963.

NOTE ON DR MUIR'S PAPER, "A PROBLEM OF SYLVESTER'S IN ELIMINATION."

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I IN part reproduce this very interesting paper for the sake of a remark which appears to me important. I write (a, b, c, f, g, h) in place of Muir's (A, B, C, A', B', C'), and take as usual (A, B, C, F, G, H) and K to denote

$$(bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch)$$

and the discriminant $abc-af^2-bg^2-ch^2+2fgh$

I then write

$$\begin{aligned} U &= bz^2-2fyz+cy^2, & P &= fx^2+ayz-hzx-gxy, & L &= bcx^2+afyz-bgzx-chxy,\\ V &= cx^2-2gzx+az^2, & Q &= gy^2-hyz+bzx-fxy, & M &= cay^2-afyz+bgzx-chxy,\\ W &= ay^2-2hxy+bx^2, & R &= hz^2-gyz-fzx+cxy, & N &= abz^2-afyz-bgzx+chxy. \end{aligned}$$

The equations $U=0$, $V=0$, $W=0$, imply $P=0$, $Q=0$, $R=0$, but observe that P, Q, R are not the sums of mere numerical multiples of U, V, W; we, in fact, have identically

$$\begin{aligned} 2yzP &= -x^2U+y^2V+z^2W,\\ 2zxQ &= x^2U-y^2V+z^2W,\\ 2xyR &= x^2U+y^2V-z^2W. \end{aligned}$$

If then $U=0$, $V=0$, $W=0$, we have also $P=0$, $Q=0$, $R=0$, and we can from the six equations dialytically eliminate x^2, y^2, z^2, yz, zx, xy, thus obtaining a result, Determinant $=0$, which is $K^2=0$; this is, in fact, Sylvester's process for the elimination.

But L, M, N are sums of mere numerical multiples of U, V, W, viz. we have

$$\begin{aligned} 2L &= -aU + bV + cW, \\ 2M &= aU - bV + cW, \\ 2N &= aU + bV - cW, \end{aligned}$$

so that the original equations $U=0$, $V=0$, $W=0$ are equivalent to and may be replaced by $L=0$, $M=0$, $N=0$.

Muir shows that we have identically

$$\begin{aligned} L - fP &= x\,(Ax + Hy + Gz), \\ M - gQ &= y\,(Hx + By + Fz), \\ N - hR &= z\,(Gx + Fy + Cz), \end{aligned}$$

where observe that the first of these equations is

$$\left.\begin{aligned} &(fx^2 - ayz)\,(bz^2 - 2fyz + cy^2) \\ -&(fy^2 - byz)\,(cx^2 - 2gzx + az^2) \\ -&(fz^2 - cyz)\,(ay^2 - 2hxy + bx^2) \end{aligned}\right\} = 2xyz\,(Ax + Hy + Gz);$$

and similarly for the second and third equations.

He thence infers that the elimination may be performed by eliminating x, y, z from the equations

$$\begin{aligned} Ax + Hy + Gz &= 0, \\ Hx + By + Fz &= 0, \\ Gx + Fy + Cz &= 0, \end{aligned}$$

viz. that the result is

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0,$$

that is, $K^2 = 0$ as before.

The natural inference is that K being $=0$, the three linear equations in (x, y, z) are equivalent to two equations giving for the ratios $x : y : z$ rational values which should satisfy the original equations $U=0$, $V=0$, $W=0$: the fact is that there are no such values, but that, K being $=0$, the three equations are equivalent to a single equation: for observe that, combining for instance the first and second equations, these will be equivalent to each other if only

$$\frac{A}{H} = \frac{H}{B} = \frac{G}{F},$$

that is,

$$AB - H^2 = 0, \quad GH - AF = 0, \quad HF - BG = 0,$$

which are $cK=0$, $fK=0$, $gK=0$, all satisfied by $K=0$; and similarly for the first and third, and the second and third equations. It will be remembered that the true form of the result is not $K=0$ but $K^2=0$, and this seems to be an indication that the three equations should be, as they have been found to be, equivalent to a single equation.

The problem may be further illustrated as follows: instead of the original equations $U=0$, $V=0$, $W=0$, consider the like equations with the inverse coefficients (A, B, C, F, G, H), viz.

$$Bz^2 - 2Fyz + Cy^2 = 0,$$

$$Cx^2 - 2Gzx + Az^2 = 0,$$

$$Ay^2 - 2Hxy + Bx^2 = 0,$$

so that the result of the elimination should be

$$(ABC - AF^2 - BG^2 - CH^2 + 2FGH)^2 = 0.$$

Here considering in connexion with the triangle $x=0$, $y=0$, $z=0$ (say the vertices hereof are the points A, B, C) the conic

$$(a, b, c, f, g, h)(x, y, z)^2 = 0,$$

the first equation represents the pair of tangents from the point A to the conic, the second the pair of tangents from the point B to the conic, and the third the pair of tangents from the point C to the conic. The first and second pairs of tangents intersect in four points, and if one of the third pair of tangents passes through one of the four points, then it is at once seen that the conic must touch one of the sides $x=0$, $y=0$, $z=0$ of the triangle, viz. we must have $bc-f^2=0$, $ca-g^2=0$, or $ab-h^2=0$. But we have $a=BC-F^2$, &c., or writing

$$K_1 = ABC - AF^2 - BG^2 - CH^2 + 2FGH,$$

then these equations are $K_1A=0$, $K_1B=0$, $K_1C=0$, all satisfied by $K_1=0$. We may regard $K_1=0$ as the condition in order that the conic $(a, b, c, f, g, h)(x, y, z)^2=0$ may be a *point-pair*: the analytical reason for this is not clear, but we see at once that, if the conic be a point-pair, then the three pairs of tangents are the lines drawn from the points A, B, C respectively to the two points of the point-pair, so that the three pairs of tangents have in common these two points. Regarding $K_1=0$ as the condition in order to the existence of a single common point, and recollecting that the true result of the elimination is $K_1^2=0$, the form perhaps indicates what we have just seen is the case, that there are in fact two common points of intersection: but at any rate the foregoing geometrical considerations lead to $K_1=0$, as the condition for the coexistence of the three equations.

I remark in conclusion that I do not know that there is any general theory of the case where a result of elimination presents itself in the form $\Omega^2=0$, as distinguished from the ordinary form $\Omega=0$.