write $z=\cos\theta + i\sin\theta$, the trigonometrical functions are then given as rational functions of z, and the equation is converted into a quartic equation in z; the result is therefore obtained by equating to zero the discriminant of a quartic function. The equation, in fact, becomes

$$A\,\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)+B\,\frac{1}{2i}\left(z^2-\frac{1}{z^2}\right)+C\,\frac{1}{2}\left(z+\frac{1}{z}\right)+D\,\frac{1}{2i}\left(z-\frac{1}{z}\right)+E=0,$$

that is

$$A(z^4+1)-Bi(z^4-1)+C(z^3+z)-Di(z^3-z)+2Ez^2=0;$$

or, multiplying by 12 to avoid fractions, this is

$$(a,\ b,\ c,\ d,\ e \,\between\, z,\ 1)^4=0,$$

where

$$a=12(A-Bi),\quad b=3(C-Di),\quad c=4E,$$

$$e=12(A+Bi),\quad d=3(C+Di);$$

and substituting in

$$(ae-4bd+3c^2)^3-27(ace-ad^2-b^2e+2bcd-c^3)^2=0,$$

the equation divides by 1728, and the final result is

$$\{12(A^2+B^2)-3(C^2+D^2)+4E^2\}^3$$
$$-\{27A(C^2-D^2)+54BCD-[72(A^2+B^2)+9(C^2+D^2)]E+8E^3\}^2=0.$$

It is to be noticed, that in developing the equation according to the powers of E, the terms in E^6, E^4 each disappear, so that the highest power is E^3; the degree in the coordinates, or order of the curve, is on this account sometimes lower than it would otherwise have been.

[Vol. XII., July to December, 1869, p. 69.]

2920. (Proposed by Professor CAYLEY.)—Imagine a tetrahedron $BB'CC'$ in which the opposite sides BB', CC' are at right angles to each other and to the line joining their middle points M, N; and in which moreover $\overline{CN}^2 + NM^2 + \overline{MB}^2 = 0$, (or, what is the same thing, the sides CB, CB', $C'B$, $C'B'$ are each $= 0$; the tetrahedron is of course imaginary; viz. the lines CC', BB' and points M, N may be real; but the distances $MB = MB'$ and $NC = NC'$ may be one real and the other imaginary, or both imaginary, but they cannot be both real) the points B, B' and C, C' are said to be "skew antipoints." Then it is required to prove that

1. A given system of skew antipoints may be taken to be the nodes (conical points) of a tetranodal cubic surface, passing through the circle at infinity, and which is in fact a Parabolic Cyclide.

2. The equation of the surface may be expressed in the form

$$x(x+\beta)(x+\gamma) + (x+\beta)y^2 + (x+\gamma)z^2 = 0.$$

3. The section through either of the lines $(y = 0,\ x + \gamma = 0)$ and $(z = 0,\ x + \beta = 0)$ is made up of this line and a circle; the two systems of circles being the curves of curvature of the surface; it is required to verify this *à posteriori;* viz. by means of the equation of the surface to transform the differential equation of the curves of curvature in such manner that the transformed equation shall have the integrals

$$y = C(x+\gamma), \quad z = C'(x+\beta).$$

In the following Contents, the Problems are referred to each by its No. and the Proposer's name, and the Subject is briefly indicated. An asterisk shows that no solution was given. A line —— shows that there is no No.