752.

ON THE FINITE GROUPS OF LINEAR TRANSFORMATIONS OF A VARIABLE; WITH A CORRECTION.

[From the *Mathematische Annalen*, t. XVI. (1880), pp. 260—263; 439, 440.]

IN the paper "Ueber endliche Gruppen linearer Transformationen einer Veränderlichen," *Math. Ann.* t. XII. (1877), pp. 23—46, Prof. Gordan gave in a very elegant form the groups of 12, 24 and 60 homographic transformations $\frac{ax+b}{cx+d}$. The groups of 12 and 24 are in the like form, the group of 24 thus containing as part of itself the group of 12; but the group of 60 is in a different form, not containing as part of itself the group of 12. It is, I think, desirable to present the group of 60 in the form in which it contains as part of itself Gordan's group of 12: and moreover to identify the group of 60 with the group of the 60 positive permutations of 5 letters: or (writing *abc* for the cyclical permutation *a* into *b*, *b* into *c*, *c* into *a*, and so in other cases) say with the group of the 60 positive permutations 1, *abc*, *ab.cd* and *abcde*.

Any two forms of a group are, it is well known, connected as follows, viz. if 1, α, β, ... are the functional symbols of the one form, then those of the other form are 1, $\vartheta\alpha\vartheta^{-1}$, $\vartheta\beta\vartheta^{-1}$, ... (where in the case in question ϑ is a functional symbol of the like homographic form, $\vartheta x = \frac{Ax+B}{Cx+D}$). But instead of obtaining the new form in this manner, I found it easier to use the values of the rotation-symbol

$$\cos\frac{\pi}{q} + \sin\frac{\pi}{q}(i\cos X + j\cos Y + k\cos Z)$$

for the axes of the icosahedron or dodecahedron, given in my paper "Notes on polyhedra," *Quart. Math. Jour.* t. VII. (1866), pp. 304—316, [375]; viz. if for any axes, λ, μ, ν denote the parameters of rotation $\tan\frac{\pi}{q}\cos X$, $\tan\frac{\pi}{q}\cos Y$, $\tan\frac{\pi}{q}\cos Z$, then, by a formula which is in fact equivalent to that given in my note "On the correspondence of Homographies and Rotations," *Math. Annalen*, t. XV. (1879), pp. 238—240, [660], the corresponding homographic function of x is

$$\frac{(-\nu-i)\,x+\lambda+i\mu}{(\lambda-i\mu)\,x+\nu-i},$$

where i denotes $\sqrt{-1}$ as usual.

The new formulæ for the group of 60, or icosahedron group, of homographic functions $\frac{\alpha x+\beta}{\gamma x+\delta}$ are contained in the following table, where the four columns show the values of the coefficients α, β, γ, δ respectively: and where in the outside column, the substitution is represented as a permutation-symbol on the five letters *abcde*: moreover for shortness Θ is written to denote $\sqrt{5}$.

THE GROUP OF 60.

	α	β	γ	δ	
1	1	0	0	1	1
2	−1	0	0	1	*ab . cd*
3	0	1	1	0	*ac . bd*
4	0	−1	1	0	*ad . bc*
5	2	$-3+\Theta+i(\ 1-\Theta)$	$-3+\Theta+i(-1+\Theta)$	−2	*bc . de*
6	2	$-3+\Theta+i(-1+\Theta)$	$-3+\Theta+i(\ 1-\Theta)$	−2	*ae . bc*
7	2	$3-\Theta+i(-1+\Theta)$	$3-\Theta+i(\ 1-\Theta)$	−2	*ad . ce*
8	2	$3-\Theta+i(\ 1-\Theta)$	$3-\Theta+i(-1+\Theta)$	−2	*ad . be*
9	2	$-1-\Theta+i(\ 1-\Theta)$	$-1-\Theta+i(-1+\Theta)$	−2	*ae . cd*
10	2	$-1-\Theta+i(-1+\Theta)$	$-1-\Theta+i(\ 1-\Theta)$	−2	*ab . de*
11	2	$1+\Theta+i(-1+\Theta)$	$1+\Theta+i(\ 1-\Theta)$	−2	*be . cd*
12	2	$1+\Theta+i(\ 1-\Theta)$	$1+\Theta+i(-1+\Theta)$	−2	*ab . ce*
13	2	$-1-\Theta+i(-3-\Theta)$	$-1-\Theta+i(\ 3+\Theta)$	−2	*ac . be*
14	2	$-1-\Theta+i(\ 3+\Theta)$	$-1-\Theta+i(-3-\Theta)$	−2	*bd . ce*
15	2	$1+\Theta+i(\ 3+\Theta)$	$1+\Theta+i(-3-\Theta)$	−2	*ae . bd*
16	2	$1+\Theta+i(-3-\Theta)$	$1+\Theta+i(\ 3+\Theta)$	−2	*ac . de*
17	$-i$	i	1	1	*abc*
18	−1	i	1	i	*acb*
19	1	$-i$	1	i	*adc*
20	$-i$	$-i$	1	−1	*acd*
21	i	i	1	−1	*adb*
22	1	i	1	$-i$	*abd*
23	−1	$-i$	1	$-i$	*bcd*
24	i	$-i$	1	1	*bdc*

	α	β	γ	δ	
25	$-1-\Theta+i(\ 3+\Theta)$	2	-2	$-1-\Theta+i(-3-\Theta)$	*aec*
26	$1+\Theta+i(\ 3+\Theta)$	2	-2	$1+\Theta+i(-3-\Theta)$	*ace*
27	$1+\Theta+i(-3-\Theta)$	2	-2	$1+\Theta+i(\ 3+\Theta)$	*bed*
28	$-1-\Theta+i(-3-\Theta)$	2	-2	$-1-\Theta+i(\ 3+\Theta)$	*bde*
29	$-3+\Theta+i(\ 1-\Theta)$	2	2	$3-\Theta+i(\ 1-\Theta)$	*bec*
30	$-3+\Theta+i(-1+\Theta)$	2	2	$3-\Theta+i(-1+\Theta)$	*bce*
31	$3-\Theta+i(-1+\Theta)$	2	2	$-3+\Theta+i(-1+\Theta)$	*aed*
32	$3-\Theta+i(\ 1-\Theta)$	2	2	$-3+\Theta+i(\ 1-\Theta)$	*ade*
33	2	$-1-\Theta+i(-1+\Theta)$	$1+\Theta+i(-1+\Theta)$		*cde*
34	2	$1+\Theta+i(\ 1-\Theta)$	$-1-\Theta+i(\ 1-\Theta)$		*ced*
35	2	$-1-\Theta+i(\ 1-\Theta)$	$1+\Theta+i(\ 1-\Theta)$		*aeb*
36	2	$1+\Theta+i(-1+\Theta)$	$-1-\Theta+i(-1+\Theta)$		*abe*
37	$-1-\Theta+i(-3-\Theta)$	2	2	$1+\Theta+i(-3-\Theta)$	*abcde*
38	$-1-\Theta+i(\ 1-\Theta)$	2	2	$1+\Theta+i(\ 1-\Theta)$	*acebd*
39	$-1-\Theta+i(-1+\Theta)$	2	2	$1+\Theta+i(-1+\Theta)$	*adbec*
40	$-1-\Theta+i(\ 3+\Theta)$	2	2	$1+\Theta+i(\ 3+\Theta)$	*aedcb*
41	$1+\Theta+i(\ 3+\Theta)$	2	2	$-1-\Theta+i(\ 3+\Theta)$	*adceb*
42	$1+\Theta+i(-1+\Theta)$	2	2	$-1-\Theta+i(-1+\Theta)$	*acbde*
43	$1+\Theta+i(\ 1-\Theta)$	2	2	$-1-\Theta+i(\ 1-\Theta)$	*aedbc*
44	$1+\Theta+i(-3-\Theta)$	2	2	$-1-\Theta+i(-3-\Theta)$	*abecd*
45	$-1-\Theta+i(-1+\Theta)$	2	-2	$-1-\Theta+i(\ 1-\Theta)$	*acbed*
46	$-3+\Theta+i(-1+\Theta)$	2	-2	$-3+\Theta+i(\ 1-\Theta)$	*abdce*
47	$3-\Theta+i(-1+\Theta)$	2	-2	$3-\Theta+i(\ 1-\Theta)$	*aecdb*
48	$1+\Theta+i(-1+\Theta)$	2	-2	$1+\Theta+i(\ 1-\Theta)$	*adebc*
49	$1+\Theta+i(\ 1-\Theta)$	2	-2	$1+\Theta+i(-1+\Theta)$	*aecbd*
50	$3-\Theta+i(\ 1-\Theta)$	2	-2	$3-\Theta+i(-1+\Theta)$	*acdeb*
51	$-3+\Theta+i(\ 1-\Theta)$	2	-2	$-3+\Theta+i(-1+\Theta)$	*abedc*
52	$-1-\Theta+i(\ 1-\Theta)$	2	-2	$-1-\Theta+i(-1+\Theta)$	*adbce*
53	2	$-3+\Theta+i(-1+\Theta)$	$3-\Theta+i(-1+\Theta)$	2	*aebdc*
54	2	$-1-\Theta+i(\ 3+\Theta)$	$1+\Theta+i(\ 3+\Theta)$	2	*abced*
55	2	$1+\Theta+i(-3-\Theta)$	$-1-\Theta+i(-3-\Theta)$	2	*adecb*
56	2	$3-\Theta+i(\ 1-\Theta)$	$-3+\Theta+i(\ 1-\Theta)$	2	*acdbe*
57	2	$-3+\Theta+i(\ 1-\Theta)$	$3-\Theta+i(\ 1-\Theta)$	2	*abdec*
58	2	$-1-\Theta+i(-3-\Theta)$	$1+\Theta+i(-3-\Theta)$	2	*adcbe*
59	2	$1+\Theta+i(\ 3+\Theta)$	$-1-\Theta+i(\ 3+\Theta)$	2	*aebcd*
60	2	$3-\Theta+i(-1+\Theta)$	$-3+\Theta+i(-1+\Theta)$	2	*acedb*

This contains (as one of five groups of 12) the group of the positive permutations of *abcd*; and, completing this into a group of 24, we have

GROUPS OF 12 AND 24.

	α	β	γ	δ	
1	1	0	0	1	1
2	-1	0	0	1	*ab . cd*
3	0	1	1	0	*ac . bd*
4	0	-1	1	0	*ad . bc*
5	$-i$	i	1	1	*abc*
6	-1	i	1	i	*acb*
7	1	$-i$	1	i	*adc*
8	$-i$	$-i$	1	-1	*acd*
9	i	i	1	-1	*adb*
10	1	i	1	$-i$	*abd*
11	-1	$-i$	1	$-i$	*bcd*
12	i	$-i$	1	1	*bdc*
13	i	0	0	1	*adbc*
14	$-i$	0	0	1	*acbd*
15	0	i	1	0	*cd*
16	0	i	-1	0	*ab*
17	1	-1	1	1	*acdb*
18	$-i$	-1	1	i	*bd*
19	i	1	1	i	*abcd*
20	1	1	1	-1	*bc*
21	-1	-1	1	-1	*abdc*
22	i	-1	1	$-i$	*ac*
23	$-i$	1	1	$-i$	*adcb*
24	-1	1	1	1	*ad*

The groups of 60 and 24 thus each of them contain the group of 12,

$$\pm x,\quad \pm\frac{1}{x},\quad \pm i\,\frac{1-x}{1+x},\quad \pm i\,\frac{1+x}{1-x},\quad \pm\frac{x+i}{x-i},\quad \pm\frac{x-i}{x+i}.$$

It may be remarked that, to verify the periodicities of the forms contained in the group of 60, we have as the conditions that

$\dfrac{\alpha x+\beta}{\gamma x+\delta}$ may be periodic of the order 2, $\dfrac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma}=0$, that is, $\alpha+\delta=0$,

" " " 3, " $=1$,

" " " 5, " $=\frac{1}{2}(3+\sqrt{5})$.

For instance, in the form

$$\frac{[-1-\Theta+i(-3-\Theta)]\,x+2}{2x+[1+\Theta+i(-3-\Theta)]},$$

we have

$$\alpha\delta=-(1+\Theta)^2-(3+\Theta)^2,\quad =-20-8\Theta,\quad \beta\gamma=4,$$

$$\alpha+\delta=-2i(3+\Theta):$$

and therefore

$$\frac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma}=\frac{-4(3+\Theta)^2}{-8(3+\Theta)},\quad =\frac{3+\Theta}{2}=\tfrac{1}{2}(3+\sqrt{5}),$$

as it should be.

Cambridge, 11 *Nov.* 1879.

CORRECTION*, pp. 439, 440.

I erroneously assumed that the symbol *adcb* could be taken as corresponding to the linear transformation ix: but this was obviously wrong, for it gave *bd* as corresponding to the transformation $-ix$, and these are not of the same order, but of the orders 4 and 2 respectively. The proper symbol is *adbc*, as given above, and the remaining eleven symbols are then at once obtained.

Cambridge, 17 *Feb.* 1880.

[* The correction in the Table of the Groups of 12 and 24 has been inserted in the Table as now printed on p. 240; it applies to the second half of the column of symbols on the extreme right-hand.]