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ON SCHUBERT'S METHOD FOR THE CONTACTS OF A LINE
WITH A SURFACE.

I WISH to reproduce in part § 33, "Coincidenz von Schnittpunkten einer Geraden mit einer Fläche" of Schubert's very interesting work *Calcul der abzählende Geometrie*, Leipzig, 1879, explaining in the first instance (but not altogether in the manner or from the point of view of the author) the general principles of the theory.

We have to do with *conditions* relating to a *subject*; the subject is a geometrical form or entity of any kind depending upon a certain number of constants; and the condition is onefold, twofold, &c., according as it imposes a onefold, twofold, &c., relation upon these constants. The number of constants is the Postulandum of the subject, and the manifoldness of the condition is called also its Postulation. A condition is incomplete when its postulation is less than postulandum of subject, complete when its postulation is equal to postulandum of subject; two or more incomplete conditions, making up a complete condition, are supplementary to each other. The case where the postulation exceeds the postulandum, or say that of a more than complete condition, is not in general considered; it may however sometimes present itself. For instance, the subject may be a line with n points upon it; the number of constants is here $= n + 4$. A condition that the line shall meet a given line, or that a certain one of the n points shall lie on a given plane, is a onefold condition; the condition that such point shall lie upon a given line is a twofold condition; and so in other cases.

Conditions are denoted by letters, and simultaneous conditions by a product; for instance, the subject is a line carrying the n points 1, 2, ..., n; g is the condition that the line meets a given line; p_1 the condition that the point 1 lies on a given plane; then gp_1 is the twofold condition that the line meets a given line and that

the point 1 lies on a given plane; p_1^2 is the twofold condition that the point 1 lies on each of two given planes (in fact, on their line of intersection). The letters p, g, e are used as the initials of Punkt, Gerade, Ebene.

The letter or combination of letters denoting an incomplete condition, or, say, the incomplete condition itself, has no numerical value; but for a complete condition there exists a definite number of subjects satisfying the condition, and the condition is regarded as having this number as its value. A more than complete condition has the value 0.

Conditions of the same postulation may be connected by the sign +; for instance,

subject a line,

g_e the condition that it lies in a given plane,

g_p the condition that it passes through a given point,

then $g_e + g_p$ is the condition that the line shall either lie in the given plane or else pass through the given point.

I abstain from attempting any definition in regard to the sign −.

Conditions of the same postulation may be connected by an equation or equations; for instance,

subject a point,

p the condition that the point shall lie in a given plane,

p_g the condition that the point shall lie in a given line,

then $p^2 = p_g$.

This equation has (so far) no numerical signification; it has the logical signification that the condition that a point shall lie on each of two given planes is equivalent to the condition that the point shall lie on a given line.

Second example. Subject a line,

g the condition that the line meets a given line,

g_e the condition that it lies in a given plane,

g_p the condition that it passes through a given point,

then $g^2 = g_e + g_p$.

This equation has (so far) no numerical signification, and I regard it as having no logical signification. Schubert, however, gives it a logical signification by means of his "Princip der speciellen Lage" (Principle of Special Situation), viz. the condition of the line meeting each of two given lines is, in the particular case where the two given lines meet, equivalent to the condition, that the line shall either lie in the plane of the two given lines or else pass through their point of intersection.

Third example. Subject a line bearing upon it the points 1 and 2,

   ϵ the condition of the coincidence of the two points,

   p „   that the point 1 shall lie on a given plane,

   q „   „   2   „   „

   g „   that the line shall meet a given line,

then $\epsilon = p + q - g$.

This equation has (so far) no numerical signification, and it does not appear to have any logical signification. In fact, in the actual form of the equation we have a sign $-$ which has not had given to it any logical interpretation; and if we write the equation in the form $\epsilon + g = p + q$, there seems to be no logical signification in the assertion, the condition that either the points shall coincide, or else the line meet a given line, is equivalent to the condition that either the first point, or else the second point, shall lie in a given plane.

Any equation connecting complete conditions is a numerical equation; and to render a condition complete, we have only to join to it a supplementary condition X of the proper postulation. Thus, in the last example the postulandum is $= 6$; ϵ, p, q, g are onefold conditions, and joining to each of them one and the same fivefold condition X, we have $X\epsilon = Xp + Xq - Xg$. And, taking X to be an arbitrary fivefold condition, the original equation $\epsilon = p + q - g$ has in fact the meaning

$$X\epsilon = Xp + Xq - Xg.$$

For instance, the fivefold condition X may be that the line shall belong to a given regulus (scroll or developable surface), and that the points 1, 2 upon the line shall be the intersections of the line with given surfaces S_1, S_2 respectively. The subject is the line of the given regulus with its two points; and the meaning of the equation is that the number of subjects with two coincident points is equal to the number of subjects with the point 1 on a given plane, *plus* the number of subjects with the point 2 on a given plane, *minus* the number of subjects for which the line meets a given line. Although for the moment concerned only with the meaning of the theorem, not with its truth, I stop to show *à posteriori* that the theorem is in fact true: take k for the order of the regulus; m_1, m_2 for the orders of the surfaces S_1, S_2 respectively; then it is to be shown that $X\epsilon$, Xp, Xq, Xg are each $= km_1m_2$ (values which satisfy the equation). First $X\epsilon$: the points 1 and 2 here coincide at a point of the curve of the order m_1m_2, which is the intersection of S_1 and S_2; the regulus meets this curve in km_1m_2 points, and through each of these we have a line of the regulus having upon it the two coincident points; that is, $X\epsilon = km_1m_2$. Next Xp: the point 1 is here on the plane curve of the order m_1, which is the intersection of S_1 with the corresponding given plane; the regulus meets this plane curve in km_1 points; through each of these we have a line of the regulus intersecting S_2 in m_2 points, any one of which may be taken for the point 2; that is, the number of subjects is Xp, $= km_1 \cdot m_2$. Then Xq: in precisely the same manner we have $Xq = km_2 \cdot m_1$. Lastly Xg: the given line meets the regulus in k points, and

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through each of these there is a line of the regulus meeting S_1 in m_1 points, any one of which may be taken for the point 1, and meeting S_2 in m_2 points, any one of which may be taken for the point 2; the number of the subjects Xg is thus $Xg, = k \cdot m_1 \cdot m_2$.

The general theorem $X\epsilon = Xp + Xq - Xg$ is proved by means of Chasles' theorem of united points as follows: the subject is a line, or say, for convenience, an *axis* ξ, bearing upon it the two points 1 and 2; we consider in conjunction therefore a given line λ, and through this draw the planes P_1, P_2 passing through the points 1 and 2 respectively.

Suppose that when 2 lies in a given plane there are α' positions of the axis, and on each of these β' positions of the point 1; and, similarly, that when 1 lies on a given plane there are α positions of the axis, and on each of these β positions of the point 2; then, 1 lying in a given plane, the number of subjects is $\alpha\beta$, or we have $Xp = \alpha\beta$; and, similarly, $Xq = \alpha'\beta'$. Take now for the point P_1 an arbitrary plane through λ; then, 1 lying on this plane, the number of the points 2 is $= \alpha\beta$, or, since each of these determines with λ a position of the plane P_2, the number of these planes is $= \alpha\beta$, that is, it is $= Xp$; and, similarly, taking P_2 an arbitrary plane through λ, the number of the planes P_1 is $\alpha'\beta'$, that is, it is $= Xq$; viz. the two planes P_1, P_2 through the line λ have an (Xp, Xq) correspondence; hence, by Chasles' theorem, the number of united planes is $= Xp + Xq$.

But we have a united plane, 1°, if the points 1 and 2 coincide, that is, if the condition $X\epsilon$ be satisfied, and the number of these united planes is $X\epsilon$; 2°, if the axis ξ meet the arbitrary line λ, that is, if the condition Xg be satisfied, and the number of these united planes is $= Xg$; hence the whole number is $= X\epsilon + Xg$; or we have $Xp + Xq = X\epsilon + Xg$, that is, $X\epsilon = Xp + Xq - Xg$, which is the theorem in question.

The conclusion is that the equation $\epsilon = p + q - g$, which in this, its original form, has neither a numerical nor a logical signification, is to be understood as meaning the numerical equation $X\epsilon = Xp + Xq - Xg$, the truth of which numerical equation has just been proved. Or we may, without explicit introduction of the condition X, understand the equation $\epsilon = p + q - g$ as a numerical equation as follows, viz. taking for the subject a line with two points, *which line and points are regarded as satisfying a given fivefold condition*, then

ϵ is the (additional onefold) condition that the two points shall coincide,

p „   „   „   that the point 1 shall lie in given plane,

q „   „   „   that the point 2 shall lie in given plane,

g „   „   „   that line shall meet given line.

The conditions ϵ, p, q, g are thus in effect complete conditions, having values which may be connected by an equation; there, in fact, exists between them the relation

$$\epsilon = p + q - g.$$

The like remarks would apply to the before-mentioned equation (subject a point) $p^2 = p_g$: either adding to it a onefold condition X, and so taking it in the form $Xp^2 = Xp_g$, or understanding it in its original form $p^2 = p_g$ as belonging to a point which satisfies already a onefold condition, the equation is true as a numerical equation; and this in fact follows at once from its truth as a logical equation. But observe the difference: the equation in question $p^2 = p_g$ has, the equation $\epsilon = p + q - g$ has not, a logical signification.

I regard as the fundamental notion of the theory the existence of equations between conditions such as the foregoing equation $\epsilon = p + q - g$; equations which in their original form have not (of necessity) any logical signification, and have not any numerical signification; but which, when we adjoin to them a supplementary condition X of the proper postulation, become numerical equations, which are true, independently of the form of the supplementary condition X and whatever this condition may be. And this being so, it seems to follow at once that such equations may be treated and worked with as ordinary algebraical equations. For instance, let M be any condition of less postulation than X: then if from the equation $\epsilon = p + q - g$, assumed to be true, we deduce $M\epsilon = Mp + Mq - Mg$, this (like the original equation $\epsilon = p + q - g$) is in its actual form an equation without logical or numerical signification; but if we adjoin to it a supplementary condition K, such that postulation of $K + $ do. of $M = $ do. of X (or, what is the same thing, that the condition KM shall be supplementary to the several conditions contained in the original equation $\epsilon = p + q - g$), then the equation in question, $M\epsilon = Mp + Mq - Mg$, is to be interpreted as meaning

$$KM\epsilon = KMp + KMq - KMg,$$

that is,

$$X\epsilon = Xp + Xq - Xg,$$

which is numerically true. We thus see that the original equation $\epsilon = p + q - g$ implies the new equation

$$M\epsilon = Mp + Mq - Mg,$$

which is its algebraical consequence. And if we regard, for instance, $A + B$ as the condition that either the condition A shall be satisfied or else the condition B shall be satisfied, then $A + B$ is a condition, and as such we have

$$(A + B)\epsilon = (A + B)p + (A + B)q - (A + B)\epsilon.$$

It is going a step further to say that if we have, for instance, an equation $M = A + B - C$ between conditions M, A, B, C, then that, instead of

$$M\epsilon = Mp + Mq - M\epsilon,$$

we may write

$$(A + B - C)\epsilon = (A + B - C)p + (A + B - C)q - (A + B - C)\epsilon;$$

this is, in fact, treating $A + B - C$ as being to all intents and purposes a condition such as M, or an alternative condition $A + B$. It is, in fact, assumed that the step is permissible; and we thus make such deductions as

$$(\epsilon + p + q - g)(\epsilon - p - q + g) = 0;$$

that is,
$$\epsilon^2 - (p + q - g)^2 = 0,$$
or
$$\epsilon^2 = (p + q - g)^2, \ = p^2 + 2pq + q^2 - 2pg - 2qg + g^2\,;$$

viz. this is an equation such as the original equation $\epsilon = p + q - g$, acquiring a numerical signification when we adjoin to it a supplementary condition X of the proper postulation.

The section above referred to deals with the question to determine the number of lines which satisfy the several relations of contact in regard to a given surface F of the order n, without point-singularities, that is, the surface represented by the *general* equation $(*\!\!\;\!\!\;x, y, z, w)^n = 0$.

The chief results are contained in the following table, the notation of which will be explained:

1. $\epsilon_2 g_s$ $=$ $n(n-1)$,

2. $\epsilon_2 b_2 g_e$ $=$ n,

3. $\epsilon_3 g_e$ $=$ $3n(n-2)$,

4. $\epsilon_3 g_p$ $=$ $n(n-1)(n-2)$,

5. $\epsilon_3 b_3^2$ $=$ $2n$,

6. $\epsilon_{22} g_e$ $=$ $\frac{1}{2} n(n-2)(n-3)(n+3)$,

7. $\epsilon_{22} g_p$ $=$ $\frac{1}{2} n(n-1)(n-2)(n-3)$,

8. $\epsilon_{22} b_3^2$ $=$ $n(n-3)(n+2)$,

9. $\epsilon_{22} b_2 c_2$ $=$ $n(n^3 - 2n^2 + 2n - 6)$,

10. $\epsilon_4 g$ $=$ $2n(n-3)(3n-2)$,

11. $\epsilon_4 b_4$ $=$ $n(11n - 24)$,

12. $\epsilon_{32} g$ $=$ $n(n-3)(n-4)(n^2 + 6n - 4)$,

13. $\epsilon_{32} b_3$ $=$ $n(n-4)(3n^2 + 5n - 24)$,

14. $\epsilon_{32} b_2$ $=$ $n(n-2)(n-4)(n^2 + 2n + 12)$,

15. $\epsilon_{222} g$ $=$ $\frac{1}{3} n(n-3)(n-4)(n-5)(n^2 + 3n - 2)$,

16. $\epsilon_{222} b_2$ $=$ $\frac{1}{2} n(n-2)(n-4)(n-5)(n^2 + 5n + 12)$,

17. ϵ_5 $=$ $5n(n-4)(7n-12)$,

18. ϵ_{42} $=$ $2n(n-4)(n-5)(n+6)(3n-5)$,

19. ϵ_{33} $=$ $\frac{1}{2} n(n-4)(n-5)(n^3 + 3n^2 + 29n - 60)$,

20. ϵ_{322} $=$ $\frac{1}{2} n(n-4)(n-5)(n-6)(n^3 + 9n^2 + 20n - 60)$,

21. ϵ_{2222} $=$ $\frac{1}{12} n(n-4)(n-5)(n-6)(n-7)(n^3 + 6n^2 + 7n - 30)$,

22. $\epsilon_{222} b_1$ $=$ $\frac{1}{3} n(n-4)(n-5)(n-6)(n^3 + 3n^2 - 2n - 12)$,

23. $\epsilon_3 b_1^2$ $=$ $(n-3)(n^2 + 2)$,

24. $\epsilon_2 b_1 c_1 d_1 =$ $n^2(n-4)(2n^2 - 3n - 3)$.

In the foregoing formulæ the suffixes of the ϵ refer to the contacts, viz. ϵ_2 denotes a 2-pointic intersection, ϵ_{32} a 3-pointic and a 2-pointic intersection. The letters b, c, d refer to the points of contact or intersection, thus $\epsilon_{32}b_3$, b_3 is the point of 3-pointic intersection; $\epsilon_{222}b_1$, b_1 is one of the points of simple intersection; b_1 is also the condition that the point in question lies on a given plane; g, g_s, g_e, g_p have their ordinary signification explained a little further on. Thus (15) $\epsilon_{222}g$ denotes the number of triple tangents which can be drawn to meet a given line; or, what is the same thing, it is the order of the regulus formed by the triple tangents.

The following are elementary formulæ used in the investigation of the foregoing results.

Subject a line having upon it a point,

			Postul.
p the condition that point is in a given plane			1
p_g	„	„ line	2
g	„	line meets a given line	1
g_e	„	„ is in a given plane	2
g_p	„	„ passes through a given point	2
g_s	„	„ lies in a given plane and passes through a given point of that plane	3
G	„	„ coincides with a given line	4

We have (p. 22 *et seq.*)

		Postul.	
p_g	$= p^2$	2	(*logical*)
p_g	$= p^2 + g_e$	2	
g^2	$= g_e + g_p$	2	
g_s	$= g g_e$	3	(*logical*)
g_s	$= g g_p$	3	(*logical*)
$p g_p$	$= p^3 + g_s$	3	(*demons. infrà*)
p^4	$= 0$	4	
$g_e g_p$	$= 0$	4	
g_e^2	$= G$	4	
g_p^2	$= G$	4	
$p^3 g$	$= p^2 g_e$	4	(*demons. infrà*)
$p g_s$	$= p^2 g_p$	4	„
$p g_s$	$= p^2 g_e + G$	4	„
$p^3 g_e$	$= 0$	5	„
$p^3 g_p$	$= p G$	5	„
$p^2 g_s$	$= p G$	5	„

$pg_p = p^3 + g_s$; we have $0 = g_e + p^2 - pg$, $0 = g_e + g_p - g^2$, and thence

$$
\begin{aligned}
0 = \; & (p+g)\{g_e + p(p-g)\} - p\{g_e + g_p - g^2\} \\
= \; & pg_e + gg_e + p^3 - pg^2 \\
& -pg_e - pg_p \quad\quad + pg^2 \\
= \; & gg_e - pg_p + p^3 \\
= \; & g_s - pg_p + p^3.
\end{aligned}
$$

$p^3 g = p^2 g_e$: from $pg = p^2 + g_e$, we have $p^2 g = p^4 + p^2 g_e = p^2 g_e$, since $p^4 = 0$,

$pg_s = p^2 g_e + G$ „　$g_s = gg_e$ 　„　$pg_e = pgg_e = (p^2 + g_e)g_e = p^2 g_e + G$,

$pg_s = p^2 g_p$ 　„　$g_s = gg_p$ 　„　$pg_s = pgg_p = (p^2 + g_e)g_p = p^2 g_p$, since $g_e g_p = 0$;

and in a similar manner we prove the last three equations.

For the demonstration of the formulæ of the table we take the subject to be a line bearing upon it the points $1, 2, \ldots, n$, which are its intersections with a given surface of the order n. The symbols $p_1, p_2, \ldots$ refer to these points respectively; thus, p_1 is the condition that the point 1 may lie on a given plane; and then, writing

$$
\begin{aligned}
\epsilon \; &= p_1 + p_2 - g, \\
\epsilon' \; &= p_1 + p_3 - g, \\
\epsilon'' \; &= p_3 + p_4 - g,
\end{aligned}
$$

it appears that ϵ will denote the condition of the coincidence of the points 1 and 2; ϵ' that of the points 1 and 3, &c. Hence also, $\epsilon\epsilon'$ will denote the twofold condition of the coincidence of the points 1, 2, 3; and so in other cases. But, according to the notation above explained, ϵ is also denoted by ϵ_2, $\epsilon\epsilon'$ by ϵ_3, $\epsilon\epsilon''$ by ϵ_{22}, &c.

We thus have

$$
\begin{aligned}
\epsilon_2 \; &= p_1 + p_2 - g, \\
\epsilon_3 \; &= (p_1 + p_2 - g)(p_1 + p_3 - g), \\
2\epsilon_{22} \; &= (p_1 + p_2 - g)(p_3 + p_4 - g), \\
\epsilon_4 \; &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_1 + p_4 - g), \\
\epsilon_{32} \; &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_4 + p_5 - g), \\
6\epsilon_{222} \; &= (p_1 + p_2 - g)(p_3 + p_4 - g)(p_5 + p_6 - g), \\
\epsilon_5 \; &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_1 + p_4 - g)(p_1 + p_5 - g), \\
\epsilon_{42} \; &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_1 + p_4 - g)(p_5 + p_6 - g), \\
2\epsilon_{33} \; &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_4 + p_5 - g)(p_4 + p_6 - g), \\
2\epsilon_{322} \; &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_4 + p_5 - g)(p_6 + p_7 - g), \\
24\epsilon_{2222} \; &= (p_1 + p_2 - g)(p_3 + p_4 - g)(p_5 + p_6 - g)(p_7 + p_8 - g).
\end{aligned}
$$

We can now, by a mere analytical process of development and reduction, express each of the foregoing values as a linear function of

$$p_1^2 p_2^2, \quad p_1^2 p_2 p_3, \quad p_1 p_2 p_3 p_4, \quad \text{and } G.$$

(Schubert says, as a linear function of these four symbols and $p_1 p_2 g_e$; but in fact $p_1 p_2 g_e$ is $= p_1^2 p_2^2$.)

Observe, first, that we may, p. 287, in all the general equations instead of p write p_1, p_2, &c.; and, further, that any symbol containing for instance p_1^3 is $= 0$. For the symbols now belong to the intersections of the line with a given surface; p_1^3 is the condition that a certain one of these intersections shall lie in three given planes, that is, that it shall coincide with a given arbitrary point; this cannot be the case, for the arbitrary point is not on the surface F; and therefore $p_1^3 = 0$.

We thus have $p_1 g = p_1^2 + g_e$, thence $p_1^2 g = p_1^3 + p_1 g_e$, that is, $p_1^2 g = p_1 g_e$; and thence further $p_1^3 g = p_1^2 g_e$, that is, $p_1^2 g_e = 0$.

Again, from $p_2 g = p_2^2 + g_e$, $p_1 g = p_1^2 + g_e$, we have

$$p_1^2 (p_2^2 + g_e) = p_1 p_2 (p_1^2 + g_e),$$

which, in virtue of $p_1^2 g_e = 0$ and $p_1^3 p_2 = 0$, becomes

$$p_1^2 p_2^2 = p_1 p_2 g_e.$$

As a simple instance of the reductions, take

$$\epsilon_2 g_8, \ = (p_1 + p_2 - g) g_8.$$

Here

$$p_1 g_8, \ = p_2 g_8, \ = p_1^2 g_e + G, \ = G, \text{ since } p_1^2 g_e = 0;$$

and

$$g g_8 = g^2 g_e = (g_e + g_p) g_e = g_e^2 + g_e g_p = G, \text{ since } g_e^2 = 0, \ g_e g_p = G;$$

whence the value is

$$\epsilon_2 g_8 = G + G - G, \ = G.$$

As a more complicated example, take

$$\epsilon_5, \ = (p_1 + p_2 - g)(p_1 + p_3 - g)(p_1 + p_4 - g)(p_1 + p_5 - g).$$

Observe that, after the multiplication is effected we may, in any way we please, interchange the suffixes, $p_1^2 p_3 p_4 = p_1^2 p_2 p_3$, $p_3^2 p_4^2 = p_1^2 p_2^2$, &c.; the suffixes serve only to distinguish from each other symbols in the same product (thus p_1^4 is different from $p_1 p_2 p_3 p_4$), but there is nothing to distinguish one point of intersection from another. Thus the foregoing expression containing the terms $(p_2 + p_3 + p_4 + p_5)(p_1 - g)^3$, these may be combined into the single term $4 p_2 (p_1 - g)^3$; expanding in powers of $p_1 - g$ and reducing in this manner, the value of ϵ_5 is, in fact, found to be

$$= (p_1 - g)^4 + 4 p_2 (p_1 - g)^3 + 6 p_2 p_3 (p_1 - g)^2 + 4 p_2 p_3 p_4 (p_1 - g) + p_1 p_2 p_3 p_4.$$

Developing this in powers of g, omitting the terms containing p_2^3 which vanish, and further reducing, the value is

$$6 p_1^2 p_2 p_3 + 5 p_1 p_2 p_3 p_4 + g(-12 p_1^2 p_2 - 16 p_1 p_2 p_3) + g^2 (6 p_1^2 + 18 p_1 p_2) - 8 p_1 g^3 + g^4.$$

We have

$$g^4 = 2G, \quad p_1g^3 = p_1g_s = p_1^2g_e + G, \ = G.$$

Next for the terms in g^2, from $p_1g = p_1^2 + g_e$ we have

$$p_1^2g = \qquad\qquad p_1g_e,$$

$$p_1p_2g = p_1^2p_2 \ + p_1g_e,$$

and thence

$$p_1^2g^2 = \qquad\qquad p_1g_s;$$

$$p_1p_2g^2 = p_1^2p_2g + p_1g_s,$$

or, since $p_1g_s = G$ as before, the whole term is $= 18p_1^2p_2g + 24G$. The terms in g thus become $= g(6p_1^2p_2 - 16p_1p_2p_3)$, and from the same equation $p_1g = p_1^2 + g_e$ we find

$$p_1^2p_2g = p_1^2p_2^2 \text{ and } p_1p_2p_3g = p_1^2p_2p_3 + p_1^2p_2^2.$$

The value is thus finally found to be

$$= -10p_1^2p_2^2 - 10p_1^2p_2p_3 + 5p_1p_2p_3p_4 + 10G.$$

The whole series of like results is

		$p_1^2p_2^2$	$p_1^2p_2p_3$	$p_1p_2p_3p_4$	G
1.	ϵ_2 g_s				+ 1
2.	,, b_2g_e	+ 1			− 1
3.	ϵ_3 g_e	3			− 3
4.	,, g_p				+ 1
5.	,, b_3^2	− 2	+ 1		+ 1
6.	$2\epsilon_{22}$ g_e	+ 4			− 3
7.	2 ,, g_p				+ 1
8.	,, b_2^2	− 3	+ 2		+ 1
9.	ϵ_2 b_2c_2	− 2		+ 1	+ 1
10.	ϵ_4 g	− 2	+ 4		− 2
11.	,, b_4	− 6		+ 1	+ 4
12.	ϵ_{32} g	− 3	+ 6		− 2
13.	,, b_3	− 7	− 1	+ 2	+ 4
14.	,, b_2	− 6	− 3	+ 3	+ 4
15.	$6\epsilon_{222}$ g	− 4	+ 8		− 2
16.	2 ,, b_2	− 7	− 4	+ 4	+ 4
17.	ϵ_5	− 10	− 10	+ 5	+ 10
18.	ϵ_{42}	− 10	− 16	+ 8	+ 10
19.	$2\epsilon_{32}$	− 9	− 18	+ 9	+ 10
20.	$2\epsilon_{322}$	− 9	− 24	+ 12	+ 10
21.	$24\epsilon_{2222}$	− 8	− 32	+ 16	+ 10
22.	$6\epsilon_{222}$ b_1	− 6	− 12	+ 8	+ 4
23.	$n\epsilon_3$ b_1^2	− 3	+ 3		+ 1
24.	ϵ_2 $b_1c_1d_1$	− 1	− 1	+ 2	

But in these formulæ $p_1^2 p_2^2$, $p_1^2 p_2 p_3$, $p_1 p_2 p_3 p_4$, G have numerical values which are different according to the number of points of intersection presenting themselves in the several formulæ; viz. this number being called i, we have for the formulæ in

ϵ_2	ϵ_3	ϵ_{22}	ϵ_4	ϵ_{32}	ϵ_{222}	ϵ_5	ϵ_{42}	ϵ_{33}	ϵ_{322}	ϵ_{2222}	$\epsilon_{222}b_1$	$\epsilon_3 b_1^2$	$\epsilon_2 b_1 c_1 d_1$
$i=2$	3	4	4	5	6	5	6	6	7	8	7	4	5,

and the values of the symbols are

$$p_1^2 p_2^2 \ \ = n^2 (n-2)(n-3) \qquad \ldots (n-i+1),$$

$$p_1^2 p_2 p_3 = n^2 (n-1)(n-3) \qquad \ldots (n-i+1),$$

$$p_1 p_2 p_3 p_4 = n^2 (2n^2 - 6n + 3)(n-4)\ldots(n-i+1),$$

$$G \qquad \ \ = n\,(n-1)(n-2) \qquad \ldots (n-i+1).$$

Thus, suppose $i=4$, the subject is a line bearing the points 1, 2, 3, 4, which are intersections of the line with the surface F; we have then G as the condition in order that this line (or, say, the line of the subject) may coincide with a given line, which given line intersects the surface in n points; any four of these (their order being attended to) may be regarded as being the points 1, 2, 3, 4; or there are $n(n-1)(n-2)(n-3)$ subjects satisfying the prescribed condition (that the line of the subject may coincide with the given line). Hence here $G = n(n-1)(n-2)(n-3)$; and so in general $G = n(n-1)(n-2)\ldots(n-i+1)$.

Next, for $p_1^2 p_2^2$. Here p_1^2 is the condition that the point 1 shall lie in each of two given planes, that is, in a given line, say L_1; and, similarly, p_2^2 is the condition that 2 may lie in a given line L_2. We take any one of the n intersections of L_1 with F for the point 1, and any one of the n intersections of L_2 with F for the point 2; this determines the line of the subject, but the $i-2$ points 3, 4, ..., i are then any $i-2$ of the remaining $n-2$ intersections of this line with F; that is, $p_1^2 p_2^2 = n^2 (n-2)(n-3)\ldots(n-i+1)$ as above.

Again, for $p_1^2 p_2 p_3$. Here p_1^2 is the condition that 1 shall lie in a given line L_1; we therefore take for 1 any one of the n intersections of L_1 with F; p_2 is the condition that 2 may lie in a given plane P_2, it lies therefore in the curve of intersection of P_2 with F; and, similarly, 3 lies in the curve of intersection of a plane P_3 with F; the two planes intersect in a line meeting F in n points σ, and the two cones, vertex 1, which stand upon the plane curves respectively, intersect in the n lines joining 1 with the n points σ, and in $n^2 - n$ other lines. The line of the subject is then any one of these $n^2 - n$ lines, or, since the vertex is any one of n points, the line is any one of $n(n^2 - n)$, $= n^2(n-1)$ lines; the remaining points 4, 5, ..., i are any $i-3$ of the remaining $n-3$ intersections of the line with F; hence the formula

$$p_1^2 p_2 p_3 = n^2 (n-1)(n-3)(n-4)\ldots(n-i+1).$$

For $p_1 p_2 p_3 p_4$. We have here 1, 2, 3, 4 lying in given plane sections of the surface F, and we have consequently to find the number of lines which can be drawn to meet each of these four sections. Observing that any two of the sections meet in the n

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intersections with F of the line of intersection of their planes, the order of the scroll generated by the lines which meet three of the sections is $2n^3 - 3n^2$; this scroll meets the fourth section in $n(2n^3 - 3n^2)$, $= 2n^4 - 3n^3$ points; or we have this number of lines meeting each of the four sections. But among these are included $3n^2(n-1)$ lines which have to be rejected, viz. the sections 1 and 4 meet in n points, each of which is the vertex of cones through the sections 1 and 2 respectively; these cones meet in n lines, which are to be disregarded, and in $n^2 - n$ other lines, and we have thus $n(n^2 - n)$, $= n^2(n-1)$ lines; and similarly from the intersections of 2 and 4, and from the intersections of 3 and 4, $n^2(n-1)$ and $n^2(n-1)$ lines, in all $3n^2(n-1)$ lines. Hence the number of lines meeting the four sections is

$$2n^4 - 3n^3 - 3n^3 + 3n^2, \quad = 2n^4 - 6n^3 + 3n^2;$$

taking any one of these for the line of the subject, the remaining points 5, 6, ..., i are any $i-4$ of the remaining $n-4$ intersections, or we have the required formula

$$p_1 p_2 p_3 p_4 = n^2 (2n^2 - 6n + 3)(n-4)\ldots(n-i+1).$$

The four numbers $p_1^2 p_2^2$, $p_1^2 p_2 p_3$, $p_1 p_2 p_3 p_4$, G for any line of the table being now known, we can at once calculate the required values $\epsilon_2 g_8$, &c., as the case may be; for instance,

$$
\begin{aligned}
i = 5, \quad \epsilon_5 = &- 10 p_1^2 p_2^2 &&= -10n^2 (n-2)(n-3)(n-4) \\
&- 10 p_1^2 p_2 p_3 && \quad -10n^2 (n-1)(n-3)(n-4) \\
&+ 5 p_1 p_2 p_3 p_4 && \quad + 5n^2 (2n^2 - 6n + 3)(n-4) \\
&+ 10 G && \quad +10n(n-1)(n-2)(n-3)(n-4) \\
&= && \quad 5n(n-4)(7n - 12).
\end{aligned}
$$

In fact, throwing out $n(n-4)$, the remaining terms give

$$
\begin{aligned}
&- 10n^3 + 50n^2 - 60n \\
&- 10n^3 + 40n^2 - 30n \\
&+ 10n^3 - 30n^2 + 15n \\
&+ 10n^3 - 60n^2 + 110n - 60 \\
\hline
& 35n - 60, \quad = 5(7n - 12).
\end{aligned}
$$

And we obtain in like manner the other formulæ of the table.

The remainder of § 33 contains investigations of less systematically connected theorems, and I quote the results only.

25. If on the surface F_n there is a curve order r, then of the tangent planes of F_n along this curve there pass $r(n-1)$ through an arbitrary point of space; *aliter*, class of torse is $= r(n-1)$.

In particular, for curve of 4-pointic contact, $r = n(11n - 24)$, class of torse is

$$= n(n-1)(11n - 24).$$

No. of tangent planes through line, or class of surface, $= n(n-1)^2$.

26. $\epsilon_3 b_3 g = \epsilon_3 b_3^2 + \epsilon_3 g_e = 2n + 3n\,(n-2),\ = n\,(3n-4).$

$\epsilon_3 b_3 g,\ = n\,(3n-4)$, is the order of curve of contact of the 3-pointic (chief) tangents which meet a given line.

Parabolic tangents are coincident chief tangents.

No. of 4-pointic parabolic tangents $= 2n\,(n-2)\,(11n-24).$

27. Order of parabolic curve $= 4n\,(n-2).$

Order of regulus formed by parabolic tangents
$$= 2n\,(n-2)\,(3n-4).$$

The parabolic curve and curve of contacts of an ϵ_4 tangent meet in
$$4n\,(n-2)\,(11n-24)$$
points, i.e., they *touch* in $2n\,(n-2)\,(11n-24)$ points.

28. Umbilici. No. is $= 2n\,(5n^2 - 14n + 11).$

29. No. of points at which the chief tangents being distinct are each of them 4-pointic, or, what is the same thing, No. of actual double points of curve ϵ_4,
$$= 5n\,(7n^2 - 28n + 30),$$
$n = 3$, No. is $15\,(63 - 84 + 30),\ = 135$, viz. this is the number of points of intersection of two of the 27 lines; or, what is the same thing, the number of triple tangent planes is $= 45.$

30. No. of parabolic tangents which have besides a 2-pointic contact is
$$= 2n\,(n-2)\,(n-4)\,(3n^2 + 5n - 24).$$

31. No. of double tangent planes such that line through points of contact is at one of these points 3-pointic
$$= n\,(n-2)\,(n-4)\,(n^3 + 3n^2 + 13n - 48).$$

32. No. of points where one chief tangent is 4-pointic, the other 3-pointic and (at another point of the surface) 2-pointic is
$$= n\,(n-4)\,(27n^3 - 13n^2 - 264n + 396).$$

33. No. of points where chief tangents being distinct are each of them at another point of the surface 2-pointic is
$$= n\,(n-4)\,(4n^5 - 4n^4 - 95n^3 + 99n^2 + 544n - 840).$$

34. The curve of contacts b_3 of an ϵ_{32} tangent has with the parabolic curve 2-pointic intersections only, and these are at the points for which the chief tangent is (at another point of the surface) 2-pointic.

35. The curve of contacts b_3 of an ϵ_{32} tangent has, with the curve of contacts of an ϵ_4 tangent, 2-pointic intersections at the contacts of an ϵ_5 tangent; and has also simple intersections with the same curve, 1° at the contacts b_4 of an ϵ_{42} tangent, 2° at the points where the chief tangents are ϵ_4 and ϵ_{32}.