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NOTE ON LANDEN'S THEOREM.

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LANDEN'S theorem, as given in the paper "An Investigation of a General Theorem for finding the length of any Arc of any Conic Hyperbola by means of two Elliptic Arcs, with some other new and useful Theorems deduced therefrom," *Phil. Trans.*, t. LXV. (1775), pp. 283—289, is, as appears by the title, a theorem for finding the length of a hyperbolic arc in terms of the length of two elliptic arcs; this theorem being obtained by means of the following differential identity, viz., if

$$t = gx\sqrt{\frac{m^2 - x^2}{m^2 - gx^2}},$$

where

$$g = \frac{m^2 - n^2}{n^2},$$

then

$$\sqrt{\frac{m^2 - gx^2}{m^2 - x^2}}\,dx = \left\{\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\frac{(m-n)^2 - t^2}{(m+n)^2 - t^2}} + \tfrac{1}{4}\sqrt{\frac{(m+n)^2 - t^2}{(m-n)^2 - t^2}}\right\} dt,$$

(this is exactly Landen's form, except that he of course writes $\dot{x}$, $\dot{t}$ in place of dx, dt respectively): viz., integrating each side, and interpreting geometrically in a very ingenious and elegant manner the three integrals which present themselves, he arrives at his theorem for the hyperbolic arc; but with this I am not now concerned.

Writing for greater convenience $m = 1$, $n = k'$, and therefore $g = k^2$, if as usual $k^2 + k'^2 = 1$, the transformation is

$$t = k^2 x\sqrt{\frac{1 - x^2}{1 - k^2x^2}},$$

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leading to

$$\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = \left\{\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\frac{(1-k')^2-t^2}{(1+k')^2-t^2}} + \sqrt{\frac{(1+k')^2-t^2}{(1-k')^2-t^2}}\right\} dt.$$

The form in which the transformation is usually employed (see my *Elliptic Functions*, pp. 177, 178) is

$$y = (1+k')\,x\sqrt{\frac{1-x^2}{1-k^2x^2}},$$

leading to

$$\frac{(1+k')\,dx}{\sqrt{1-x^2\,.\,1-k^2x^2}} = \frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}},$$

where

$$\lambda = \frac{1-k'}{1+k'}.$$

If, to identify the two forms, we write $y = \dfrac{t}{1-k'}$ and in the last equation introduce t in place of y, the last equation becomes

$$\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}} = \frac{dt}{\sqrt{\{(1-k')^2-t^2\}\{(1+k')^2-t^2\}}}.$$

Comparing with Landen's form, in order that the two may be identical, we must have

$$1-k^2x^2 = \left\{\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\frac{(1-k')^2-t^2}{(1+k')^2-t^2}} + \tfrac{1}{4}\sqrt{\frac{(1+k')^2-t^2}{(1-k')^2-t^2}}\right\}$$
$$\times\sqrt{(1-k')^2-t^2}\,\sqrt{(1+k')^2-t^2},$$

viz., this is

$$1-k^2x^2 = \tfrac{1}{4}\,\{\sqrt{(1-k')^2-t^2} + \sqrt{(1+k')^2-t^2}\}^2,$$

that is,

$$1-k^2x^2 = \tfrac{1}{2}\,[1+k'^2-t^2+\sqrt{\{(1-k')^2-t^2\}\{(1+k')^2-t^2\}}],$$

where the function under the radical sign is

$$(1-k'^2)^2 - 2\,(1+k'^2)\,t^2 + t^4\;(=T\ \text{suppose});$$

and this must consequently be a form of the original integral equation

$$t = k^2x\sqrt{\frac{1-x^2}{1-k^2x^2}}.$$

In fact, squaring and solving in regard to x^2 with the assumed sign of the radical, we have

$$x^2 = \frac{k^2+t^2-\sqrt{T}}{2k^2},$$

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corresponding to an equation given by Landen. And we thence have

$$1 - k^2x^2 = \tfrac{1}{2}\{2 - k^2 - t^2 + \sqrt{T}\}, \quad = \tfrac{1}{2}\{1 + k'^2 - t^2 + \sqrt{T}\},$$

which is the required expression for $1 - k^2x^2$.

The trigonometrical form $\sin(2\phi' - \phi) = c \sin\phi$ of the relation between y and x does not occur in Landen; it is employed by Legendre, I believe, in an early paper, *Mém. de l'Acad. de Paris*, 1786, and in the *Exercices*, 1811, and also in the *Traité des Fonctions Elliptiques*, 1825, and by means of it he obtains an expression for the arc of a hyperbola in terms of two elliptic functions, $E(c, \phi)$, $E(c', \phi')$, showing that the arc of the hyperbola is expressible by means of two elliptic arcs,—this, he observes, "est le beau théorème dont Landen a enrichi la géométrie." We have, then (1828), Jacobi's proof, by two fixed circles, of the addition-theorem (see my *Elliptic Functions*, p. 28), and the application of this (p. 30) to Landen's theorem is also due to Jacobi, see the "Extrait d'une lettre adressée à M. Hermite," *Crelle*, t. XXXII. (1846), pp. 176—181; the connection of the demonstrations, by regarding the point, which is alone necessary for Landen's theorem as the limit of the smaller circle in the figure for the addition-theorem is due to Durège (see his *Theorie der elliptischen Functionen*, Leipzig, 1861, pp. 168, *et seq.*).