763.

ON THE THEOREMS OF THE 2, 4, 8, AND 16 SQUARES.

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A SUM of 2 squares multiplied by a sum of 2 squares is a sum of 2 squares; a sum of 4 squares multiplied by a sum of 4 squares is a sum of 4 squares; a sum of 8 squares multiplied by a sum of 8 squares is a sum of 8 squares; but a sum of 16 squares multiplied by a sum of 16 squares is *not* a sum of 16 squares. These theorems were considered in the paper, Young, "On an extension of a theorem of Euler, with a determination of the limit beyond which it fails," *Trans. R. I. A.*, t. XXI. (1848), pp. 311—341; and the later history of the question is given in the paper by Mr S. Roberts, "On the Impossibility of the general Extension of Euler's Theorem &c.," *Quart. Math. Jour.* t. XVI. (1879), pp. 159—170; as regards the 16-question, it has been throughout assumed that there is only one type of synthematic arrangement (what this means will appear presently); but as regards this type, it is, I think, well shown that the signs cannot be determined. It will appear in the sequel, that there are in fact four types (the last three of them possibly equivalent) of synthematic arrangement; and for a complete proof, it is necessary to show in regard to each of these types that the signs cannot be determined. The existence of the four types has not (so far as I am aware) been hitherto noticed; and it hence follows, that no complete proof of the non-existence of the 16-square theorem has hitherto been given.

For the 2 squares the theorem is of course

$$(x_1^2+x_2^2)(y_1^2+y_2^2)=(x_1y_1+x_2y_2)^2+(x_1y_2-x_2y_1)^2.$$

For the 4 squares (for which the nature of the theorem is better seen) it is

$$\begin{aligned}(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)= \quad &(x_1y_1+x_2y_2+x_3y_3+x_4y_4)^2\\ &+(x_1y_2-x_2y_1+x_3y_4-x_4y_3)^2\\ &+(x_1y_3-x_3y_1-x_2y_4+x_4y_2)^2\\ &+(x_1y_4-x_4y_1+x_2y_3-x_3y_2)^2;\end{aligned}$$

or, as this may be written,

$$(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)-(x_1y_1+x_2y_2+x_3y_3+x_4y_4)^2$$
$$= (12+34)^2$$
$$+(13-24)^2$$
$$+(14+23)^2;$$

where 12 is used to denote $x_1y_2-x_2y_1$, &c., and the truth of the theorem depends on the identity $12.34-13.24+14.23=0$. Clearly, the first step for forming the equation is to arrange the duads in a synthematic form

12.34
13.24
14.23,

and then to determine the signs: such an arrangement exists in the case of 8, and the signs can be determined; it exists also in the case of 16, but the signs *cannot* be determined to satisfy all the necessary relations.

In the case of 8, we have the synthematic arrangement

12.34.56.78
13.24.57.68
14.23.58.67
15.26.37.48
16.25.38.47
17.28.35.46
18.27.36.45,

being the *only type* of synthematic arrangement. This is, in fact, important as regards the 16-question, and it will appear that the case is so; but in the 8-question, starting from this arrangement, we have to show that there exists an equation which, for convenience, I write as follows:

$$(x_1^2+\ldots+x_8^2)(y_1^2+\ldots+y_8^2)-(x_1y_1+\ldots+x_8y_8)^2$$
$$= (12+34+56+78)^2$$
$$+(13+24+57+68)^2$$
$$+(14+23+58+67)^2$$
$$+(15+26+37+48)^2$$
$$+(16+25+38+47)^2$$
$$+(17+28+35+46)^2$$
$$+(18+27+36+45)^2,$$

but in which it is to be understood that each duad is affected by a factor ± 1 which is to be determined; say the factor of 12 is ϵ_{12}, that of 34, ϵ_{34}; and so in other cases. It is however assumed that ϵ_{12}, ϵ_{34}, ϵ_{56}, ϵ_{78}; ϵ_{13}, ϵ_{14}, ϵ_{15}, ϵ_{16}, ϵ_{17}, ϵ_{18} are each $=+1$.

We have then on the right-hand side triads of terms such as, 2 into

$$\epsilon_{12}\epsilon_{34}\, 12\,.\,34 + \epsilon_{13}\epsilon_{24}\, 13\,.\,24 + \epsilon_{14}\epsilon_{23}\, 14\,.\,23,$$

which triad ought to vanish identically, as reducing itself to a multiple of

$$12\,.\,34 - \; 13\,.\,24 + 14\,.\,23\,;$$

viz. we ought to have

$$\epsilon_{12}\epsilon_{34} = - \; \epsilon_{13}\epsilon_{24} \; = \; \epsilon_{14}\epsilon_{23}\,;$$

or, *using now and henceforward when occasion requires*, 12, 34, &c. to denote ϵ_{12}, ϵ_{34}, &c. respectively, we have

$$12\,.\,34 = +k,$$
$$13\,.\,24 = -k,$$
$$14\,.\,23 = +k,$$

where k, $=\pm 1$, has to be determined (in the actual case we have $12=+1$, $34=+1$, $13=1$, $14=1$; and therefore the first equation gives $k=1$, and the other two then give $24=-1$, $23=+1$).

We have in this way triads of values corresponding to the different tetrads

1234
1256
1278
1357
1368
1458
1467
2358
2367
2457
2468
3456
3478
5678,

which can be formed with the several lines of the formula. Thus we have from the first line 1234, 1256, 1278; then from the second line (not 1324 which in the form 1234 has been taken already) 1357, 1368, ...; and finally from the last line 5678.

We might consider each line as giving 6 tetrads, but the tetrads would then be obtained 3 times over; the number of tetrads is thus $6\times 7\div 3$, $=14$ as above. And observe, that the systems of values for the coefficients $\epsilon=\pm 1$ are obtained directly from the tetrads, without the employment of any other formula.

We thus obtain the system of signs as follows:

12	$+1$	
13	$+1$	
14	$+1$	
15	$+1$	
16	$+1$	
17	$+1$	
18	$+1$	
23	$+1$	
24	-1	
25	$+1$	
26	-1	
27	$+1$	
28	-1	
34	$+1$	
35	a	$-\theta$
36	b	θ
37	$-a$	θ
38	$-b$	$-\theta$
45	c	θ
46	d	θ
47	$-d$	$-\theta$
48	$-c$	$-\theta$
56	$+1$	
57	a	$-\theta$
58	c	θ
67	d	θ
68	b	θ
78	$+1$	

viz. the original assumptions $12 = +1$, &c., and the tetrads 1234, 1256, 1278 give all the signs ± 1 up to $34 = +1$; from the tetrad 1357 we have

$$\begin{array}{llll} 13.57 & + & 1 & a, \\ 15.37 & - & 1 & a, \\ 17.35 & + & 1 & a, \end{array}$$

that is, $35 = a$, $37 = -a$, $57 = a$, where a, $= \pm 1$, is still undetermined; and similarly, the tetrads 1368, 1458, 1467 give the remaining signs b, c, d. The tetrad 2358 then gives

$$\begin{array}{llll} 23.58 & + & 1 & c, \\ 25.38 & - & 1 & -b, \\ 28.35 & + & -1 & a, \end{array}$$

that is, $-a = b = c$; and similarly the tetrads 2367, 2457, 2468 give $-a = b = d$, $-a = c = d$, $b = c = d$ respectively; the four tetrads thus give $-a = b = c = d$, say each of these $= \theta$. But retaining for the moment a, b, c, d, the tetrad 3456 then gives

$$\begin{array}{llll} 34.56 & + & 1 & 1, \\ 35.46 & - & a & d, \\ 36.45 & + & b & c, \end{array}$$

that is, $1 = -ad = bc$, and similarly the last two tetrads 3478 and 5678 give $1 = -ac = bd$ and $1 = -ab = cd$ respectively; substituting the values in terms of θ, the several equations give only $\theta^2 = 1$, that is, $\theta = \pm 1$ at pleasure; and the series of signs for the 8-formula, containing this one arbitrary sign $\theta = \pm 1$, is thus determined.

Passing to the case of 16, we have in like manner to form a synthematic arrangement of the numbers 1, 2, ..., 16 in 15 lines, each containing the 16 numbers in 8 duads (no duad twice repeated), and this containing all the 120 duads. And, using for the moment letters instead of numbers, the necessary condition is, that $ab\,.\,cd$ occurring in one line, $ac\,.\,bd$ must occur in another line, and $ad\,.\,bc$ in a third line. Observe that as well the order of the letters in a duad as the order of the duads is thus far immaterial; so that a line containing $bd\,.\,ca$ may be considered as containing $ac\,.\,bd$.

Considering any such combination $ab\,.\,cd$, the line which contains it may be taken to be the first line; and the line which contains $ac\,.\,bd$ may be taken to be the second line. And then writing 1, 2, 3, 4 in place of a, b, c, d respectively, the first line will contain $12\,.\,34$, and the second line will contain $13\,.\,24$. Let e be any other symbol occurring in the first line, say in the duad ef, and in the second line say in the duad eg; then g must occur in the first line in some duad gh, or the first line will contain $ef\,.\,gh$, and then the second line as containing eg will contain also fh, that is, it will contain $eg.fh$. And then writing 5, 6, 7, 8 in place of e, f, g, h respectively, the first line will contain 56.78 and the second line will contain 57.68. And continuing the like reasoning, it appears that the first line and the second line may be taken to be

1 2. 3 4. 5 6. 7 8. 9 10. 11 12. 13 14. 15 16,

and

1 3. 2 4. 5 7. 6 8. 9 11. 10 12. 13 15. 14 16,

respectively. There will then be a line containing 1 4 which may be taken for the third line, a line containing 1 5 which may be taken for the fourth line, and so on; viz. the successive lines may be taken to begin with 1 2, 1 3, 1 4, ..., 1 16 respectively.

Proceeding to form the synthematic arrangement, and starting with the first and second lines and first column as above, it appears that in each of the remaining lines there are three duads which occur of necessity, and putting these in the second, third, and fourth places (the order of the duads in any line being immaterial), it is seen that the second, third, and fourth columns can be filled up in one, and only one way; see the annexed first-half:

First-half common to all.

1	2	3	4	5	6	7	8
1	3	2	4	5	7	6	8
1	4	2	3	5	8	6	7
1	5	2	6	3	7	4	8
1	6	2	5	3	8	4	7
1	7	2	8	3	5	4	6
1	8	2	7	3	6	4	5
1	9	2	10	3	11	4	12
1	10	2	9	3	12	4	11
1	11	2	12	3	9	4	10
1	12	2	11	3	10	4	9
1	13	2	14	3	15	4	16
1	14	2	13	3	16	4	15
1	15	2	16	3	13	4	14
1	16	2	15	3	14	4	13

Four forms of second-half.

I.

9	10	11	12	13	14	15	16
9	11	10	12	13	15	14	16
9	12	10	11	13	16	14	15
9	13	10	14	11	15	12	16
9	14	10	13	11	16	12	15
9	15	10	16	11	13	12	14
9	16	10	15	11	14	12	13
5	13	6	14	7	15	8	16
5	14	6	13	7	16	8	15
5	15	6	16	7	13	8	14
5	16	6	15	7	14	8	13
5	9	6	10	7	11	8	12
5	10	6	9	7	12	8	11
5	11	6	12	7	9	8	10
5	12	6	11	7	10	8	9

II.

9	10	11	12	13	14	15	16
9	11	10	12	13	15	14	16
9	12	10	11	13	16	14	15
9	14	10	13	11	16	12	15
9	13	10	14	11	15	12	16
9	16	10	15	11	14	12	13
9	15	10	16	11	13	12	14
5	14	6	13	7	16	8	15
5	13	6	14	7	15	8	16
5	16	6	15	7	14	8	13
5	15	6	16	7	13	8	14
5	10	6	9	7	12	8	11
5	9	6	10	7	11	8	12
5	12	6	11	7	10	8	9
5	11	6	12	7	9	8	10

III.

9	10	11	12	13	14	15	16
9	11	10	12	13	15	14	16
9	12	10	11	13	16	14	15
9	15	10	16	11	13	12	14
9	16	10	15	11	14	12	13
9	13	10	14	11	15	12	16
9	14	10	13	11	16	12	15
5	15	6	16	7	13	8	14
5	16	6	15	7	14	8	13
5	13	6	14	7	15	8	16
5	14	6	13	7	16	8	15
5	11	6	12	7	9	8	10
5	12	6	11	7	10	8	9
5	9	6	10	7	11	8	12
5	10	6	9	7	12	8	11

IV.

9	10	11	12	13	14	15	16
9	11	10	12	13	15	14	16
9	12	10	11	13	16	14	15
9	16	10	15	11	14	12	13
9	15	10	16	11	13	12	14
9	14	10	13	11	16	12	15
9	13	10	14	11	15	12	16
5	16	6	15	7	14	8	13
5	15	6	16	7	13	8	14
5	14	6	13	7	16	8	15
5	13	6	14	7	15	8	16
5	12	6	11	7	10	8	9
5	11	6	12	7	9	8	10
5	10	6	9	7	12	8	11
5	9	6	10	7	11	8	12

And it is to be noticed that in this first-half the upper part, or first seven lines, give in fact the synthematic arrangement for the 8-question; so that (as remarked above) in this 8-question there is but one form of synthematic arrangement.

Proceeding to fill up the remaining columns, the duad 59 cannot be placed in any line which contains a 5 or a 9; that is, it must be placed in some one of the last 4 lines; and placing it successively in each of these, it appears that the columns can be filled up in one, and only one, way; we have thus the above "four forms of second-half," each of which, taken in conjunction with the common first-half, gives a synthematic arrangement of the 16 numbers.

Each of these synthematic arrangements may be converted into a square, the first line of which is formed with the numbers 1 to 16 in order, and the other fifteen lines of which are derived from the fifteen lines of the synthematic arrangement respectively: thus the line

1 2. 3 4. 5 6. 7 8. 9 10. 11 12. 13 14. 15 16

gives the second line of

1 2. 3 4. 5 6. 7 8. 9 10. 11 12. 13 14. 15 16,

2 1. 4 3. 6 5. 8 7. 10 9. 12 11. 14 13. 16 15,

and so in other cases. And conversely, by comparing with the first line of the square each of the other fifteen lines respectively, we have the fifteen lines of the synthematic arrangement; we thus obtain the four squares presently given. These squares are not required in the sequel, but they serve to put in a clearer light the construction of the synthematic arrangements; by converting in like manner into a square the formula p. 332 of Young's paper, it appears that his arrangement is in fact the first of the foregoing four arrangements. The squares are

I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11
7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7
11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5
13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
14	13	16	15	10	9	12	11	6	5	8	7	2	1	4	3
15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2
16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

II.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	14	13	16	15	10	9	12	11
6	5	8	7	2	1	4	3	13	14	15	16	9	10	11	12
7	8	5	6	3	4	1	2	16	15	14	13	12	11	10	9
8	7	6	5	4	3	2	1	15	16	13	14	11	12	9	10
9	10	11	12	14	13	16	15	1	2	3	4	6	5	8	7
10	9	12	11	13	14	15	16	2	1	4	3	5	6	7	8
11	12	9	10	16	15	14	13	3	4	1	2	8	7	6	5
12	11	10	9	15	16	13	14	4	3	2	1	7	8	5	6
13	14	15	16	10	9	12	11	6	5	8	7	1	2	3	4
14	13	16	15	9	10	11	12	5	6	7	8	2	1	4	3
15	16	13	14	12	11	10	9	8	7	6	5	3	4	1	2
16	15	14	13	11	12	9	10	7	8	5	6	4	3	2	1

III.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	15	16	13	14	11	12	9	10
6	5	8	7	2	1	4	3	16	15	14	13	12	11	10	9
7	8	5	6	3	4	1	2	13	14	15	16	9	10	11	12
8	7	6	5	4	3	2	1	14	13	16	15	10	9	12	11
9	10	11	12	15	16	13	14	1	2	3	4	7	8	5	6
10	9	12	11	16	15	14	13	2	1	4	3	8	7	6	5
11	12	9	10	13	14	15	16	3	4	1	2	5	6	7	8
12	11	10	9	14	13	16	15	4	3	2	1	6	5	8	7
13	14	15	16	11	12	9	10	7	8	5	6	1	2	3	4
14	13	16	15	12	11	10	9	8	7	6	5	2	1	4	3
15	16	13	14	9	10	11	12	5	6	7	8	3	4	1	2
16	15	14	13	10	9	12	11	6	5	8	7	4	3	2	1

IV.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	16	15	14	13	12	11	10	9
6	5	8	7	2	1	4	3	15	16	13	14	11	12	9	10
7	8	5	6	3	4	1	2	14	13	16	15	10	9	12	11
8	7	6	5	4	3	2	1	13	14	15	16	9	10	11	12
9	10	11	12	16	15	14	13	1	2	3	4	8	7	6	5
10	9	12	11	15	16	13	14	2	1	4	3	7	8	5	6
11	12	9	10	14	13	16	15	3	4	1	2	6	5	8	7
12	11	10	9	13	14	15	16	4	3	2	1	5	6	7	8
13	14	15	16	12	11	10	9	8	7	6	5	1	2	3	4
14	13	16	15	11	12	9	10	7	8	5	6	2	1	4	3
15	16	13	14	10	9	12	11	6	5	8	7	3	4	1	2
16	15	14	13	9	10	11	12	5	6	7	8	4	3	2	1

The foregoing investigation of the synthematic arrangements is exhaustive: it thereby appears that there are at most four types, viz. that every synthematic arrangement is of the type of one or other of the four arrangements above written down. The real nature of these is perhaps more clearly seen by means of the corresponding squares; and it will be observed, that there is in the first square a repetition of parts without transposition, which does not occur in the other three squares; this seems to suggest, that while the first square (and therefore the first synthematic arrangement) is really of a distinct type, the other three squares (or synthematic arrangements) *may possibly* belong to one and the same type. If this were so, it would be sufficient to prove the 16-theorem (viz. the non-existence of the 16-square formula) for the first and for any one of the other three synthematic arrangements; but I provisionally assume that the four types are really distinct, and propose therefore to prove the theorem for each of the four arrangements separately.

The process is the same as for the 8-theorem; we require the tetrads 1234, &c., contained in the synthematic arrangements. In any one of these, each line gives $\frac{1}{2}8.7, =28$ tetrads, and the 15 lines give therefore $15.28, =420$ tetrads: but we thus obtain each tetrad 3 times, or the number of the tetrads is $420 \div 3, =140$.

For the four arrangements respectively, these are as follows: the word "same" means same as in column I.

I.				II.				III.				IV.			
1	2	3	4	same				same				same			
		5	6												
		7	8												
		9	10												
		11	12												
		13	14												
		15	16												
1	3	5	7												
		6	8												
		9	11												
		10	12												
		13	15												
		14	16												
1	4	5	8												
		6	7												
		9	12												
		10	11												
		13	16												
		14	15												
1	5	9	13	1	5	9	14	1	5	9	15	1	5	9	16
		10	14			10	13			10	16			10	15
		11	15			11	16			11	13			11	14
		12	16			12	15			12	14			12	13
1	6	9	14	1	6	9	13	1	6	9	16	1	6	9	15
		10	13			10	14			10	15			10	16
		11	16			11	15			11	14			11	13
		12	15			12	16			12	13			12	14
1	7	9	15	1	7	9	16	1	7	9	13	1	7	9	14
		10	16			10	15			10	14			10	13
		11	13			11	14			11	15			11	16
		12	14			12	13			12	16			12	15
1	8	9	16	1	8	9	15	1	8	9	14	1	8	9	13
		10	15			10	16			10	13			10	14
		11	14			11	13			11	16			11	15
		12	13			12	14			12	15			12	16

I.	II.	III.	IV.
2 3 5 8 6 7 9 12 10 11 13 16 14 15	same	same	same
2 4 5 7 6 8 9 11 10 12 13 15 14 16	same	same	same
2 5 9 14 10 13 11 16 12 15	2 5 9 13 10 14 11 15 12 16	2 5 9 16 10 15 11 14 12 13	2 5 9 15 10 16 11 13 12 14
2 6 9 13 10 14 11 15 12 16	2 6 9 14 10 13 11 16 12 15	2 6 9 15 10 16 11 13 12 14	2 6 9 16 10 15 11 14 12 13
2 7 9 16 10 15 11 14 12 13	2 7 9 15 10 16 11 13 12 14	2 7 9 14 10 13 11 16 12 15	2 7 9 13 10 14 11 15 12 16
2 8 9 15 10 16 11 13 12 14	2 8 9 16 10 15 11 14 12 13	2 8 9 13 10 14 11 15 12 16	2 8 9 14 10 13 11 16 12 15
3 4 5 6 7 8 9 10 11 12 13 14 15 16	same	same	same

I.				II.				III.				IV.			
3	5	9	15	3	5	9	16	3	5	9	13	3	5	9	14
		10	16			10	15			10	14			10	13
		11	13			11	14			11	15			11	16
		12	14			12	13			12	16			12	15
3	6	9	16	3	6	9	15	3	6	9	14	3	6	9	13
		10	15			10	16			10	13			10	14
		11	14			11	13			11	16			11	15
		12	13			12	14			12	15			12	16
3	7	9	13	3	7	9	14	3	7	9	15	3	7	9	16
		10	14			10	13			10	16			10	15
		11	15			11	16			11	13			11	14
		12	16			12	15			12	14			12	13
3	8	9	14	3	8	9	13	3	8	9	16	3	8	9	15
		10	13			10	14			10	15			10	16
		11	16			11	15			11	14			11	13
		12	15			12	16			12	13			12	14
4	5	9	16	4	5	9	15	4	5	9	14	4	5	9	13
		10	15			10	16			10	13			10	14
		11	14			11	13			11	16			11	15
		12	13			12	14			12	15			12	16
4	6	9	15	4	6	9	16	4	6	9	13	4	6	9	14
		10	16			10	15			10	14			10	13
		11	13			11	14			11	15			11	16
		12	14			12	13			12	16			12	15
4	7	9	14	4	7	9	13	4	7	9	16	4	7	9	15
		10	13			10	14			10	15			10	16
		11	16			11	15			11	14			11	13
		12	15			12	16			12	13			12	14
4	8	9	13	4	8	9	14	4	8	9	15	4	8	9	16
		10	14			10	13			10	16			10	15
		11	15			11	16			11	13			11	14
		12	16			12	15			12	14			12	13
5	6	7	8	same				same				same			
		9	10												
		11	12												
		13	14												
		15	16												

I.	II.	III.	IV.
5 7 9 11 10 12 13 15 14 16	same	same	same
5 8 9 12 10 11 13 16 14 15			
6 7 9 12 10 11 13 16 14 15			
6 8 9 11 10 12 13 15 14 16			
7 8 9 10 11 12 13 14 15 16			
9 10 11 12 13 14 15 16			
9 11 13 15 14 16			
9 12 13 16 14 15			
10 11 13 16 14 15			
10 12 13 15 14 16			
11 12 13 14 15 16			
13 14 15 16			

As regards the signs, observe that the first line may always be written

$$ab + cd + ef + \&c.,$$

with the signs all of them $+$; and then writing $a, b, c, \ldots = 1, 2, 3, \ldots, 16$ respectively, the first line will be

$$1\ 2 + 3\ 4 + 5\ 6 + 7\ 8 + 9\ 10 + 11\ 12 + 13\ 14 + 15\ 16,$$

with the signs all of them $+$; that is, we may assume ϵ_{12}, ϵ_{34}, &c., or say

$$1\ 2,\ 3\ 4,\ 5\ 6,\ 7\ 8,\ 9\ 10,\ 11\ 12,\ 13\ 14,\ 15\ 16,$$

all of them $= +1$. And in the other lines, the signs of all the terms of any line may be reversed at pleasure, that is, we may assume ϵ_{13}, ϵ_{14}, &c., or say 1 3, 1 4, 1 5, 1 6, 1 7, 1 8, 1 9, 1 10, 1 11, 1 12, 1 13, 1 14, 1 15, 1 16, all of them $= +1$.

Making these assumptions, then for any one of the synthematic arrangements the several tetrads give as before relations between the signs; among these are included the results already obtained for the 8-question, and taking as before

$$-a = b = c = d = \theta,$$

we have the signs of the several terms belonging to the 8-question given as $= \pm 1$ or $\pm\theta$ as before. The remaining tetrads up to 1 8 12 13 then serve to express all the remaining signs in terms of the as yet undetermined signs $e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z, \alpha, \beta$, for instance

$$\begin{array}{l} 1\ \ 3\,.\,9\ 11 + \ 1\ \ e, \\ 1\ \ 9\,.\,3\ 11 - \ 1\ \ e, \\ 1\ 11\,.\,3\ \ 9 + \ 1\ \ e, \end{array}$$

that is, $3\ 9 = e$, $3\ 11 = -e$, $9\ 11 = e$; and then the tetrads up to 2 8 9 15 serve to express these signs in terms of the undetermined signs $\lambda, \mu, \nu, \rho, \sigma, \tau$; for instance

$$\begin{array}{l} 2\ \ 3\,.\,9\ 12 + \ 1\ \ i, \\ 2\ \ 9\,.\,3\ 12 - \ 1 - f, \\ 2\ 12\,.\,3\ \ 9 + -1\ \ e, \end{array}$$

that is, $-e = f = i$; and in like manner 2 3 10 11, 2 4 9 11 and 2 4 10 12 give respectively $-e = f = j$, $-e = i = j$, $f = i = j$; that is, we have $-e = f = i = j$, $= \lambda$ suppose. And in this way we have, for each of the four synthematic arrangements the signs of all the terms expressed in terms of the undetermined signs $\theta, \lambda, \mu, \nu, \rho, \sigma, \tau$, as shown in the following table; where observe that the results apply to the four synthematic arrangements *separately*, viz. the e, f, g, &c., and the $\theta, \lambda, \mu, \nu, \rho, \sigma, \tau$ in each column are altogether independent of the like symbols in the other three columns.

Signs for the four synthematic arrangements:

		I.		II.	III.	IV.
1	2	+ 1		same	same	same
	3	+ 1				
	4	+ 1				
	5	+ 1				
	6	+ 1				
	7	+ 1				
	8	+ 1				
	9	+ 1				
	10	+ 1				
	11	+ 1				
	12	+ 1				
	13	+ 1				
	14	+ 1				
	15	+ 1				
	16	+ 1				
2	3	+ 1		same	same	same
	4	− 1				
	5	+ 1				
	6	− 1				
	7	+ 1				
	8	− 1				
	9	+ 1				
	10	− 1				
	11	+ 1				
	12	− 1				
	13	+ 1				
	14	− 1				
	15	+ 1				
	16	− 1				
3	4	+ 1		same	same	same
	5	$-\theta$				
	6	θ				
	7	θ				
	8	$-\theta$				
	9	e	$-\lambda$			
	10	f	λ			
	11	$-e$	λ			
	12	$-f$	$-\lambda$			
	13	g	$-\mu$			
	14	h	μ			
	15	$-g$	μ			
	16	$-h$	$-\mu$			

		I.		II.		III.		IV.	
4	5	θ		same		same		same	
	6	θ							
	7	$-\theta$							
	8	$-\theta$							
	9	i	λ						
	10	j	λ						
	11	$-j$	$-\lambda$						
	12	$-i$	$-\lambda$						
	13	k	μ						
	14	l	μ						
	15	$-l$	$-\mu$						
	16	$-k$	$-\mu$						
5	6	$+1$		$+1$		$+1$		1	
	7	$-\theta$		$-\theta$		$-\theta$		$-\theta$	
	8	θ		θ		θ		θ	
	9	m	$-\nu$	m	ν	m	$-\nu$	m	ν
	10	n	ν	n	ν	n	ν	n	ν
	11	o	$-\rho$	o	ρ	o	$-\rho$	o	ρ
	12	p	ρ	p	ρ	p	ρ	p	ρ
	13	$-m$	ν	$-n$	$-\nu$	$-o$	ρ	$-p$	$-\rho$
	14	$-n$	$-\nu$	$-m$	$-\nu$	$-p$	$-\rho$	$-o$	$-\rho$
	15	$-o$	ρ	$-p$	$-\rho$	$-m$	ν	$-n$	$-\nu$
	16	$-p$	$-\rho$	$-o$	$-\rho$	$-n$	$-\nu$	$-m$	$-\nu$
6	7	θ		θ		θ		θ	
	8	θ		θ		θ		θ	
	9	q	ν	q	ν	q	ν	q	ν
	10	r	ν	r	$-\nu$	r	ν	r	$-\nu$
	11	s	ρ	s	ρ	s	ρ	s	ρ
	12	t	ρ	t	$-\rho$	t	ρ	t	$-\rho$
	13	$-r$	$-\nu$	$-q$	$-\nu$	$-t$	$-\rho$	$-s$	$-\rho$
	14	$-q$	$-\nu$	$-r$	ν	$-s$	$-\rho$	$-t$	ρ
	15	$-t$	$-\rho$	$-s$	$-\rho$	$-r$	$-\nu$	$-q$	$-\nu$
	16	$-s$	$-\rho$	$-t$	ρ	$-q$	$-\nu$	$-r$	ν
7	8	$+1$		$+1$		$+1$		$+1$	
	9	u	$-\sigma$	u	$-\sigma$	u	$-\sigma$	u	σ
	10	v	σ	v	σ	v	σ	v	σ
	11	w	$-\tau$	w	τ	w	$-\tau$	w	τ
	12	x	τ	x	τ	x	τ	x	τ
	13	$-w$	τ	$-x$	$-\tau$	$-u$	σ	$-v$	$-\sigma$
	14	$-x$	$-\tau$	$-w$	$-\tau$	$-v$	$-\sigma$	$-u$	$-\sigma$
	15	$-u$	σ	$-v$	$-\sigma$	$-w$	τ	$-x$	$-\tau$
	16	$-v$	$-\sigma$	$-u$	$-\sigma$	$-x$	$-\tau$	$-w$	$-\tau$

		I.		II.		III.		IV.	
8	9	y	σ	y	σ	y	σ	y	σ
	10	z	σ	z	$-\sigma$	z	σ	z	$-\sigma$
	11	α	τ	α	τ	α	τ	α	τ
	12	β	τ	β	$-\tau$	β	τ	β	$-\tau$
	13	$-\beta$	$-\tau$	$-\alpha$	$-\tau$	$-z$	$-\sigma$	$-y$	$-\sigma$
	14	$-\alpha$	$-\tau$	$-\beta$	τ	$-y$	$-\sigma$	$-z$	σ
	15	$-z$	$-\sigma$	$-y$	$-\sigma$	$-\beta$	$-\tau$	$-\alpha$	$-\tau$
	16	$-y$	$-\sigma$	$-z$	σ	$-\alpha$	$-\tau$	$-\beta$	τ
9	10	$+1$		$+1$		$+1$		$+1$	
	11	e	$-\lambda$	e	$-\lambda$	e	$-\lambda$	e	$-\lambda$
	12	i	λ	i	λ	i	λ	i	λ
	13	m	$-\nu$	q	ν	u	$-\sigma$	y	σ
	14	q	ν	m	ν	y	σ	u	σ
	15	u	$-\sigma$	y	σ	m	$-\nu$	q	ν
	16	y	σ	u	σ	q	ν	m	ν
10	11	j	λ	j	λ	j	λ	j	λ
	12	f	λ	f	λ	f	λ	f	λ
	13	r	ν	n	ν	z	σ	v	σ
	14	n	ν	r	$-\nu$	v	σ	z	$-\sigma$
	15	z	σ	v	σ	r	ν	n	ν
	16	v	σ	z	$-\sigma$	n	ν	r	$-\nu$
11	12	$+1$		$+1$		1		$+1$	
	13	w	$-\tau$	α	τ	o	$-\rho$	s	ρ
	14	α	τ	w	τ	s	ρ	o	ρ
	15	o	$-\rho$	s	ρ	w	$-\tau$	α	τ
	16	s	ρ	o	ρ	α	τ	w	τ
12	13	β	τ	x	τ	t	ρ	p	ρ
	14	x	τ	β	$-\tau$	p	ρ	t	$-\rho$
	15	t	ρ	p	ρ	β	τ	x	τ
	16	p	ρ	t	$-\rho$	x	τ	β	$-\tau$
13	14	$+1$		$+1$		$+1$		$+1$	
	15	g	$-\mu$	g	$-\mu$	g	$-\mu$	g	$-\mu$
	16	k	μ	k	μ	k	μ	k	μ
14	15	l	μ	l	μ	l	μ	l	μ
	16	h	μ	h	μ	h	μ	h	μ
15	16	$+1$		$+1$		$+1$		$+1$	

We have now for the four arrangements respectively, by means of hitherto unused tetrads, the following determinations of sign: these being in each case inconsistent with each other.

First arrangement.

$$\begin{array}{rrrrl} 3 & 5 & 9 & 15 & +-\theta\,.-\sigma \\ 3 & 9 & 5 & 15 & --\lambda\,.\ \ \rho \\ 3 & 15 & 5 & 9 & +\ \ \mu\,.-\nu \end{array} \qquad \text{that is,} \quad \theta\sigma = \lambda\rho = -\mu\nu,$$

$$\begin{array}{rrrrl} 3 & 5 & 10 & 16 & +-\theta\,.\ \ \sigma \\ 3 & 10 & 5 & 16 & -\ \ \lambda\,.-\rho \\ 3 & 16 & 5 & 10 & +-\mu\,.\ \ \nu \end{array} \qquad -\theta\sigma = \lambda\rho = -\mu\nu,$$

$$\begin{array}{rrrrl} 3 & 5 & 11 & 13 & +-\theta\,.-\tau \\ 3 & 11 & 5 & 13 & -\ \ \lambda\,.\ \ \nu \\ 3 & 13 & 5 & 11 & +-\mu\,.-\rho \end{array} \qquad \theta\tau = -\lambda\nu = \mu\rho,$$

$$\begin{array}{rrrrl} 3 & 5 & 12 & 14 & +-\theta\,.\ \ \tau \\ 3 & 12 & 5 & 14 & --\lambda\,.-\nu \\ 3 & 14 & 5 & 12 & +\ \ \mu\,.\ \ \rho \end{array} \qquad -\theta\tau = -\lambda\nu = \mu\rho.$$

Second arrangement.

$$\begin{array}{rrrrl} 3 & 5 & 9 & 16 & +-\theta\,.\ \ \sigma \\ 3 & 9 & 5 & 16 & --\lambda\,.-\rho \\ 3 & 16 & 5 & 9 & +-\mu\,.\ \ \nu \end{array} \qquad \theta\sigma = \lambda\rho = \mu\nu,$$

$$\begin{array}{rrrrl} 3 & 5 & 10 & 15 & +-\theta\,.\ \ \sigma \\ 3 & 10 & 5 & 15 & -\ \ \lambda\,.-\rho \\ 3 & 15 & 5 & 10 & +\ \ \mu\,.\ \ \nu \end{array} \qquad -\theta\sigma = \lambda\rho = \mu\nu,$$

$$\begin{array}{rrrrl} 3 & 5 & 11 & 14 & +-\theta\,.\ \ \tau \\ 3 & 11 & 5 & 14 & -\ \ \lambda\,.-\nu \\ 3 & 14 & 5 & 11 & +\ \ \mu\,.\ \ \rho \end{array} \qquad -\theta\tau = \lambda\nu = \mu\rho,$$

$$\begin{array}{rrrrl} 3 & 5 & 12 & 13 & +-\theta\,.\ \ \tau \\ 3 & 12 & 5 & 13 & --\lambda\,.-\nu \\ 3 & 13 & 5 & 12 & +-\mu\,.\ \ \rho \end{array} \qquad \theta\tau = \lambda\nu = \mu\rho.$$

Third arrangement.

$$\begin{array}{llll} 3\ \ 5\ \ 9\ 13 & +-\theta\,.-\sigma & & \\ 3\ \ 9\ \ 5\ 13 & --\lambda\,.\ \ \rho & & \theta\sigma=-\lambda\rho=\ \ \mu\nu, \\ 3\ 13\ \ 5\ \ 9 & +-\mu\,.-\nu & & \end{array}$$

$$\begin{array}{llll} 3\ \ 5\ 10\ 14 & +-\theta\,.\ \ \sigma & & \\ 3\ 10\ \ 5\ 14 & -\ \ \lambda\,.-\rho & & \theta\sigma=-\lambda\rho=-\mu\nu, \\ 3\ 14\ \ 5\ 10 & +\ \ \mu\,.\ \ \nu & & \end{array}$$

$$\begin{array}{llll} 3\ \ 5\ 11\ 15 & +-\theta\,.-\tau & & \\ 3\ 11\ \ 5\ 15 & -\ \ \lambda\,.\ \ \nu & & \theta\tau=-\lambda\nu=-\mu\rho, \\ 3\ 15\ \ 5\ 11 & +\ \ \mu\,.-\rho & & \end{array}$$

$$\begin{array}{llll} 3\ \ 5\ 12\ 16 & +-\theta\,.\ \ \tau & & \\ 3\ 12\ \ 5\ 16 & --\lambda\,.-\nu & & \theta\tau=-\lambda\nu=\ \ \mu\rho. \\ 3\ 16\ \ 5\ 12 & +-\mu\,.\ \ \rho & & \end{array}$$

Fourth arrangement.

$$\begin{array}{llll} 3\ \ 5\ \ 9\ 14 & +-\theta\,.\ \ \sigma & & \\ 3\ \ 9\ \ 5\ 14 & --\lambda\,.-\rho & & \theta\sigma=\ \ \lambda\rho=-\mu\nu, \\ 3\ 14\ \ 5\ \ 9 & +\ \ \mu\,.\ \ \nu & & \end{array}$$

$$\begin{array}{llll} 3\ \ 5\ 10\ 13 & +-\theta\,.\ \ \sigma & & \\ 3\ 10\ \ 5\ 13 & -\ \ \lambda\,.-\rho & & \theta\sigma=-\lambda\rho=\ \ \mu\nu, \\ 3\ 13\ \ 5\ 10 & +-\mu\,.\ \ \nu & & \end{array}$$

$$\begin{array}{llll} 3\ \ 5\ 11\ 16 & +-\theta\,.\ \ \tau & & \\ 3\ 11\ \ 5\ 16 & -\ \ \lambda\,.-\nu & & \theta\tau=-\lambda\nu=\ \ \mu\rho, \\ 3\ 16\ \ 5\ 11 & +-\mu\,.\ \ \rho & & \end{array}$$

$$\begin{array}{llll} 3\ \ 5\ 12\ 15 & +-\theta\,.-\tau & & \\ 3\ 12\ \ 5\ 15 & --\lambda\,.-\nu & & \theta\tau=-\lambda\nu=-\mu\rho. \\ 3\ 15\ \ 5\ 12 & +-\mu\,.\ \ \rho & & \end{array}$$

And it hence finally appears, that we cannot, in any one of the four arrangements, determine the signs so as to give rise to a 16-square theorem; that is, the product of a sum of 16 squares into a sum of 16 squares cannot be made equal to a sum of 16 squares.