953.

ON THE NINE-POINTS CIRCLE.

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IF from the angles A, B, C of a triangle we draw tangents to a conic Ω, meeting the opposite sides in the points α, α'; β, β'; γ, γ' respectively, then it is known that these six points lie in a conic. In particular, if the conic Ω reduce itself to a point-pair OO', then we have the theorem that, if from the angles A, B, C, we draw to the point O lines meeting the opposite sides in the points α, β, γ respectively; and to the point O' lines meeting the opposite sides in the points α', β', γ' respectively, then the six points α, α'; β, β'; γ, γ' lie in a conic. We may enquire the conditions under which this conic becomes a circle. It may be remarked that one of the points say O' remains arbitrary: for if through the points α', β', γ', we draw a conic (or in particular a circle) meeting the three sides respectively in the remaining points α, β, γ, then (by a converse of the general theorem) the lines $A\alpha$, $B\beta$, $C\gamma$ will meet in a point O.

Using trilinear coordinates (x, y, z) and writing $x : y : z = a : b : c$ for the point O, and $x : y : z = a' : b' : c'$ for the point O', it is at once seen that the equation of the conic through the six points is

$$aa'x^2 + bb'y^2 + cc'z^2 - (bc' + b'c)\,yz - (ca' + c'a)\,zx - (ab' + a'b)\,xy = 0\,;$$

in fact, writing herein successively $x = 0$, $y = 0$, $z = 0$, we see that the equation is satisfied by $x = 0$, $(by - cz)(b'y - c'z) = 0$; by $y = 0$, $(cz - ax)(c'z - a'x) = 0$; and by $z = 0$, $(ax - by)(a'x - b'y) = 0$ respectively. And it is to be observed that the equation may also be written

$$(aa'x + bb'y + cc'z)(x + y + z) - (b + c)(b' + c')\,yz - (c + a)(c' + a')\,zx - (a + b)(a' + b')\,xy = 0.$$

Suppose now that x, y, z represent areal coordinates, viz. that (x, y, z) are proportional to the perpendicular distances of the point from the sides, each divided by the perpendicular distance of the opposite angle from the same side; or, what is the same thing, coordinates such that the equation of the line infinity is $x+y+z=0$. Then if A, B, C denote the angles of the triangle, the general equation of a circle is

$$(yz\sin^2 A + zx\sin^2 B + xy\sin^2 C) + (\lambda x + \mu y + \nu z)(x+y+z) = 0,$$

where λ, μ, ν are arbitrary coefficients.

Hence, putting this

$$= \Theta\{-(b+c)(b'+c')\,yz - (c+a)(c'+a')\,zx$$
$$-(a+b)(a'+b')\,xy + (aa'x + bb'y + cc'z)(x+y+z)\},$$

we must have

$$\Theta(b+c)(b'+c') = -\sin^2 A,$$
$$\Theta(c+a)(c'+a') = -\sin^2 B,$$
$$\Theta(a+b)(a'+b') = -\sin^2 C;$$

and then

$$\Theta aa' = \lambda, \quad \Theta bb' = \mu, \quad \Theta cc' = \nu,$$

which last equations determine the values of λ, μ, ν.

Taking a', b', c' at pleasure, we have

$$2a = \frac{1}{\Theta}\left(\ \frac{\sin^2 A}{b'+c'} - \frac{\sin^2 B}{c'+a'} - \frac{\sin^2 C}{a'+b'}\right),$$

$$2b = \frac{1}{\Theta}\left(-\frac{\sin^2 A}{b'+c'} + \frac{\sin^2 B}{c'+a'} - \frac{\sin^2 C}{a'+b'}\right),$$

$$2c = \frac{1}{\Theta}\left(-\frac{\sin^2 A}{b'+c'} - \frac{\sin^2 B}{c'+a'} + \frac{\sin^2 C}{a'+b'}\right),$$

viz. a, b, c having these values, the conic through the six points α, β, γ, α', β', γ' is the circle having for its equation

$$yz\sin^2 A + zx\sin^2 B + xy\sin^2 C + \Theta(aa'x + bb'y + cc'z)(x+y+z) = 0;$$

and we may obviously without loss of generality give to Θ any specific value, say $\Theta = 1$.

If $a' = b' = c'$, $=1$, then we have

$$-4a = \frac{1}{\Theta}(-\sin^2 A + \sin^2 B + \sin^2 C),$$

$$-4b = \frac{1}{\Theta}(\ \sin^2 A - \sin^2 B + \sin^2 C),$$

$$-4c = \frac{1}{\Theta}(\ \sin^2 A + \sin^2 B - \sin^2 C),$$

or writing for convenience $\Theta = -\tfrac{1}{2}$, the values of a, b, c are

$$\tfrac{1}{2}(-\sin^2 A + \sin^2 B + \sin^2 C), \quad \tfrac{1}{2}(\sin^2 A - \sin^2 B + \sin^2 C), \quad \tfrac{1}{2}(\sin^2 A + \sin^2 B - \sin^2 C)$$

respectively. But we have

$$A + B + C = \pi,$$

and thence

$$\begin{aligned} & \sin^2 A + \sin^2 B - \sin^2 C, \\ = & \sin^2 A + \sin^2 B - \sin^2 (A + B) \\ = & 2 \sin A \sin B (\sin A \sin B - \cos A \cos B), \\ = & -2 \sin A \sin B \cos (A + B), \\ = & 2 \sin A \sin B \cos C, \end{aligned}$$

and we thus have

$$a,\ b,\ c = \sin B \sin C \cos A,\ \ \sin C \sin A \cos B,\ \ \sin A \sin B \cos C,$$

(or, what is the same thing, $a : b : c = \cot A : \cot B : \cot C$), and the equation of the circle is

$$yz \sin^2 A + zx \sin^2 B + xy \sin^2 C$$
$$- \tfrac{1}{2} (x \sin B \sin C \cos A + y \sin C \sin A \cos B + z \sin A \sin B \cos C) (x + y + z) = 0.$$

We thus have $x : y : z = 1 : 1 : 1$ for the point O', and $x : y : z = \cot A : \cot B : \cot C$ for the point O; viz. O' is the point of intersection of the lines from the angles to the mid-points of the opposite sides respectively; and O is the point of intersection of the perpendiculars from the angles on the opposite sides respectively: and the foregoing equation is consequently that of the Nine-points Circle.