Archives of Mechanics • Archiwum Mechaniki Stosowanej • 26, 4, pp. 545–556, Warszawa 1974

Change of the order of solution and interaction of simple waves for two independent variables

M. BURNAT (WARSZAWA)

THE SOLUTION $u = u(x)$, $x \in D$ of the system (1.1) is of rank k if its set of values $u(D)$ represents a k-dimensional manifold. We consider in the paper the classes of solutions for which the change of the rank Δk, when we pass from one region to the other, fulfills the inequality $|\Delta k| \leqslant 1$. In our consideration we use the method of simple wave interaction. As the examples of the system (1.1) we use the systems of gas dynamics and plasticity theory.

Rozwiązanie układu (1.1) $u = u(x)$, $x \in D$ jest rzędu k, jeśli zbiór jego wartości $u(D)$ jest k, wymiarową rozmaitością. W pracy rozpatrzono przy użyciu metody współdziałania fal prostych klasy takich rozwiązań układu (1.1), dla których zmiana rzędu Δk przy przejściu z jednego obszaru do drugiego spełnia nierówność $|\Delta k| \leqslant 1$. Rozpatrzono przykłady układów (1.1), występujących w dynamice gazów i teorii plastyczności.

Решение системы (1.1) $u = u(x)$, $x \in D$ имеет ранг k, если множество его значении $u(D)$ является k-мерным многообразием. В работе рассматриваются классы таких решений системы (1.1), для которых перемена ранга Δk при переходе с одной области в другую удовлетворяет неравенство: $|\Delta k| \leqslant 1$. Применяется метод взаимодействия простых волн. В качестве примеров системы (1.1) рассмотриваются системы динамики газов и теории пластичности.

1. Introduction

IN THE CASE of the hyperbolic systems

$$a_j^{si}(u^1, u^2) u_{x^i}^j = 0, \quad s, i, j = 1, 2,$$

the following fact is well known: if the solution is constant in a certain region, then in an adjacent region the solution must be degenerated—that is, the Jacobian matrix rank must satisfy the inequality

$$R_u(x) = r\|u_{x^i}^j(x)\| < 2.$$

In other words, the rank of solution $u(x)$ may be varied only by one involving passing from a region in which $u(x) = \text{const}$ to a region where $u(x) \neq \text{const}$.

This phenomenon is of considerable interest from the physical point of view, since it simplifies the construction of solutions in the regions adjoining the regions of $u(x) = \text{const}$ (cf. e.g. [1]).

In connection with the above observation, a number of interesting questions arises as regards the system of equations of the form

$$a_j^{si}(u^1, \ldots, u^l) u_{x^i}^j = 0, \quad s, j = s, \ldots, l, \quad i = 1, \ldots, n \tag{1.1}$$

concerning the problem as to under which conditions their solutions have analogous properties (cf. [1], pp. 75–78). In what follows, the point space $x = (x^1, \ldots, x^n)$ will be denoted by R^n, and the point space $u = (u^1, \ldots, u^l)$—by H^l.

Let us consider the region $G \subset R^n$ which is cut by the $(n-1)$-dimensional manifold N_{n-1} into two regions D_ω and D. It is assumed that

$$u(x) \in C(G) \cap C^1(\overline{D}_\omega) \cap C^1(\overline{D}) \tag{1.2}$$

and

$$R_u(x) \underset{x \in \overline{D}_\omega}{=} \text{const} = \omega. \tag{1.3}$$

The question arises as to what may be locally stated on $R_u(x)$ for $x \in D$, provided the solution $u(x)$ satisfies the conditions (1.2) and (1.3). This concerns the upper and lower bound estimates of $R_u(x)$ for $x \in D$ depending on the value of ω.

The assumption (1.3) means that the set of values of the solution $u(D_\omega) \subset H^l$ is locally a ω-dimensional manifold in H^l. Thus our general question may be formulated as follows: what can be stated concerning dimensions of manifolds $\underline{M}$, $\overline{M} \subset H^l$ satisfying, in the case of solutions having the properties (1.2), (1.3), the condition

$$\underline{M} \subset u(D) \subset \overline{M}.$$

Our considerations are aimed at the proof of existence of the manifolds $\underline{M}$ and $\overline{M}$ the dimensions of which differ as little as possible from each other.

Let us now formulate the particular problem which is of fundamental importance for our further considerations. The set of Eqs. (1.1) or a class of solutions possesses the ω-property if for each solution satisfying the conditions (1.2), (1.3) and each neighbourhood $V \subset N_{n-1}$ there exists a certain neighbourhood $I \subset V$ such that two manifolds $\underline{M}$, $\overline{M}$ exist satisfying the condition

$$\underline{M} \subseteq u(\ _I) \subseteq \overline{M},$$

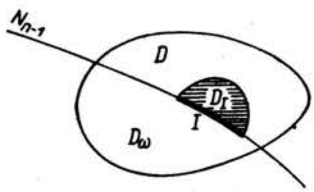

FIG. 1.

where $D_I \subset D$ is a certain one-sided neighbourhood of I (Fig. 1), and

$$\omega - 1 \leqslant \dim M, \quad \dim \overline{M} \leqslant \omega + 1.$$

In a slightly weaker approach, the ω-property may be defined by the inequality

$$\omega - 1 \leqslant R_u(x) \leqslant \omega + 1, \quad x \in D_I.$$

Hence, the ω-property is satisfied if the jump of the rank does not exceed unity.

It is necessary to answer the question, when the system or the class of solutions has the ω-property? We do not know very much about the problems just stated except the result obtained by LAX [2], who established the fact that a strongly hyperbolic system (1.1) has the 0-property for $n = 2$.

In order to clarify the formulation of ω-properties, the following simple lemma proves to be important:

LEMMA 1. *If for the mapping of class* C^1, $u = u(x)$, $x \in G$, $G \to H^l$, *G being the region (or closed region) in which the condition is satisfied,*

$$R_u(x) \underset{x \in G}{=} \text{const} = d,$$

then for each $(n-1)$*-dimensional manifold* $L_{n-1} \subset G$ *such that the set* $u(L_{n-1})$ *is a manifold of fixed dimension we have*

$$\dim u(L_{n-1}) = d \quad or \quad d-1.$$

From the assumption (1.3) it follows that in each neighbourhood $V \subseteq N_{n-1}$ there exists such a neighbourhood $I \subset V$ that $u(I)$ represents a manifold of fixed dimension in H^l. If then it is additionally assumed that

$$R_u(x) \underset{x \in \bar{D}}{=} \text{const} = d, \tag{1.4}$$

so applying the Lemma 1 first to $\bar{D}_\omega$ and next to $\bar{D}$ we obtain

$$\dim u(I) = \omega, \omega-1, \dim u(I) = d, d-1,$$

which immediately yields the only three possibiliteies: $d = \omega-1$, $d = \omega$, $d = \omega+1$. The additional assumption (1.4) proves that an arbitrary mapping has the ω-property.

From our considerations it follows that the essential features of ω-properties are as follows: nothing is assumed on the behaviour of the rank of solution in D, and the construction of manifolds $\underline{M}$, $\bar{M}$ defines to a high degree the set of values of the solution in D. Our principal aim will be to determine which systems of the form of (1.1) or which classes of solutions have the ω-property. The case $n = 2$ will be considered in this paper, the cases of arbitrary numbers of independent variables will be dealt with in a subsequent paper.

In what follows, the method of interaction of simple waves is used, called also the method of Riemann invariants (cf. [3–12]). The method is confined to the case of $n = 2$.

It is shown that the ω-property expressed in the language of wave interaction occurs when the solutions constitute "the interaction of independent simple waves". The property does not occur if the solutions may constitute "the interaction of simple waves which are not independent", which means that the interaction is not regular in a certain sense.

In particular, the result by LAX [2] is formulated in the language of wave interaction. In the cases considered in which the ω-property occurs, the manifolds $\underline{M}$ and $\bar{M}$ are uniquely determined by the set $u(I)$.

All the functions, curves, surfaces and manifolds dealt with in the paper will be assumed to be of the C^1 class.

2. Simple waves

The solution $u = u(x)$, $x \in D \subset R^n$ will be called a k-fold wave if its hodograph (set of values) is a locally k-dimensional manifold. This condition may be written by means of the equation

$$R_u(x) \underset{x \in D}{=} \text{const} = k.$$

It is easily verified that a solution $u = u(x)$, $x \in D$ represents a k-fold wave if, and only if, it may be represented, in the neighbourhood of every point belonging to D, in the form:

$$u^j = f^j\left(\mu^1(x), \ldots, \mu^k(x)\right), \quad j = 1 \ \ldots, l,$$

where

$$r\|f^j_{\mu^\sigma}\| = k, \quad r\|\mu^\sigma_{x^i}\| = k.$$

Two sets of characteristic vectors will be considered in connection with the system (1.1): the characteristic vectors $\lambda(u) = \left(\lambda_1(u), \ldots, \lambda_n(u)\right)$ in the space R^n determined by

(2.1) $$\det\|a^{si}_j(u^1, \ldots, u^l)\lambda_i\| = 0$$

and the characteristic vectors $\gamma(u) = \left(\gamma^1(u), \ldots, \gamma^l(u)\right)$ in the space H^l determined by the condition

$$r\|a^{si}_j(u^1, \ldots, u^l)\gamma^j\| < n.$$

Two characteristic vectors λ and γ are called to be knotted and denote it by $\lambda \leftrightarrow \gamma$ if

$$a^{si}_j(u)\lambda_i\gamma^j = 0, \quad s = 1, \ldots, l.$$

At least one vector λ corresponds to each vector γ and vice versa. It proves convenient to introduce the space

$$\Lambda[\gamma] = \{\lambda: \lambda \leftrightarrow \gamma\}.$$

Curves $\Gamma \subset H^l$ of the class C^1 tangent to the characteristic vectors γ will be called characteristic curves in the space H^l. It is easily seen (cf. [3, 6, 12]) that each simple wave may be written in the form

(2.2) $$u^j = u^j(x) = f^j\left(\mu(x)\right), \quad j = 1, \ldots, l.$$

Here $u = f(s)$ is a parametric representation of the characteristic curve Γ.

Our considerations will be confined to such curves Γ for which

$$\dim \Lambda[\dot{f}(s)] \underset{s}{\equiv} \text{const} = d(\Gamma).$$

The following theorem holds true (cf. [3, 6, 12]):

THEOREM 1. *If $u = f(s)$ is the parametric representation of the characteristic curve Γ, then the mapping $u = u(x)$, $x \in D$, $D \overset{u}{\to} H^l$ is a simple wave provided*

$$u(x) = \text{const} = f(s_0)$$

for x belonging to the $(n-\varrho)$-dimensional planes $\pi_{n-\varrho}$, $1 \leqslant \varrho \leqslant d(\Gamma)$ described by the equations

$$\overset{\sigma}{\lambda}_i(s_0)(x^i - \overset{\circ}{x}{}^i) = 0, \quad \sigma = 1, \ldots, \varrho.$$

Here

$$\overset{\sigma}{\lambda}(s) \in \Lambda[\dot{f}(\overset{\circ}{s})].$$

In order to prove the theorem it is sufficient to observe that $u(x) = f[\mu(x)]$ represents a solution if, and only if, the condition

$$\operatorname{grad}\mu(x) \in \Lambda[\dot{f}(\mu(x))]$$

is satisfied. The condition is equivalent to

$$a_j^{si} u^j_{x^i} = a_j^{si} \dot{f}^j(u(x)) \mu_{x^i} = 0. \tag{2.3}$$

From the Theorem 1 it follows that each characteristic curve Γ is a set of values of an infinite family of simple waves. It may be verified that, conversely, the set of values of every simple wave is a characteristic curve Γ.

Integral elements of the system (1.1) at $u \in H^l$ will be called the matrices $L^j_i(u)$ satisfying the conditions

$$a_j^{si}(u) L^j_i(u) = 0, \quad s = 1, \ldots, l.$$

Solution $u(x)$ will be called to be constructed at a point p of the integral element $L^j_i(u)$ if

$$u^j_{x^i}(p) = L^j_i(u(p)).$$

The following simple theorem holds:

THEOREM 2. *Simple waves are constructed of integral elements of the form*

$$L^j_i = \gamma^j \lambda_i, \tag{2.4}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ *is the characteristic vector in* E^n, *and* $\gamma = (\gamma^1, \ldots, \gamma^l)$*-characteristic vector in* H^l, *and* $\gamma \leftrightarrow \lambda$.

The proof follows immediately from the correspondence definition and from the Eq. (2.3).

Obviously, a matrix of the form of (2.4) as also any linear combination of such matrices, is also an integral element. The elements of the form (2.4) will be called simple integral elements.

3. Interaction of simple waves for $n = 2$. Lax theorem

The notion of interaction of simple waves will be discussed first on the example of the hyperbolic system (1.1) in which $n = l = 2$. Such a system has in the R^2-plane two linearly independent characteristic vectors $\overset{+}{\lambda}(u)$, $\overset{-}{\lambda}(u)$, and two linearly independent charactistic vectors $\overset{+}{\gamma}(u)$, $\overset{-}{\gamma}(u)$ in the space H^2, and $\overset{+}{\gamma} \leftrightarrow \overset{+}{\lambda}$, $\overset{-}{\gamma} \leftrightarrow \overset{-}{\lambda}$. Every po int of H^2 is crossed by exactly two characteristic curves $\overset{+}{\Gamma}$, $\overset{-}{\Gamma}$ which satisfy the condition $d(\overset{+}{\Gamma}) = d(\overset{-}{\Gamma}) = 1$, so that the simple waves (Theorem 1) are constant along straight lines.

As the base of characteristic vectors $\overset{+}{\lambda}$, $\overset{-}{\lambda}$ let us introduce the co-base of characteristic vectors $\overset{+}{c}$, $\overset{-}{c}$, that is

$$(\overset{+}{c}, \overset{-}{\lambda}) = 0, \quad (\overset{-}{c}, \overset{+}{\lambda}) = 0.$$

By $\overset{+}{C}$, $\overset{-}{C}$ denote the characteristic curves in R^2 tangent to the respective vectors $\overset{+}{c}(u(x))$ and $\overset{-}{c}(u(x))$.

A solution $u = u(x)$, $x \in D$ will be said to represent *a regular interaction of simple waves* if in the neighbourhood of each characteristic curve $\overset{+}{C}$, $\overset{-}{C} \subset D$ a simple wave exists which assumes at $\overset{+}{C}$, $\overset{-}{C}$ the values of the solution $u(x)$.

The geometric and physical sense of interaction is illustrated in Fig. 2. Characteristic curves $\overset{+}{C}$, $\bar{C}$ are drawn through an arbitrary point $P \in D$; they cut a region $G(P)$ from the region D. Simple waves occurring in the definition of interaction exist within the regions $\underset{1}{G}$ and $\underset{2}{G}$ (Fig. 2). The solution in $G(P)$ may be considered as a result of interaction of simple waves defined in $\underset{1}{G}$ and $\underset{2}{G}$.

It is easily seen that the solution is a regular interaction of simple waves if, and only if, it is a double wave. It may be demonstrated (cf. e.g. [8]) that for $n = l = 2$ every solution is with accuracy up to a set of zero measure either a simple or a double wave. Each solution of the system (1.1), $n = l = 2$, is then, with accuracy up to a set of zero measure, either a simple wave or a regular interaction of simple waves. This fact is equivalent to the following theorem:

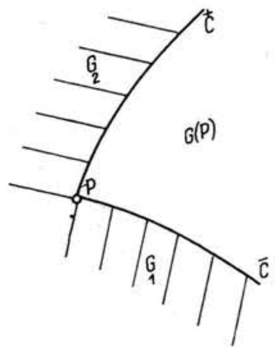

FIG. 2.

THEOREM 3. *Mapping $u = u(x)$ determined in the region D, $D \overset{u}{\to} H^l$ is a solution of the Eq.* (1.1) *if, and only if, for each of the characteristics $\overset{+}{C} \subset D$ and $\bar{C} \subset D$ the condition*

$$u(\overset{+}{C}) \subset \overset{+}{\Gamma}, \quad u(\bar{C}) \subset \bar{\Gamma} \tag{3.1}$$

is satisfied.

The theorem makes possible simple construction of the set of values of the solution of an arbitrary Cauchy problem, which simplifies its determination and qualitative analysis. This fact is of particular importance for the supersonic gasdynamics [1].

Theorem 3 may, also in the language of integral elements, be formulated as follows:

THEOREM 3_α. *All integral elements of Eqs.* (1.1), $n = l = 2$, *have the form*

$$L_i^j = \overset{+}{\alpha}\overset{+}{\gamma}{}^j\overset{+}{\lambda}_i + \bar{\alpha}\bar{\gamma}^j\bar{\lambda}_i, \tag{3.2}$$

where

$$\overset{+}{\gamma} = (\overset{+}{\gamma}{}^1, \overset{+}{\gamma}{}^2), \quad \bar{\gamma} = (\bar{\gamma}^1, \bar{\gamma}^2), \quad \overset{+}{\lambda} = (\overset{+}{\lambda}_1, \overset{+}{\lambda}_2), \quad \bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)$$

and $-\infty < \overset{+}{\alpha}, \bar{\alpha} < +\infty$.

The solution is a simple wave if it is constructed of the integral elements for which $\overset{+}{\alpha} = 0$, $\bar{\alpha} \neq 0$, or $\overset{+}{\alpha} \neq 0$, $\alpha = 0$. The solution is an interaction of simple waves if $\overset{+}{\alpha} \neq 0$, $\bar{\alpha} \neq 0$.

Let us now consider the hyperbolic system (1.1) for $n = 2$ and arbitrary $l > 2$. Then we have to deal with a system of pairwise linearly independent characteristic vectors

$$\overset{1}{\underset{0}{\lambda}}(u), \ldots, \overset{k}{\underset{0}{\lambda}}(u), \quad k \leqslant l.$$

An infinite number of vectors $\gamma(u)$ may appear in H^l [8]—that is, with a single $\underset{0}{\lambda}$ may be knotted a family of vectors depending on several parameters.

Three types of interaction of simple waves are introduced for the system (1.1), $n = 2$, $l > 2$.

The solution $u(x)$ is an interaction of simple waves if it is constructed of integral elements having the form:

$$\sum_{\sigma=1}^{q} \alpha_{\sigma} \underset{\sigma}{\gamma}^{j} \overset{\sigma}{\lambda}_{i}, \tag{3.3}$$

where $\underset{\sigma}{\gamma} \leftrightarrow \overset{\sigma}{\lambda}$, q—a finite number. The class of such solutions is denoted by F.

Let $\mathfrak{G}_r$, $r \geqslant 2$, denote a r-dimensional manifold in H^l, which at every point of $u \in \mathfrak{G}_r$ is tangent to r linearly independent characteristic vectors $\underset{1}{\gamma}(u), \ldots, \underset{r}{\gamma}(u)$. It is additionally assumed that there exists the system of pairwise linearly independent vectors $\overset{1}{\lambda}(u), \ldots, \overset{k}{\lambda}(u)$ such that $\underset{\sigma}{\gamma} \leftrightarrow \overset{\sigma}{\lambda}$. Hence each manifold $\mathfrak{G}_r$ defines the following set of integral elements

$$L_i^j(u) = \sum_{\sigma=1}^{r} \alpha_{\sigma} \underset{\sigma}{\gamma}^{j}(u) \overset{\sigma}{\lambda}_{i}(u), \quad u \in \mathfrak{G}_r. \tag{3.4}$$

A solution $u(x)$ is said to be the interaction of independent simple waves if such a manifold $\mathfrak{G}_r$ exists that $u(x)$ is constructed of the integral elements (3.4). The set of such solutions corresponding to the manifold $\mathfrak{G}_r$ prescribed is denoted by $F(\mathfrak{G}_r)$.

Two-dimensional manifolds $\mathfrak{G}_2$ are termed the characteristic surfaces and denoted by $\mathfrak{H}_2$.

A solution $u(x)$ is a regular interaction of two simple waves if there exists a characteristic surface $\mathfrak{H}_2$ such that $u(x) \in F(\mathfrak{H}_2)$.

Solutions of the class $F(\mathfrak{H}_2)$ are called regular interactions of simple waves because it may be seen that a theorem analogous to the Theorem 3 also holds in that case and, consequently, the solutions $u \in F(\mathfrak{H}_2)$ are regular interactions in the sense illustrated in Fig. 2.

Let us first discuss the properties of solutions of the class $F(\mathfrak{H}_2)$. First of all let us observe that in the case of the Eqs. (1.1) and $n = l = 2$, the only characteristic surface is the plane H^2, and that all solutions are of the class $F(H^2)$.

Let $u(x)$, $x \in D$ be a solution of the Eqs. (1.1), $n = 2$, $l > 2$, $u(x) \in F(\mathfrak{H}_2)$. The base of the characteristic vectors $\overset{1}{\lambda}$, $\overset{2}{\lambda}$ in the R^2-space is complemented by the co-base $\overset{1}{c}$, $\overset{2}{c}$,

$$(\overset{1}{c}, \overset{2}{\lambda}) = 0, \quad (\overset{2}{c}, \overset{1}{\lambda}) = 0.$$

$\overset{1}{C}, \overset{2}{C} \subset D$ will denote the characteristic curves tangent to $\overset{1}{c}(u(x))$ and $\overset{2}{c}(u(x))$, respectively. The following theorem constitutes a simple generalization of Theorem 3:

THEOREM 4. *Mapping* $u = u(x)$, $x \in D$, $D \overset{u}{\to} \mathfrak{H}_2$ *belongs to* $F(\mathfrak{H}_2)$ *if, and only if, the characteristics* $\overset{1}{C}, \overset{2}{C} \subset D$ *satisfy the conditions*:

$$u(\overset{1}{C}) \subset \overset{1}{\Gamma}, \quad u(\overset{2}{C}) \subset \overset{2}{\Gamma}.$$

The proof follows from the observation that in the case of a matrix (3.4)

$$L\overset{1}{c} = \beta \underset{1}{\gamma}, \quad L\overset{2}{c} = \delta \underset{2}{\gamma},$$

β, δ being real numbers.

It may also be proved that the double waves of class $F(\mathfrak{H}_2)$ are interactions of simple waves in the sense illustrated by Fig. 2. The proof is the same as in the case of the Eqs. (1), $n = l = 2$. The following theorem may be formulated:

THEOREM 5. *Classes* $F(\mathfrak{H}_2)$ *have the* ω*-property for* $\omega = 0{,}1$.

Proof. With $\omega = 0$ the set $u(I)$ reduces to a point, $u(I) = P$. From Theorem 4 it follows that

$$\underline{M} = P, \quad \overline{M} = \Gamma(P),$$

$\Gamma(p)$ denoting one of the characteristics $\overset{1}{\Gamma}, \overset{2}{\Gamma} \subset \mathfrak{H}_2$ passing through P.

If $\omega = 1$, then $u(x)$ is a simple wave in D_ω—that is,

$$R_u(x) = \text{const} = 1, \quad x \in \overline{D}_\omega.$$

From Lemma 1 it follows that we are dealing with two possibilities:

(1) $u(I)$ is the point P,

(2) $u(I) = \Gamma(P)$.

In the first case

$$\underline{M} = P, \quad \overline{M} = \Gamma(P)$$

and in the second case

$$\underline{M} = \Gamma(P), \quad \overline{M} = \mathfrak{H}_2;$$

thus the theorem is proved.

For each characteristic surface $\mathfrak{H}_2$ the family $F(\mathfrak{H}_2)$ contains an infinite number of solutions depending on two arbitrary functions of a single variable. This results from the fact that, due to the Theorem 4, construction of the solutions $u \in F(\mathfrak{H}_2)$ reduces, independently of l, to the solution of a hyperbolic system of first order with two dependent and independent variables, [8]. Also derivative from the theorem is the fact that for a given $\mathfrak{H}_2$ each non-characteristic Cauchy problem

$$u^j(x^1(s), x^2(s)) = \varphi^j(s), \quad j = 1, \ldots, l$$

such that $\varphi(s) \in \mathfrak{H}_2$ has a solution in the class $F(\mathfrak{H}_2)$. The set of values of that solution is easily determined, simplifies the construction and enables analysis of the solution of the class $F(\mathfrak{H}_2)$.

A question arises as to how the surfaces $\mathfrak{H}_2$ should be constructed and how many surfaces of that kind exist for a given hyperbolic system? If the system is strongly hyperbolic, then it has l pairwise linearly independent characteristic vectors $\overset{1}{\underset{0}{\lambda}}, \ldots, \overset{l}{\underset{0}{\lambda}}$, and

$$r\,||a_j^{si}\overset{\sigma}{\underset{0}{\lambda}}_i|| = l-1, \quad \sigma = 1, \ldots, l. \tag{3.5}$$

Thus for each $\overset{\sigma}{\underset{0}{\lambda}}$ there exists exactly one characteristic vector $\underset{\sigma}{\gamma} \in H^1$ such that $\underset{\sigma}{\gamma} \leftrightarrow \overset{\sigma}{\lambda}$, the vectors $\underset{1}{\gamma}, \ldots, \underset{l}{\gamma}$ being linearly independent. It follows that the problem of existence of the surface $\mathfrak{H}_2$ is reduced to the problem of integrability of the Pfaff forms

$$\overset{\mu}{n}_j(u)\,du^j = 0, \quad \mu = 1, \ldots, l-2 \tag{3.6}$$

such that for certain $\sigma' \neq \sigma''$, $1 \leqslant \sigma', \sigma'' \leqslant l$, $\overset{\mu}{n}_j\underset{\sigma'}{\gamma}^j = \overset{\mu}{n}_j\underset{\sigma''}{\gamma}^j = 0$. Each pair $\sigma' \neq \sigma'$, corresponds one system (3.6) and if it is integrable a family of characteristic surfaces $\mathfrak{H}_2$. For instance, in a strongly hyperbolic system

$$ua_x + va_y + ka(u_x + v_y) = 0,$$
$$uv_x + vu_y + aa_x\,1/k = 0,$$
$$uv_x + vv_y + aa_y\,1/k = 0,$$

describing a stationary, isentropic, plane supersonic gas flow, each of the systems (3.6) reduces to a single form, and only one of them proves to be integrable. Consequently, in the space H^3 of points (a, u, v) the characteristic surfaces $\mathfrak{H}_2^A$ are given by the equation

$$a^2 = A - k(u^2+v^2), \quad A = \text{const (Bernoulli's law)}.$$

Classes $F(\mathfrak{H}_2^A)$ contain the potential supersonic flows.

If the system is not strongly hyperbolic, characteristic vectors λ will appear which correspond to the characteristic vector families γ depending on $\alpha > 1$ parameters. The more multi-parameter vector families γ a system has, the more surfaces $\mathfrak{H}_2$ exist.

A limiting case is a offered by the "simple hyperbolic systems" which possess only two linearly independent vectors $\overset{1}{\underset{0}{\lambda}}, \overset{2}{\underset{0}{\lambda}}$ (cf. [8]), and hence the vector families γ depend on maximal numbers of parameters. In such systems (cf. [8]), exactly one surface $\mathfrak{H}_2$ passes through every non-characteristic curve $K \subset H^l$, and infinitely many such surfaces pass through every characteristic curve $\Gamma \subset H^l$. Moreover, to each solution $u(x)$ may be described such a surface $\mathfrak{H}_2$ that $u(x) \in F(\mathfrak{H}_2)$.

As an example of a "simple hyperbolic system" let us consider the case of flow of a perfectly plastic material

$$\sigma_x - k(\vartheta_x \cos 2\vartheta + \vartheta_y \sin 2\vartheta) = 0,$$
$$\sigma_y - k(\vartheta_x \sin 2\vartheta - \vartheta_y \cos 2\vartheta) = 0,$$
$$(u_y + v_x)\sin 2\vartheta + (u_x - v_y)\cos 2\vartheta = 0,$$
$$u_x + v_y = 0.$$

The observation that classes $F(\mathfrak{H}_2)$ exhaust all the solutions of the system considerably simplifies the solutions of the boundary-value problems of plasticity [8].

Let us now pass to the discussion of the Lax theorem and the problem of interaction of simple waves which are not necessarily regular.

Theorems 4 and 3 do not apply, in general, to the classes F and $F(\mathfrak{G}_r)$, $r > 2$. Let us prove the following theorem:

THEOREM 6 (cf. [10]). *The system* (1.1), $n = 2$, $l \geqslant 2$ *is strongly hyperbolic in the region* $T \subset H^l$ *if, and only if, all its solutions are of the class* $F(\mathfrak{G}_l)$, *where* $\mathfrak{G}_l = T$.

If the system is hyperbolic, then all its solutions are of the class F.

Proof. The system (1.1), $n = 2, l \geqslant 2$ is strongly hyperbolic if, and only if, the following conditions are satisfied:

(1) There exist l pairwise linearly independent characteristic vectors $\overset{1}{\lambda}(u), \ldots, \overset{l}{\lambda}(u)$, $u \in T$, and

(2) Vectors $\underset{1}{\gamma}(u), \ldots, \underset{l}{\gamma}(u)$, $u \in T$ such that $\underset{\sigma}{\gamma} \leftrightarrow \overset{\sigma}{\lambda}$ are linearly independent.

It follows that if the system is strongly hyperbolic in T, then T represents the manifold $\mathfrak{G}_l$.

Denote by $\mathfrak{M}$ a $2l$-dimensional space of matrices M_i^j, $j = 1, \ldots, l$; $i = 1, 2$. Let $J(u) \subset \mathfrak{M}$ denote the subspace of integral elements of the system. Then,

$$\dim J(u) = l.$$

Let $Q_1(u) \subset J(u)$ denote the set of integral elements of the form (3.4) for $\mathfrak{G}_l = T$. From the linear independence of vectors $\underset{1}{\gamma}, \ldots, \underset{l}{\gamma}$, it follows that

$$\dim Q_1(u) = l,$$

and hence $Q_1 = J$, which concludes the first part of the proof. The second part follows in a similar way from the fact that for an arbitrary hyperbolic system (1.1), $n = 2$, $l \geqslant 2$, the characteristic vectors stretch H^l in spite of T not being, in general, the manifold $\mathfrak{G}_l$.

Theorem 6 yields the conclusion that the Lax theorem may be formulated as follows:

THEOREM 7. *For arbitrary system* (1.1) *the classes* $F(\mathfrak{G}_r)$, $l \leqslant r$, *have the o-property.*

The proof is the simple consequence of the fact that for each $\mathfrak{G}_r$ the construction of $u(x) \in F(\mathfrak{G}_r)$ reduces to the solution of some strongly hyperbolic system with r unknown functions. If the manifold $\mathfrak{G}_r$ is given:

$$u = U(\mu^1, \ldots, \mu^r)$$

in the way that $U_{k^i} = \underset{i}{\gamma}(\mu) \leftrightarrow \underset{i}{\lambda}(\mu)$, then each solution $u \in F(\mathfrak{G}_r)$ is of the form:

$$u(x) = U\big(\mu^1(x), \ldots, \mu^l(x)\big),$$

where $\operatorname{grad}\mu^s(x) = \alpha^s(x)\,\lambda^s\big(\mu(x)\big)$. Indeed, only in that case we have

$$u^j_{x^i} = U^j_{u^s}\mu^s_{x^i} = \sum_{s=1}^{r} \alpha^s \underset{s}{\gamma^j} \underset{s}{\lambda_i^s}.$$

Hence for the construction of the solution $u \in F(\mathfrak{G}_r)$, we have to solve the following strongly hyperbolic system

$$\left(\underset{s}{c}(\mu), \operatorname{grad} \mu^s\right) = 0, \quad s = 1, \ldots, r,$$

where $\left(\underset{1}{c}(\mu), \overset{2}{\lambda}(\mu)\right) = \left(\underset{2}{c}(\mu), \overset{1}{\lambda}(\mu)\right) = 0$, what ends the proof of the Theorem.

It is easily observed that the theorem does not hold for the class F. To this end, let us consider a simple system

$$\begin{aligned} u^1_{x^1} + u^1 u^1_{x^2} - u^2 u^3_{x^1} &= 0, \\ u^2_{x^1} - u^2 u^2_{x^2} - u^1 u^3_{x^1} &= 0, \\ u^3_{x^3} &= 0. \end{aligned} \tag{3.7}$$

The system is strongly hyperbolic within the regions $T_\varepsilon \subset H^3$ defined by the condition $(u^1)^2 + (u^2)^2 > \varepsilon$. All T_ε are the $\mathfrak{G}_3$-manifolds. The system is hyperbolic along the line $u^1 = 0$, $u^2 = 0$ but ceases to be strongly hyperbolic. It follows from the observation of the system (3.7) to have the following characteristic vectors:

$$\underset{1}{\gamma} = (1, 0, 0) \leftrightarrow \underset{1}{\lambda} = (-u^1, 1),$$

$$\underset{2}{\gamma} = (0, 1, 0) \leftrightarrow \underset{2}{\lambda} = (u^2, 1),$$

$$\underset{3}{\gamma} = (u^2, u^1, 1) \leftrightarrow \underset{3}{\lambda} = (1, 0).$$

Thus all the solutions are of the class F. An example of solution of the system (3.7) such that

$$u(x) \in F(T_\varepsilon) \cap F, \quad \varepsilon > 0, \tag{3.8}$$

$u(x)$ not being, however, the interaction of independent simple waves, is the solution

$$u^1 = \underset{1}{\mu}(x^1, x^2), \quad u^2 = \underset{2}{\mu}(x^1, x^2), \quad u^3 = 0 \tag{3.9}$$

which is determined in the neighbourhood of the segment (1/2, 3/2) of the x^1-axis as follows:

$$\underset{1}{\mu}(x^1, x^2) = \text{const} = \alpha \text{ along the straight lines } x^1\alpha - x^2 = 0,$$

$$\underset{2}{\mu}(x^1, x^2) = \text{const} = \alpha \text{ along the straight lines } (x^1 - 2)\alpha + x^2 = 0.$$

The solution is constructed of the integral elements

$$L^j_i = -\underset{1}{\gamma}^j \overset{1}{\lambda}_i + \underset{2}{\gamma}^j \overset{2}{\lambda}_i,$$

where

$$\underset{1}{\gamma} = (1, 0, 0) \leftrightarrow \overset{1}{\lambda}(x) = \left(-\underset{1}{\mu}(x), 1\right) = a \operatorname{grad} \underset{1}{\mu},$$

$$\underset{2}{\gamma} = (0, 1, 0) \leftrightarrow \overset{2}{\lambda}(x) = \left(\underset{2}{\mu}(x), 1\right) = b \operatorname{grad} \underset{2}{\mu}.$$

The vectors $\overset{1}{\lambda}(x^1, x^2)$, $\overset{2}{\lambda}(x^1, x^2)$ are linearly independent for $x^2 \neq 0$, and linearly dependent for $x^2 = 0$. It instantaneously follows that the solution (3.9) satisfies the condition (3.8) but is not an interaction of independent simple waves.

It may also be verified that the solution of the Eqs. (3.7) given by

$$u^1 = \begin{cases} \underset{1}{\mu}(x^1, x^2), & x^2 \geqslant 0 \\ 0 & x^2 < 0 \end{cases}, \quad u^2 = \begin{cases} \underset{2}{\mu}(x^1, x^2), & x^2 \geqslant 0 \\ 0, & x^2 < 0 \end{cases}, \quad u^3 = 0,$$

satisfies the conditions (1.2), (1.3) for $\omega = 0$ but does not possess the ω-property.

References

1. R. COURANT, K. O. FRIEDRICHS, *Supersonic flow and shock waves*, New York 1948.
2. P. D. LAX, *Hyperbolic systems of conservation laws*. II. Comm. Pure and Applied Math., **10**, 1957.
3. M. BURNAT, *Theory of simple waves for non-linear systems of partial differential equations and applications to gasdynamic*, Arch. Mech. Stos., **18**, 4, 1966.
4. M. BURNAT, *The method of solution of hyperbolic systems by means of combining simple waves*, Fluid Dynamics Transactions, **3**, 1967.
5. M. BURNAT, *Hyperbolic double waves*, Bull. Acad. Polon. Sci., Série Sci. Techn., **16**, 10, 1968.
6. M. BURNAT, *The method of characteristics and Riemann invariants for multidimensional hyperbolic systems* [in Russian], Math. Sibirian Journal, **11**, 2, 1970.
7. M. BURNAT, *The method of Riemann invariants for multidimensional nonelliptic systems*, Bull. Acad. Polon. Sci., Série Sci. Techn., **17**, 11–12, 1969.
8. M. BURNAT, *The method of Riemann invariants and its applications to the theory of plasticity*, Arch. Mech. Stos., Part I, **23**, 6, 1971. Part II, **24**, 6, 1972.
9. M. BURNAT, *Geometrical methods in fluid mechanics*, Fluid Dynamics Transactions, **6**, Part I, 1971.
10. Z. PERADZYŃSKI, *On algebraic aspects of the generalised Riemann invariants method*, Bull. Acad. Polon. Sci., Série Sci. Techn., **18**, 4, 1970.
11. Z. PERADZYŃSKI, *On certain classes of exact solutions for gasdynamics equations*, Arch. Mech. Stos., **24**, 2, 1972.
12. Z. PERADZYŃSKI, *Non-linear plane k-waves and Riemann invariants*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 9, 1971.

UNIVERSITY OF WARSAW.

Received November 23, 1972.