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ON MR WILKINSON'S RECTANGULAR TRANSFORMATION.

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CONSIDERING the three cones,

$$(p+\lambda) X^2 + (q+\lambda) Y^2 + (r+\lambda) Z^2 = 0,$$
$$(p+\mu) X^2 + (q+\mu) Y^2 + (r+\mu) Z^2 = 0,$$
$$(p+\nu) X^2 + (q+\nu) Y^2 + (r+\nu) Z^2 = 0,$$

where

$$p+q+r+\lambda+\mu+\nu=0,$$

it is easy to see that these contain a singly infinite system of rectangular axes, viz. we have in each cone one axis of a rectangular system, and for one of the cones the axis may be any line at pleasure of the cone. In fact, taking for the three axes (x, y, z), (x', y', z'), (x'', y'', z'') respectively, that is, for the first axis $X : Y : Z = x : y : z$, and so for each of the other two axes, then (x, y, z) being an arbitrary line on the first cone, we can find (x', y', z') and (x'', y'', z'') such that

$$(p+\lambda) x^2 + (q+\lambda) y^2 + (r+\lambda) z^2 = 0,$$
$$(p+\mu) x'^2 + (q+\mu) y'^2 + (r+\mu) z'^2 = 0,$$
$$(p+\nu) x''^2 + (q+\nu) y''^2 + (r+\nu) z''^2 = 0,$$
$$x'x'' + y'y'' + z'z'' = 0,$$
$$x''x + y''y + z''z = 0,$$
$$xx' + yy' + zz' = 0.$$

For, eliminating (x'', y'', z'') from the third, fourth, and fifth equations, we have, first,

$$x'' : y'' : z'' = yz' - y'z : zx' - z'x : xy' - x'y,$$

and consequently

$$(p+\nu)(yz'-y'z)^2+(q+\nu)(zx'-z'x)^2+(r+\nu)(xy'-x'y)^2=0.$$

It is to be shown that this equation is implied in the remaining first, second, and third equations; for, this being so, (x, y, z), (x', y', z') satisfy only these equations; or (x, y, z) are any values whatever satisfying the first equation. The other two equations then determine (x', y', z'), and, these being known, (x'', y'', z'') are then determined as above.

In fact, attending to the sixth equation, the equation just obtained may be written in the form

$$(p+\nu)[(y^2+z^2)(y'^2+z'^2)-x^2x'^2]+(q+\nu)[(z^2+x^2)(z'^2+x'^2)-y^2y'^2]$$
$$+(r+\nu)[(x^2+y^2)(x'^2+y'^2)-z^2z'^2]=0,$$

or, what is the same thing, in the form

$$-(x^2+y^2+z^2)[(p+\mu)x'^2+(q+\mu)y'^2+(r+\mu)z'^2]$$
$$-(x'^2+y'^2+z'^2)[(p+\lambda)x^2+(q+\lambda)y^2+(r+\lambda)z^2]=0;$$

for, comparing in the two forms, first the coefficients of $x^2x'^2$, these are

$$(q+\nu)+(r+\nu)-(p+\nu) \quad \text{and} \quad -2p+\lambda+\mu,$$

which are equal in virtue of $p+q+r+\lambda+\mu+\nu=0$; and comparing next the coefficients of $y^2z'^2$, these are

$$p+\nu \quad \text{and} \quad -(r+\mu)-(q+\lambda),$$

which are equal in virtue of the same relation: and, similarly, the coefficients of the other terms $y^2y'^2$, &c., are equal in the two equations respectively.

Take now three arguments a_0, b_0, c_0, connected by the relation $a_0+b_0+c_0=0$, and write a, a, A for the sn, cn, and dn of a_0; and similarly b, b, B and c, c, C for those of b_0 and c_0 respectively: then we may write

$$p+\lambda,\ q+\lambda,\ r+\lambda= \quad \left(1,\ \frac{\mathrm{a}}{\mathrm{bc}},\ \frac{A}{BC}\right),$$

$$p+\mu,\ q+\mu,\ r+\mu=\theta\left(1,\ \frac{\mathrm{b}}{\mathrm{ca}},\ \frac{B}{CA}\right),$$

$$p+\nu,\ q+\nu,\ r+\nu=\phi\left(1,\ \frac{\mathrm{c}}{\mathrm{ab}},\ \frac{C}{AB}\right);$$

for, starting from the first set of values, we have the second set if only

$$\mu-\lambda=\theta-1=\theta\frac{\mathrm{b}}{\mathrm{ca}}-\frac{\mathrm{a}}{\mathrm{bc}}=\theta\frac{B}{CA}-\frac{A}{BC}.$$

We thence obtain

$$\theta\left(1-\frac{\mathrm{b}}{\mathrm{ca}}\right)=1-\frac{\mathrm{a}}{\mathrm{bc}},\quad \theta\left(1-\frac{B}{CA}\right)=1-\frac{A}{BC};$$

and, in order to the identity of the two values of θ, we must have

$$\left(1-\frac{\mathrm{b}}{\mathrm{ca}}\right)\left(1-\frac{A}{BC}\right)-\left(1-\frac{\mathrm{a}}{\mathrm{bc}}\right)\left(1-\frac{B}{CA}\right)=0,$$

that is,

$$(\mathrm{abc}-\mathrm{b}^2)(ABC-A^2)-(\mathrm{abc}-\mathrm{a}^2)(ABC-B^2)=0,$$

or, reducing,

$$(\mathrm{a}^2-\mathrm{b}^2)ABC-(A^2-B^2)\,\mathrm{abc}+A^2\mathrm{b}^2-B^2\mathrm{a}^2=0.$$

But

$$\mathrm{a}^2-\mathrm{b}^2=-(a^2-b^2),\quad A^2-B^2=-k^2(a^2-b^2),\quad A^2\mathrm{b}^2-B^2\mathrm{a}^2=k'^2(a^2-b^2);$$

hence the whole equation divides by a^2-b^2, and, omitting this factor, it becomes

$$-ABC+k^2\mathrm{abc}+k'^2=0,$$

which is a known relation between the elliptic functions of the arguments a_0, b_0, c_0 connected by the equation $a_0+b_0+c_0=0$. Similarly, for ϕ, we have

$$\nu-\lambda=\phi-1=\phi\,\frac{\mathrm{c}}{\mathrm{ab}}-\frac{\mathrm{a}}{\mathrm{bc}}=\phi\,\frac{C}{AB}-\frac{A}{BC},$$

and, comparing the two values of ϕ, we have the same identical relation.

It thus appears that the three cones

$$X^2+\frac{\mathrm{a}}{\mathrm{bc}}\,Y^2+\frac{A}{BC}\,Z^2=0,$$

$$X^2+\frac{\mathrm{b}}{\mathrm{ca}}\,Y^2+\frac{B}{CA}\,Z^2=0,$$

$$X^2+\frac{\mathrm{c}}{\mathrm{ab}}\,Y^2+\frac{C}{AB}\,Z^2=0,$$

(the coefficients whereof depend on the elliptic functions sn, cn, and dn, of the arguments a_0, b_0, c_0 connected by the equation $a_0+b_0+c_0=0$) contain a singly infinite system of rectangular axes.

Considering an argument f_0, and denoting its sn, cn, dn by f, f, F respectively, we have, for an arbitrary line on the first cone, the values

$$x,\ y,\ z=M\sqrt{k'^2A\mathrm{a}},\quad M\sqrt{k^2A\mathrm{bc}}\,.\,\mathrm{f},\quad M\sqrt{-\mathrm{a}BC}\,.\,F.$$

In fact, substituting in the equation of the cone, we obtain the identity

$$k'^2+k^2\mathrm{f}^2-F^2=0;$$

and if we determine M by the condition that $x^2+y^2+z^2$ shall be $=1$, then we have

$$1=M^2\{k'^2A\mathrm{a}+k^2A\mathrm{bcf}^2-\mathrm{a}BCF^2\},$$

where the coefficient of M^2 is

$$=k'^2A\mathrm{a}+k^2A\mathrm{bc}(1-f^2)-\mathrm{a}BC(1-k^2f^2),$$

which is easily shown to be

$$= k^2 k'^2 bc\,(a^2 - f^2),$$

so that the values of x, y, z are

$$= \{\sqrt{k'^2 A\mathrm{a}},\quad \sqrt{k^2 A\mathrm{bc}}\,.\,\mathrm{f},\quad \sqrt{-\,\mathrm{a}BC}\,.\,F\} \div \sqrt{k^2 k'^2 bc\,(a^2 - f^2)},$$

and, similarly taking the arguments g_0, h_0, and denoting their elliptic functions by g, g, G, h, h, H, we have for a system of arbitrary lines in the three cones respectively, the values

$$\begin{array}{r|c|c|c|l}
x,\ y,\ z = & \sqrt{k'^2 A\mathrm{a}} & \sqrt{k^2 A\mathrm{bc}}\,.\,f & \sqrt{-\,\mathrm{a}BC}\,.\,F & \div \sqrt{k^2 k'^2 bc\,(a^2 - f^2)} \\
\hline
x',\ y',\ z' = & \sqrt{k'^2 B\mathrm{b}} & \sqrt{k^2 B\mathrm{ca}}\,.\,g & \sqrt{-\,\mathrm{b}CA}\,.\,G & \div \sqrt{k^2 k'^2 ca\,(b^2 - g^2)} \\
\hline
x'',\ y'',\ z'' = & \sqrt{k'^2 C\mathrm{c}} & \sqrt{k^2 C\mathrm{ab}}\,.\,h & \sqrt{-\,\mathrm{c}AB}\,.\,H & \div \sqrt{k^2 k'^2 ab\,(c^2 - h^2)},
\end{array}$$

these values being such that $x^2 + y^2 + z^2$, $x'^2 + y'^2 + z'^2$, $x''^2 + y''^2 + z''^2$ are each $= 1$. The radicals in the first line would be more correctly written, and may be understood as meaning $k' \sqrt{A}\sqrt{\mathrm{a}}$, $k\sqrt{A}\sqrt{\mathrm{b}}\sqrt{\mathrm{c}}$, $i\sqrt{\mathrm{a}}\sqrt{B}\sqrt{C}$, and similarly as regards the second and third lines respectively.

Taking now the arbitrary lines at right angles to each other, the condition for the second and third lines is

$$1 + k^2 \mathrm{agh} - k'^2 AGH = 0,$$

which is satisfied if $a_0 = g_0 - h_0$; similarly the condition for the third and first lines is satisfied if $b_0 = h_0 - f_0$; and we then have $a_0 + b_0 = g_0 - f_0$; that is, $-c_0 = g_0 - f_0$ or $c_0 = f_0 - g_0$, which is the condition for the first and second lines; hence the arguments a_0, b_0, c_0, f_0, g_0, h_0 being such that

$$\begin{aligned}
.\quad h_0 - g_0 + a_0 &= 0,\\
-h_0\quad .\quad + f_0 + b_0 &= 0,\\
g_0 - f_0\quad .\quad + c_0 &= 0,\\
-a_0 - b_0 - c_0\quad . \quad &= 0,
\end{aligned}$$

or, what is the same thing, a_0, b_0, c_0, f_0, g_0, h_0 being the differences of any four arguments α, β, γ, δ, the foregoing values of (x, y, z), (x', y', z'), (x'', y'', z'') will satisfy the equations

$$\begin{aligned}
x^2 + y^2 + z^2 &= 1,\\
x'^2 + y'^2 + z'^2 &= 1,\\
x''^2 + y''^2 + z''^2 &= 1,\\
x' x'' + y' y'' + z' z'' &= 0,\\
x'' x + y'' y + z'' z &= 0,\\
x\, x' + y\, y' + z\, z' &= 0,
\end{aligned}$$

for the transformation of a set of rectangular axes. These are, in fact, Mr Wilkinson's expressions, the a_0, b_0, c_0, f_0, g_0, h_0 being his $t - p$, $p - q$, $q - t$, t, p, q respectively.

Returning to the three cones, it is to be remarked that, taking in the first of them a line 1 at pleasure, then we have in the second of them *two* lines 2, 2′ each at right angles to the line 1, and such that the line 3 at right angles to the plane 12, and the line 3′ at right angles to the plane 12′, lie each of them in the third cone; or, what is the same thing, we have in the two cones respectively the rectangular lines 1 and 2, and also the rectangular lines 1 and 2′, such that the planes 12 and 12′ each of them envelope one and the same cone, the reciprocal of the third cone; where by the reciprocal cone of a given cone is meant the cone generated by the lines through the vertex at right angles to the tangent planes of the given cone. Introducing the notion of the absolute cone $X^2+Y^2+Z^2=0$, a line and plane through the vertex at right angles to each other are, in fact, reciprocal polars in regard to this absolute cone; and two lines at right angles to each other are reciprocals (or harmonics) in regard to this absolute cone; that is, the reciprocal plane of either of them passes through the other. The two cones are cones intersecting each other in four lines lying on the absolute cone; and in virtue of this relation they have the property in question, viz. taking in the first cone a line 1 at pleasure, then the reciprocal plane hereof in regard to the absolute cone meets the second cone in a pair of lines 2 and 2′ such that the planes 12 and 12′ each of them envelope one and the same cone; the reciprocal of this cone is then the third cone of the system, and as such it passes through the four lines on the absolute cone.

In verification, observe that the coefficients $p+\lambda$, $q+\lambda$, &c. of the equations of the three cones satisfy the equations

$$\left\| \begin{array}{lllll} 1, & p+\lambda, & p+\mu, & p+\nu, & (p+\lambda)(p+\mu)(p+\nu) \\ 1, & q+\lambda, & q+\mu, & q+\nu, & (q+\lambda)(q+\mu)(q+\nu) \\ 1, & r+\lambda, & r+\mu, & r+\nu, & (r+\lambda)(r+\mu)(r+\nu) \end{array} \right\| = 0.$$

This is obviously the case for each equation such as

$$|\, 1,\ p+\lambda,\ p+\mu \,| = 0;$$

and any equation containing the fifth column is at once reducible to

$$|\, 1,\ p,\ p^3+p^2(\lambda+\mu+\nu) \,| = 0,$$

that is,

$$|\, 1,\ p,\ p^3 \,| + (\lambda+\mu+\nu)\,|\, 1,\ p,\ p^2 \,| = 0;$$

or, dividing by $|\, 1,\ p,\ p^2 \,|$, this is $p+q+r+\lambda+\mu+\nu=0$, the equation connecting the coefficients.

Hence, representing the three cones by

$$\begin{aligned} p\,X^2+q\,Y^2+r\,Z^2&=0,\\ p'X^2+q'Y^2+r'Z^2&=0,\\ p''X^2+q''Y^2+r''Z^2&=0, \end{aligned}$$

and the absolute by

$$X^2+\ \ Y^2+\ \ Z^2=0,$$

the coefficients p, q, &c., are connected by the equations

$$\left\|\begin{matrix} 1, & p, & p', & p'', & pp'p'' \\ 1, & q, & q', & q'', & qq'q'' \\ 1, & r, & r', & r'', & rr'r'' \end{matrix}\right\| = 0;$$

among these are of course included the equation $|\,1,\ p,\ p'\,| = 0$, which expresses that the first and second cones intersect on the absolute; (p, q, r), (p', q', r') are any quantities satisfying this relation, and, regarding them as given, we have then two independent equations determining the ratios $p'' : q'' : r''$. The theorem is that the planes 12 and 12′ envelope one and the same quadric cone

$$\frac{X^2}{p''} + \frac{Y^2}{q''} + \frac{Z^2}{r''} = 0.$$

The equations $|\,1,\ p,\ p''\,| = 0$ and $|\,1,\ p,\ pp'p''\,| = 0$ give

$$\begin{aligned} (q-r)\,p'' \quad + (r-p)\,q'' \quad + (p-q)\,r'' \quad &= 0, \\ (q-r)\,pp'p'' + (r-p)\,qq'q'' + (p-q)\,rr'r'' &= 0, \end{aligned}$$

and thence

$$(q-r)\,p'' : (r-p)\,q'' : (p-q)\,r'' = qq' - rr' : rr' - pp' : pp' - qq';$$

or, observing that we have

$$q-r : r-p : p-q = qr' - q'r : rp' - r'p : pq' - p'q,$$

the equations may also be written

$$(qr' - q'r)\,p'' : (rp' - r'p)\,q'' : (pq' - p'q)\,r'' = qq' - rr' : rr' - pp' : pp' - qq'.$$

Starting with an arbitrary line (x, y, z) in the first cone, then the reciprocal plane thereof (in regard to the absolute cone) is the plane $Xx + Yy + Zz = 0$, which meets the second cone in two lines, say (2) and (2′), each of which is a line reciprocal to the line (1); and we have thus two planes (12) and (12′), each of which envelopes, as is to be shown, the same cone $q''r''X^2 + r''p''Y^2 + p''q''Z^2 = 0$.

Suppose, in general, that we have an arbitrary line (x, y, z) and an arbitrary plane $\alpha X + \beta Y + \gamma Z = 0$, and that it is required to find the equation of the two planes through the line (x, y, z), and the intersections of the plane $\alpha X + \beta Y + \gamma Z = 0$ with the cone $p'X^2 + q'Y^2 + r'Z^2 = 0$: the equation of the pair of planes is

$$\begin{aligned} &(\alpha X + \beta Y + \gamma Z)^2 (p'x^2 + q'y^2 + r'z^2) \\ &+ (\alpha x + \beta y + \gamma z)^2 (p'X^2 + q'Y^2 + r'Z^2) \\ &- 2\,(\alpha X + \beta Y + \gamma Z)\,(\alpha x + \beta y + \gamma z)\,(p'Xx + q'Yy + r'Zz) = 0. \end{aligned}$$

In the present case, the plane $\alpha X + \beta Y + \gamma Z = 0$ is the plane $xX + yY + zZ = 0$, which is the reciprocal of the line (x, y, z) in regard to the absolute cone, and the equation of the pair of planes is

$$\begin{aligned} &(xX + yY + zZ)^2 (p'x^2 + q'y^2 + r'z^2) \\ &+ (x^2 + y^2 + z^2)^2 (p'X^2 + q'Y^2 + r'Z^2) \\ &- 2\,(xX + yY + zZ)\,(x^2 + y^2 + z^2)\,(p'Xx + q'Yy + r'Zz) = 0, \end{aligned}$$

where the quantities (x, y, z), as belonging to a line on the first cone, satisfy the condition $px^2+qy^2+rz^2=0$. The equation may be written

$$(a, b, c, f, g, h)(yZ-zY, zX-xZ, xY-yX)^2=0,$$

where

$$a, b, c, f, g, h=q'z^2+r'y^2, r'x^2+p'z^2, p'y^2+q'x^2, -p'yz, -q'zx, -r'xy,$$

and, as before, $px^2+qy^2+rz^2=0$; viz. this is the equation of the pair of planes (12) and (12′).

The equation of the pair of tangent planes through the line (x, y, z) to the cone $q''r''X^2+r''p''Y^2+p''q''Z^2=0$ is

$$(q''r''x^2+r''p''y^2+p''q''z^2)(q''r''X^2+r''p''Y^2+p''q''Z^2)-(q''r''xX+r''p''yY+p''q''zZ)^2=0;$$

viz. omitting a factor $p''q''r''$, this equation is

$$(p'', q'', r'', 0, 0, 0)(yZ-zY, zX-xZ, xY-yX)^2=0.$$

And it is to be shown that this is equivalent to the former equation; viz. writing $yZ-zY, zX-xZ, xY-yX=\lambda, \mu, \nu$, then that the two equations

$$(q'z^2+r'y^2, r'x^2+p'z^2, p'y^2+q'x^2, -p'yz, -q'zx, -r'xy)(\lambda, \mu, \nu)^2=0,$$

$$p''\lambda^2+q''\mu^2+r''\nu^2=0,$$

are equivalent to each other.

We have identically $\lambda x+\mu y+\nu z=0$, and thence also

$$(\lambda x+\mu y+\nu z)[(p'-q'-r')\lambda x+(-p'+q'-r')\mu y+(-p'-q'+r')\nu z]=0,$$

where, on the left-hand side, the terms in $\mu\nu$, $\nu\lambda$, and $\lambda\mu$ are

$$=-2p'yz\mu\nu-2q'zx\nu\lambda-2r'xy\lambda\mu.$$

Hence the first equation may be written

$$[q'z^2+r'y^2+(p'-q'-r')x^2]\lambda^2+[r'x^2+p'z^2+(-p'+q'-r')y^2]\mu^2$$
$$+[p'y^2+q'x^2+(-p'-q'+r')z^2]\nu^2=0,$$

and it is to be shown that this is equivalent to

$$p''\lambda^2+q''\mu^2+r''\nu^2=0;$$

viz. that we have $p'' : q'' : r''=$

$$\begin{aligned} & q'z^2+r'y^2-(\ \ p'-q'-r')x^2 \\ : \ & r'x^2+p'z^2-(-p'+q'-r')y^2 \\ : \ & p'y^2+q'x^2-(-p'-q'+r')z^2, \end{aligned}$$

where $px^2+qy^2+rz^2=0$. Writing the equation in the form

$$p'' : q'' : r''=A : B : C,$$

we have

$$\begin{aligned} A &= q'z^2 + r'y^2 - p'x^2 + q'x^2 + r'x^2 \\ &= -p'x^2 + q'(x^2+z^2) + r'(x^2+y^2) \\ &= -p'x^2 + (q'+r')(x^2+y^2+z^2) - q'y^2 - r'z^2. \end{aligned}$$

By what precedes, we have an identity of the form

$$x^2 + y^2 + z^2 = \alpha (p'x^2 + q'y^2 + r'z^2) + \beta (px^2 + qy^2 + rz^2),$$

where, determining α from the equations $1 = q'\alpha + q\beta$, $1 = r'\alpha + r\beta$, we find

$$\alpha = (q - r) \div (qr' - q'r);$$

but $px^2 + qy^2 + rz^2 = 0$, and the relation thus is

$$x^2 + y^2 + z^2 = \alpha (p'x^2 + q'y^2 + r'z^2);$$

hence

$$A = \{(q' + r')\alpha - 1\} (p'x^2 + q'y^2 + r'z^2),$$

or, substituting for α its value, this is

$$\begin{aligned} A &= \left\{\frac{(q'+r')(q-r) - qr' + q'r}{qr' - q'r}\right\} (p'x^2 + q'y^2 + r'z^2), \\ &= \frac{qq' - rr'}{qr' - q'r} (p'x^2 + q'y^2 + r'z^2); \end{aligned}$$

and, forming the like values of B and C, the relations to be verified become

$$p'' : q'' : r'' = \frac{qq' - rr'}{qr' - q'r} : \frac{rr' - pp'}{rp' - r'p} : \frac{pp' - qq'}{pq' - p'q},$$

which are, in fact, the values of the ratios $p'' : q'' : r''$ obtained above; and the theorem is thus seen to be true. It may be remarked that, if the first and second cones, instead of intersecting in four lines on the absolute cone, had been arbitrary cones; then, taking in the first cone a line (1) and in the second cone a line (2), the reciprocal of (1) in regard to the absolute, the envelope of the plane (12) would have been (instead of a quadric cone) a cone of the class 8.