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Uniqueness theorem for stress equations of isochoric motions of linear elasticity

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A UNIQUENESS theorem for the stress equations of motion of the linear elasticity was established in [1] under the assumptions that the density function is strictly positive and the elastic energy is positive definite. This note gives an extension of this theorem to an incompressible elastic solid.

Introduction

A UNIQUENESS theorem for the stress equations of motion of linear elasticity was established in [1] under the assumptions that the density field is strictly positive and the elasticity field is symmetric and positive semi-definite. For an isotropic solid whose elastic properties are described by the Young modulus E and the Poisson ratio ν, these assumptions read: $\varrho > 0$, $E > 0$, $-1 < \nu < 1/2$, where ϱ stands for the density of the solid.

In this note we prove a uniqueness theorem for the stress equations of motion under the assumptions $\varrho > 0$, $E > 0$, $\nu = 1/2$, i.e. for an incompressible solid.

We believe that up to date no uniqueness theorem for an incompressible body subject to motion has been proved (cf. [2]). It was shown in [3 and 4] that for an incompressible body undergoing static deformations and for the displacement problem the stress tensor is determined to within a uniform pressure only.

Uniqueness theorem for stress equations of motion of a linear incompressible solid

Assume that an incompressible elastic body occupies three-dimensional region V bounded by a regular surface S. Let the Young modulus E and the density ϱ be strictly positive and smooth functions of position x.

In [1] it was established that the stress field of linear elastodynamics is characterized by a single tensorial equation of motion. If we set $\nu = 1/2$ in this equation, we obtain

$$(1)\qquad E^{-1}(3\ddot{\sigma}_{ij}-\ddot{\sigma}_{kk}\delta_{ij})-(\varrho^{-1}\sigma_{ik,k})_{,j}-(\varrho^{-1}\sigma_{jk,k})_{,i}-(\varrho^{-1}f_i)_{,j}-(\varrho^{-1}f_j)_{,i}=0,\qquad (x,t)\in V\times(0,\infty)\ (^1).$$

Assume now that $\sigma_{ij}(x, t)$ satisfies the following initial and boundary conditions

$$(2)\qquad \sigma_{ij}(x,0)=\sigma^0_{ij}(x),\qquad \dot{\sigma}_{ij}(x,0)=\dot{\sigma}^0_{ij}(x),\qquad x\in V,$$

$$(3)\qquad \sigma_{ik}(x,t)n_k(x)=q_i(x,t),\qquad (x,t)\in S\times(0,\infty),$$

(1) Equation (1) for $i = j$ is compatible with isochoric motions of an isotropic solid subject to the body forces f_i.

where $\sigma_{ij}^0(x)$, $\dot{\sigma}_{ij}^0(x)$, $q_i(x, t)$ are prescribed functions and $n_k(x)$ is the outward unit normal to S.

We are to prove the following

THEOREM. *If $E > 0$ and $\varrho > 0$, then there exists at most one tensor $\sigma_{ij}(x, t)$ satisfying Eq. (1) in the four-dimensional region $V\times(0, \infty)$ and subject to the conditions (2) and (3).*

Proof. It is sufficient to take into account the homogeneous equation associated with Eq. (1) ($f_i = 0$):

$$\text{(4)} \qquad E^{-1}(3\ddot{\sigma}_{ij}-\ddot{\sigma}_{kk}\delta_{ij})-(\varrho^{-1}\sigma_{ik,k})_{,j}-(\varrho^{-1}\sigma_{jk,k})_{,i} = 0,$$

and show that Eq. (4) subject to the homogeneous initial and boundary conditions

$$\text{(5)} \qquad \sigma_{ij}(x, 0) = 0, \quad \dot{\sigma}_{ij}(x, 0) = 0, \quad x \in V,$$

$$\text{(6)} \qquad \sigma_{ik}(x, t)n_k(x) = 0, \quad (x, t) \in S\times(0, \infty),$$

has only zero solution

$$\sigma_{ij}(x, t) = 0 \quad \text{for} \quad (x, t) \in V\times(0, \infty).$$

To this end we multiply Eq. (4) by $\dot{\sigma}_{ij}$ and integrate the result over the domain $V\times(0, t)$.

Using the following identities

$$\ddot{\sigma}_{ij}\dot{\sigma}_{ij} = \frac{1}{2}\frac{\partial}{\partial t}(\dot{\sigma}_{ij}\dot{\sigma}_{ij}), \quad \ddot{\sigma}_{kk}\dot{\sigma}_{ii} = \frac{1}{2}\frac{\partial}{\partial t}(\dot{\sigma}_{ii}\dot{\sigma}_{kk}),$$

$$(\varrho^{-1}\sigma_{ik,k})_{,j}\dot{\sigma}_{ij} = (\varrho^{-1}\sigma_{ik,k}\dot{\sigma}_{ij})_{,j}-\varrho^{-1}\sigma_{ik,k}\dot{\sigma}_{ij,j},$$

$$(\varrho^{-1}\sigma_{jk,k})_{,i}\dot{\sigma}_{ij} = (\varrho^{-1}\sigma_{jk,k}\dot{\sigma}_{ij})_{,i}-\varrho^{-1}\sigma_{jk,k}\dot{\sigma}_{ij,i},$$

$$\int_V (\varrho^{-1}\sigma_{ik,k}\dot{\sigma}_{ij})_{,j}dV = \int_S \varrho^{-1}\sigma_{ik,k}\dot{\sigma}_{ij}n_j dS = 0 \text{ by virtue of (6),}$$

we obtain

$$\text{(7)} \qquad \int_V dV \int_0^t d\tau\left[E^{-1}\frac{\partial}{\partial\tau}\left(\frac{3}{2}\dot{\sigma}_{ij}\dot{\sigma}_{ij}-\frac{1}{2}\dot{\sigma}_{ii}\dot{\sigma}_{kk}\right)+\varrho^{-1}(\sigma_{ik,k}\dot{\sigma}_{ij,j}+\sigma_{jk,k}\dot{\sigma}_{ij,i})\right] = 0$$

or

$$\text{(8)} \qquad \int_V dV \int_0^t d\tau\frac{\partial}{\partial\tau}[(2E)^{-1}(3\dot{\sigma}_{ij}\dot{\sigma}_{ij}-\dot{\sigma}_{ii}\dot{\sigma}_{kk})+\varrho^{-1}\sigma_{ij,j}\sigma_{ik,k}] = 0.$$

Equation (8), by virtue of the homogeneous initial conditions (5), reduces to

$$\text{(9)} \qquad \int_V dV[(2E)^{-1}(3\dot{\sigma}_{ij}\dot{\sigma}_{ij}-\dot{\sigma}_{ii}\dot{\sigma}_{kk})+\varrho^{-1}\sigma_{ij,j}\sigma_{ik,k}] = 0.$$

Define now the traceless tensor s_{ij}:

$$\text{(10)} \qquad s_{ij} \doteq \sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}, \quad s_{ii} = 0.$$

Then

$$\dot{\sigma}_{ij} = s_{ij}+\frac{1}{3}\dot{\sigma}_{kk}\delta_{ij}$$

and

$$\dot{\sigma}_{ij}\dot{\sigma}_{ij} = \dot{s}_{ij}\dot{s}_{ij} + \frac{1}{3}\dot{\sigma}_{kk}\dot{\sigma}_{ll}. \tag{11}$$

Substituting (11) into (9) we get

$$\int_V dV[3(2E)^{-1}\dot{s}_{ij}\dot{s}_{ij} + \varrho^{-1}\sigma_{ij,j}\sigma_{ik,k}] = 0. \tag{12}$$

Since $E(x) > 0$, $\varrho(x) > 0$, Eq. (12) implies

$$\dot{s}_{ij} = 0 \quad \text{for} \quad (x, t) \in V \times (0, \infty), \tag{13}$$

$$\sigma_{ij,j} = 0 \quad \text{for} \quad (x, t) \in V \times (0, \infty). \tag{14}$$

From (13) we conclude that s_{ij} must be constant in time. By virtue of (10) and (5), we have

$$s_{ij}(x, 0) = 0, \tag{15}$$

thus

$$s_{ij}(x, t) = 0, \quad (x, t) \in V \times (0, \infty), \tag{16}$$

i.e.

$$\sigma_{ij}(x, t) = 0 \quad \text{for} \quad i \neq j,\ (x, t) \in V \times (0, \infty). \tag{17}$$

On the other hand, Eqs. (10) and (16) lead to the system

$$\begin{aligned} 2\sigma_{11} - \sigma_{22} - \sigma_{33} &= 0, \\ -\sigma_{11} + 2\sigma_{22} - \sigma_{33} &= 0, \\ -\sigma_{11} - \sigma_{22} + 2\sigma_{33} &= 0, \end{aligned} \tag{18}$$

with the coefficient determinant

$$W = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0.$$

If $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ is to be treated as an unknown three-dimensional vector satisfying Eqs. (18), the components of this vector satisfy the relations

$$\sigma_{11} = \sigma_{22} = \sigma_{33}, \tag{19}$$

and it is readily seen that the set of simultaneous algebraic equations (19) is equivalent to the system (18).

Consider now Eqs. (14) which can be rewritten in the form

$$\begin{aligned} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} &= 0, \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} &= 0, \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} &= 0. \end{aligned} \tag{20}$$

By virtue of (17), Eq. (20) reduces to

$$\sigma_{11,1} = 0, \quad \sigma_{22,2} = 0, \quad \sigma_{33,3} = 0. \tag{21}$$

Thus

$$\sigma_{11} = C_1(x_2, x_3, t) \tag{22}$$

$$\sigma_{22} = C_2(x_1, x_3, t) \tag{23}$$

$$\sigma_{33} = C_3(x_1, x_2, t), \tag{24}$$

where C_i $(i = 1, 2, 3)$ are arbitrary functions. If we combine (22)–(24) with (19) we are led to

$$C_1(x_2, x_3, t) = C_2(x_1, x_3, t) \equiv C_{\mathrm{I}}(x_3, t), \tag{25}$$

$$C_1(x_2, x_3, t) = C_3(x_1, x_2, t) \equiv C_{\mathrm{II}}(x_2, t), \tag{26}$$

$$C_{\mathrm{I}}(x_3, t) = C_{\mathrm{II}}(x_2, t) \equiv C(t), \tag{27}$$

where $C_{\mathrm{I}}, C_{\mathrm{II}}$ and C are also arbitrary functions. Therefore, Eqs. (19) and (21) are satisfied if, and only if,

$$\sigma_{ik}(x, t) = C(t)\,\delta_{ik} \quad \text{for} \quad (x, t) \in V \times (0, \infty), \tag{28}$$

where $C(t)$ is a function of time only.

Inserting (28) in the boundary conditions (6) we get

$$C(t)n_i(x) = 0 \quad \text{for} \quad (x, t) \in S \times (0, \infty). \tag{29}$$

Since $|n_i| = 1$, Eq. (29) implies that

$$|C(t)| = 0, \quad \text{or} \quad C(t) \equiv 0. \tag{30}$$

Hence, by virtue of (28) and (17), we obtain

$$\sigma_{ik}(x, t) = 0 \quad \text{for} \quad (x, t) \in V \times (0, \infty).$$

This completes the proof of the theorem.

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