754.

ON THE CONNEXION OF CERTAIN FORMULÆ IN ELLIPTIC FUNCTIONS.

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IN reference to a like question in the theory of the double ϑ-functions, it is interesting to show that (if not completely, at least very nearly) the single formula

$$\Pi(u,\ a) = u\frac{\Theta' a}{\Theta a} + \tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)},$$

that is,

$$\int_0 \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u\, du}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} = u\frac{\Theta' a}{\Theta a} + \tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)},$$

leads not only to the relation

$$\log\Theta u = \tfrac{1}{2}\log\frac{2k'K}{\pi} + \tfrac{1}{2}\left(1 - \frac{E}{K}\right)u^2 - k^2\int_0 du\int_0 du \operatorname{sn}^2 u,$$

between the functions Θ, sn, but also to the addition-equation for the function sn.

Writing in the equation a indefinitely small, and assuming only that $\operatorname{sn} a$, $\operatorname{cn} a$, $\operatorname{dn} a$ then become a, 1, 1, respectively, the equation is

$$k^2 a\int_0 \operatorname{sn}^2 u\, du = u\frac{a\Theta''0}{\Theta 0} + \tfrac{1}{2}\log\frac{\Theta u - a\Theta' u}{\Theta u + a\Theta' u},$$

$$= ua\frac{\Theta''0}{\Theta 0} - a\frac{\Theta' u}{\Theta u},$$

that is,

$$\frac{\Theta' u}{\Theta u} = u\frac{\Theta''0}{\Theta 0} - k^2\int_0 du \operatorname{sn}^2 u,$$

or, integrating from $u = 0$, this is

$$\log\Theta u = C + \tfrac{1}{2}u^2\frac{\Theta''0}{\Theta 0} - k^2\int_0 du\int_0 du \operatorname{sn}^2 u,$$

which, except as regards the determination of the constants, is the required equation for $\log \Theta u$.

Next, differentiating twice the equation for $\Pi(u, a)$, and once the equation obtained for $\frac{\Theta' u}{\Theta u}$, we have

$$k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \frac{d}{du}\left(\frac{\operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u}\right) = \tfrac{1}{2}\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}(u-a) - \tfrac{1}{2}\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}(u+a),$$

and

$$\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}\, u = \frac{\Theta'' 0}{\Theta 0} - k^2 \operatorname{sn}^2 u,$$

where, for shortness, $\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}\, u$ is written to denote $\frac{\Theta'' u\, \Theta u - (\Theta' u)^2}{\Theta^2 u}$, and the like in the first equation; the right-hand side of the first equation therefore is

$$-\tfrac{1}{2}k^2 \{\operatorname{sn}^2(u-a) - \operatorname{sn}^2(u+a)\},$$

or the equation becomes

$$2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \frac{d}{du} \frac{\operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} = \operatorname{sn}^2(u+a) - \operatorname{sn}^2(u-a),$$

that is,

$$\frac{4 \operatorname{sn} u \operatorname{sn}' u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a)^2} = \operatorname{sn}^2(u+a) - \operatorname{sn}^2(u-a).$$

The numerator on the left-hand side must be a symmetrical function of u, a, and hence (even if the value of $\operatorname{sn}' u$ were unknown) it would appear that $\operatorname{sn}' u$ must be a mere constant multiple of $\operatorname{cn} u \operatorname{dn} u$; assuming, however, the actual value, $\operatorname{sn}' u = \operatorname{cn} u \operatorname{dn} u$, the formula is

$$\begin{aligned}\frac{4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a)^2} &\\ &= \operatorname{sn}^2(u+a) - \operatorname{sn}^2(u-a)\\ &= \{\operatorname{sn}(u+a) + \operatorname{sn}(u-a)\}\{\operatorname{sn}(u+a) - \operatorname{sn}(u-a)\}.\end{aligned}$$

The factor $\{\operatorname{sn}(u+a) + \operatorname{sn}(u-a)\}$ becomes $= 2\operatorname{sn} u$ for $a = 0$, and this suggests that the factor $\operatorname{sn} u$ on the left-hand side is a factor of $\{\operatorname{sn}(u+a) + \operatorname{sn}(u-a)\}$. That $\operatorname{cn} u$ is *not* a factor hereof would follow from the properties of the period K; viz. for $u = K$, $\operatorname{cn} u = 0$, but $\{\operatorname{sn}(u+a) + \operatorname{sn}(u-a)\}, = 2\operatorname{sn}(K+a)$ is not $\doteq 0$; and, similarly, that $\operatorname{dn} u$ is *not* a factor from the properties of the period iK; hence, $\operatorname{cn} u$, $\operatorname{dn} u$ belong to the other factor $\{\operatorname{sn}(u+a) - \operatorname{sn}(u-a)\}$, and by symmetry $\operatorname{cn} a$, $\operatorname{dn} a$ belong to the first-mentioned factor. And we are thus led to assume

$$\begin{aligned}\operatorname{sn}(u+a) + \operatorname{sn}(u-a) &= 2M \operatorname{sn} u \operatorname{cn} a \operatorname{dn} a,\\ \operatorname{sn}(u+a) - \operatorname{sn}(u-a) &= 2M' \operatorname{sn} a \operatorname{cn} u \operatorname{dn} u,\end{aligned}$$

where

$$\text{denom.} = 1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u,$$

and $MM' = 1$. Some further investigation is wanting to show that M and M' are constants, but assuming that they are so and each $= 1$, the formulæ give at once the ordinary expression for $\operatorname{sn}(u+a)$; that is, we have the addition-equation for the function sn.