CHAPTER XIV.

QUADRATURE, ETC. (III)

SURFACE INTEGRALS, AREALS, CORRESPONDING CURVES.

451. **Use of Second Order Infinitesimals as Elements of Area. "Surface Integrals," Centroids, etc.**

For many purposes it is found desirable, and often necessary, to use for the element of area a second order infinitesimal.

Suppose, for instance, we desire to find the mass of the area bounded by a given curve, the x-axis and a pair of ordinates, where there is a distribution of surface density over the area, not uniform, but represented at any point by $\sigma=\phi(x, y)$, say, where x, y are the coordinates of the point in question.

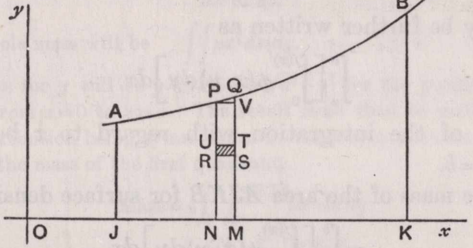

Fig. 84.

Let Ox, Oy be the coordinate axes, AB any arc of the curve whose equation is

$$y=f(x),$$

$\{a, f(a)\}$ and $\{b, f(b)\}$ the coordinates of the points A, B upon it, AJ and BK the ordinates of A and B. Let PN, QM be any contiguous ordinates of the curve, and x, $x+\delta x$ the abscissae of the points P, Q. Let R, U be contiguous points on the ordinate of P, their ordinates being y, $y+\delta y$; and we shall suppose δx, δy to be small quantities of the first order of smallness.

Draw RS, UT, PV parallel to the x-axis. Thus the area of the rectangle $RSTU$ is $\delta x\,\delta y$, and its mass may be regarded as $\phi(x, y)\,\delta x\,\delta y$ to the second order of smallness.

Then the mass of the strip $PNMV$ may be written

$$Lt_{\delta y=0}[\Sigma\phi(x, y)\,\delta y]\,\delta x,$$

and in conformity with the notation of the Integral Calculus may be expressed as

$$\left[\int\phi(x, y)\,dy\right]\delta x$$

between the limits $y=0$ and $y=f(x)$.

In performing this integration with regard to y, x is to be regarded as constant, for we are finding the limit of the sum of the masses of all elements in the elementary strip PM, parallel to the y-axis, for which x retains the same value, *i.e.* we are finding the mass of the strip PM.

If then we search for the mass of the area $AJKB$, all such strips as the above must now be summed which lie between the ordinates AJ, BK, and the result may be written

$$Lt_{\delta x=0}\Sigma\left[\int_0^{f(x)}\phi(x, y)\,dy\right]\delta x,$$

which may be further written as

$$\int_a^b\left[\int_0^{f(x)}\phi(x, y)\,dy\right]dx,$$

the limits of the integration with regard to x being from $x=a$ to $x=b$.

Thus the mass of the area $AJKB$ for surface density $\phi(x, y)$

$$=\int_a^b\left[\int_0^{f(x)}\phi(x, y)\,dy\right]dx.$$

452. **Notation.**

This will be written

$$\int_a^b\int_0^{f(x)}\phi(x, y)\,dx\,dy,$$

the elements dx, dy being written in the reverse order to that in which they occur in the previous expression, and it will be remembered that the *right-hand one refers to the first integration*, and the left-hand one to the second. It has already been stated (Art. 363) that we shall throughout the book adopt this order.

If we put $\sigma \equiv \phi(x, y) = 1$, the result of our integration will be to find the area.

Thus,
$$\text{Area} = \int_a^b \int_0^{f(x)} dx\, dy$$
$$= \int_a^b f(x) dx$$
$$= \int_a^b y\, dx, \text{ as before};$$

or, in the case of the area being bounded by two curves,

$$y = \phi(x), \quad y = \psi(x), \text{ as in Art. 395},$$
$$\text{Area} = \int_a^b \int_{\psi(x)}^{\phi(x)} dx\, dy$$
$$= \int_a^b [\phi(x) - \psi(x)] dx.$$

Ex. If the surface density of a circular disc bounded by $x^2+y^2=a^2$ be given to vary as the square of the distance from the y-axis, find the mass of the disc.

Here we have μx^2 for the density of the element $\delta x\, \delta y$, and its mass is therefore
$$\mu x^2 \delta x\, \delta y,$$
and the whole mass will be
$$\iint \mu x^2 dx\, dy.$$

The limits for y will be $y=0$ to $y=\sqrt{a^2-x^2}$ for the positive quadrant and for x from $x=0$ to $x=a$. The result must then be multiplied by 4, for the distribution being symmetrical in the four quadrants, the mass is four times the mass of the first quadrant.

Thus,
$$\text{Mass} = 4\int_0^a \int_0^{\sqrt{a^2-x^2}} \mu x^2 dx\, dy$$
$$= 4\mu \int_0^a x^2 \Big[y \Big]_0^{\sqrt{a^2-x^2}} dx$$
$$= 4\mu \int_0^a x^2 \sqrt{a^2-x^2} dx.$$

Putting $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$, we have

$$\text{Mass} = 4\mu a^4 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta$$
$$= 4\mu a^4 \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{2\Gamma(3)} = 4\mu a^4 . \frac{\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{2.2} = \frac{\pi\mu a^4}{4}.$$

453. Other Uses of Double Integration.

The same process may be used for many other purposes, of which we give a few illustrative examples, which will serve to indicate to the student the field of investigation now open to him.

Ex. Find the statical moment of a quadrant of the ellipse

$$x^2/a^2+y^2/b^2=1$$

about the y-axis, the surface density being supposed uniform.

Here each element of area $\delta x\,\delta y$ is to be multiplied by its surface density σ (which is by hypothesis constant in the case supposed), and by its distance from the y-axis; the sum of such elementary quantities is then to be found over the whole quadrant. The limits of integration will be from $y=0$ to $y=\frac{b}{a}\sqrt{a^2-x^2}$ for y; and from $x=0$ to $x=a$ for x. Thus we have

$$\text{Moment}=\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\sigma x\,dx\,dy$$

$$=\frac{\sigma b}{a}\int_0^a x\sqrt{a^2-x^2}\,dx$$

$$=\frac{\sigma b}{a}\left[-\frac{(a^2-x^2)^{\frac{3}{2}}}{3}\right]_0^a=\frac{\sigma b a^2}{3}$$

$$=M\frac{4a}{3\pi},$$

where M is the mass of the quadrant, *i.e.*

$$\frac{\pi ab}{4}\sigma.$$

454. Centroid of a Plane Area.

The formulae proved in Analytical Statics for the coordinates of the centroid of a number of masses $m_1, m_2, m_3, \ldots$ at points (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., are

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},\quad \bar{y}=\frac{\Sigma my}{\Sigma m}.$$

We may apply these to find the coordinates of the centroid of a given area on which there is any proposed distribution of surface density.

Let σ be the surface density at a given point, which may be either a constant, as for a uniform distribution, or a given function of x and y. Then the mass of the element $\delta x\,\delta y$ is $\sigma\,\delta x\,\delta y$ and

$$\bar{x}=\frac{\iint\sigma x\,dx\,dy}{\iint\sigma\,dx\,dy}.$$

Similarly, $$\bar{y} = \frac{\iint \sigma y\,dx\,dy}{\iint \sigma\,dx\,dy},$$

the limits in each case being determined so that the summation will be effected for the whole area in question.

Ex. Find the centroid of the elliptic quadrant of the example in the last article.

It was proved there that

$$\iint \sigma x\,dx\,dy = \frac{\sigma b a^2}{3} = M\frac{4a}{3\pi}$$

and $$\iint \sigma\,dx\,dy = \text{mass of quadrant} = M;$$

$$\therefore\ \bar{x} = \frac{4a}{3\pi}.$$

Also $$\iint \sigma y\,dx\,dy = \sigma\int_0^a \left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{\sigma}{2}\frac{b^2}{a^2}\int_0^a (a^2 - x^2)\,dx$$

$$= \frac{\sigma b^2}{2a^2}\left(a^3 - \frac{a^3}{3}\right) = \frac{\sigma a b^2}{3} = M\frac{4b}{3\pi};$$

$$\therefore\ \bar{y} = \frac{4b}{3\pi}.$$

Hence the coordinates of the centroid are $\frac{4a}{3\pi}$, $\frac{4b}{3\pi}$.

455. Moment of Inertia.

When every element of mass of a given body is multiplied by the square of its distance from a given line, the limit of the sum of such products is called the Moment of Inertia with regard to the line.

Ex. 1. Find the moment of inertia of the quadrant of an ellipse about the y-axis, again taking uniform surface density

Here we have to multiply each element of mass, viz. $\sigma\,\delta x\,\delta y$, by x^2, and then integrate as before.

$$\text{Moment of Inertia} = \iint \sigma x^2\,dx\,dy$$

$$= \int \sigma x^2 \Big[y\Big]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \sigma\frac{b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

$$= \sigma\frac{b}{a}\frac{\pi a^4}{16},$$ this integral having been worked out in the example of Art. 452,

$$= \pi\frac{a^3 b\sigma}{16} = M\frac{a^2}{4}, \quad \text{since } M = \frac{\pi a b\sigma}{4}.$$

Ex. 2. Find the moment of inertia of the portion of the parabola $y^2=4ax$, bounded by the axis and the latus rectum, about the x-axis, supposing the surface density at each point to vary as the n^{th} power of the abscissa.

Here the mass-element is $\mu x^n \delta x \, \delta y$, μ being a constant, and the moment of inertia is,

$$Lt \sum \mu y^2 x^n \delta x \, \delta y \quad \text{or} \quad \mu \iint y^2 x^n \, dx \, dy,$$

where the limits for y are from $y=0$ to $2\sqrt{ax}$, and for x from 0 to a.

We thus get

$$\text{Mom. of In.} = \frac{\mu}{3}\int_0^a \Big[y^3 \Big]_0^{2\sqrt{ax}} x^n \, dx = \frac{\mu}{3}\int_0^a 8a^{\frac{3}{2}} x^{n+\frac{3}{2}} \, dx$$

$$= \frac{8\mu}{3} a^{\frac{3}{2}} \left[\frac{x^{n+\frac{5}{2}}}{n+\frac{5}{2}} \right]_0^a = \frac{16\mu}{3(2n+5)} a^{n+4}$$

Again, the *Mass* of this portion of the parabola is given by

$$M = \int_0^a \int_0^{2\sqrt{ax}} \mu x^n \, dx \, dy = \mu \int_0^a \Big[y \Big]_0^{2\sqrt{ax}} x^n \, dx$$

$$= 2\mu a^{\frac{1}{2}} \int_0^a x^{n+\frac{1}{2}} \, dx = \frac{4\mu}{2n+3} a^{n+2}.$$

Thus we have

$$\text{Moment of Inertia about } Ox = \frac{4}{3}\frac{2n+3}{2n+5} Ma^2.$$

Examples.

1. In the first quadrant of the circle $x^2+y^2=a^2$ the surface density varies at each point as xy.

Find (i) the mass of the quadrant,
(ii) its centroid,
(iii) its moment of inertia about the y-axis.

2. Work out the corresponding results for the portion of the parabola $y^2=4ax$ bounded by the axis and the latus rectum, the surface density varying as $x^p y^q$.

3. Find the centroid of a fine rod of uniform sectional area and of which the line-density varies as the n^{th} power of the distance from one end. Also its moment of inertia about that end, about the other end, and about the middle point.

4. Find the centroid of the triangle bounded by the lines $y=mx$, $x=a$ and the x axis when the surface density at each point varies as the square of the distance from the origin. Also find the moment of inertia about the y-axis.

5. Find the centroid of

(i) either of the areas bounded by the circle $(x-a)^2+y^2=a^2$ and the parabola $y^2=ax$;

(ii) the centroid of the area bounded by the parabolas

$$y^2=4ax, \quad x^2=4by;$$

(iii) the centroid of the area bounded by

$$y^2=4ax, \quad y=2x,$$

the surface density being uniform in each case.

6. Find the moment of inertia of a triangle of uniform surface density
   (i) about one of its sides;
   (ii) about an axis perpendicular to its plane through an angular point.

456. Polar Coordinates. Second Order Element.

For polar curves it is desirable to use for our element of area a second order infinitesimal of different form.

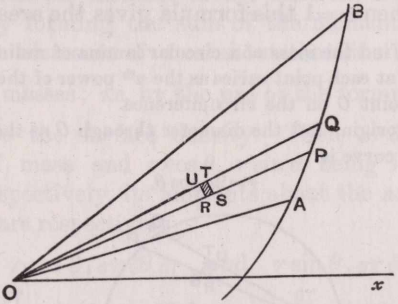

Fig. 85.

Let OP, OQ be two contiguous radii vectores of the curve $r=f(\theta)$; Ox the initial line. Let θ, $\theta+\delta\theta$ be the vectorial angles of the points P, Q on the curve. Draw two circular arcs RU, ST cutting the radii OP, OQ, with centre O and radii r $r+\delta r$ respectively, and let δr, $\delta\theta$ be small quantities of the first order of smallness.

Then area $RSTU$ = sector OST − sector ORU

$$=\tfrac{1}{2}(r+\delta r)^2\,\delta\theta-\tfrac{1}{2}r^2\,\delta\theta$$

$$=r\,\delta\theta\,\delta r \text{ to the second order.}$$

And to this order $RSTU$ may therefore be considered a rectangle of sides δr ($=RS$) and $r\,\delta\theta$ (=arc RU).

Thus, if the surface density at each point $R(r, \theta)$ be $\sigma=\phi(r, \theta)$, the mass of the element $RSTU$ is (to second order quantities) $\sigma r\,\delta\theta\,\delta r$, and the mass of the elementary sector OPQ is

$$Lt_{\delta r=0}\,[\Sigma\sigma r\,\delta r]\,\delta\theta$$

the summation being effected for all elements from $r=0$ $r=f(\theta)$,

i.e. $$\left[\int_0^{f(\theta)} \sigma r\,dr\right]\delta\theta,$$

in which integration θ is to be regarded as constant; and taking the limit of the sum of the elementary sectors for infinitesimal values of $\delta\theta$ between any specified radii vectores $\theta=\alpha$ and $\theta=\beta$, we get the mass of the sectorial area OAB

$$=\int_\alpha^\beta\left[\int_0^{f(\theta)} \sigma r\,dr\right]d\theta,$$

or, as we have agreed to write it (Art. 360),

$$\int_\alpha^\beta\int_0^{f(\theta)} \sigma r\,d\theta\,dr.$$

Obviously when $\sigma=1$ this formula gives the area of the sector.

457. Ex. 1. Find the mass of a circular lamina of radius a in which the surface density at each point varies as the n^{th} power of the distance of that point from a point O on the circumference.

Taking O as origin, and the diameter through O as the initial line, the equation of the curve is

$$r=2a\cos\theta.$$

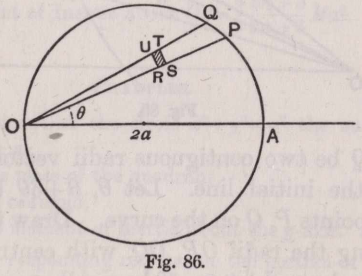

Fig. 86.

Then we have for the density at a point R distant r from O, $\sigma\equiv\mu r^n$ where μ is a constant. The mass of the element $RSTU=\mu r^n\ (r\,\delta\theta\,\delta r)$. Hence the mass of the circular lamina is

$$M\equiv 2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} \mu r^n r\,d\theta\,dr$$

$$=\frac{2\mu}{n+2}\int_0^{\frac{\pi}{2}}(2a\cos\theta)^{n+2}\,d\theta$$

$$=\frac{2\mu}{n+2}(2a)^{n+2}\frac{n+1}{n+2}\frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{2}{3}$$

or $$\frac{2\mu}{n+2}(2a)^{n+2}\frac{n+1}{n+2}\frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\frac{\pi}{2},$$

according as n is odd or even.

Ex. 2. If the moment of inertia were required about a perpendicular to the plane of the lamina through O, each elementary mass $\mu r^n(r\,\delta\theta\,\delta r)$ is to be multiplied by r^2 before integration. The result merely changes n into $n+2$ in the former work, and writing M for the value found for the mass,

$$\text{Moment of Inertia} = M\frac{n+2}{n+4}\frac{n+3}{n+4}(2a)^2.$$

458. Centroids, etc. Polars.

The distance of the centroid of an area whose boundary is defined by a polar equation, from any straight line in the plane of the area and passing through the pole, may be found, as before (Art. 454). Take the line proposed as the x-axis and a perpendicular through the pole as the y-axis. Then the distance of the centroid from the x-axis is obtained by forming the sum of the moments of the masses of the polar elements of area about that line and dividing by the sum of masses; *i.e.* by the use of the formula $\bar{y} = \dfrac{\Sigma my}{\Sigma m}$.

Let σ be the surface density. Then $\sigma r\,\delta\theta\,\delta r$ being the element of mass and $r\cos\theta$, $r\sin\theta$ being its abscissa and ordinate respectively, its moments about the axes of y and x through O are respectively

$$r\cos\theta\,.\,\sigma r\,\delta\theta\,\delta r \quad\text{and}\quad r\sin\theta\,.\,\sigma r\,\delta\theta\,\delta r.$$

Thus
$$\bar{x} = \frac{\iint r\cos\theta\,.\,\sigma r\,d\theta\,dr}{\iint \sigma r\,d\theta\,dr}, \quad \bar{y} = \frac{\iint r\sin\theta\,.\,\sigma r\,d\theta\,dr}{\iint \sigma r\,d\theta\,dr},$$

the limits to be assigned so that the summations for all elements are thereby effected.

459. Ex. 1. Find the centroid of the circular lamina of Art. 457 when the surface density is μr^n.

Obviously the centroid lies on the diameter through O. Hence $\bar{y}=0$.

To find $\bar{x}$ we have to integrate $2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r\cos\theta\,.\,\mu r^n r\,d\theta\,dr$, and then to divide by M, which has been found before (Art. 457, Ex. 1).

$$\text{This integral} = \frac{2\mu}{n+3}\int_0^{\frac{\pi}{2}}(2a\cos\theta)^{n+3}\cos\theta\,d\theta = \frac{2\mu}{n+3}(2a)^{n+3}\int_0^{\frac{\pi}{2}}\cos^{n+4}\theta\,d\theta$$

$$\left.\begin{aligned} &= \frac{2\mu}{n+3}(2a)^{n+3}\frac{n+3}{n+4}\frac{n+1}{n+2}\cdots\frac{2}{3}, \quad n \text{ odd},\\ \text{or}\quad &= \frac{2\mu}{n+3}(2a)^{n+3}\frac{n+3}{n+4}\frac{n+1}{n+2}\cdots\frac{1}{2}\frac{\pi}{2}, \quad n \text{ even.}\end{aligned}\right\}$$

$$\left.\begin{aligned}\text{Hence}\qquad \bar{x}&=\frac{n+2}{n+3}\cdot 2a\,\frac{n+3}{n+4}=\frac{n+2}{n+4}\cdot 2a\\ \text{and}\qquad \bar{y}&=0.\end{aligned}\right\}$$

If the centroid of the upper half only of the lamina had been required, we should have had the same value of $\bar{x}$ but for $\bar{y}$ we shall have to evaluate the additional integral

$$\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r\sin\theta\,.\,\mu r^n r\,d\theta\,dr$$

and divide by $\frac{1}{2}M$, where M is the mass found for the whole lamina.

$$\text{This integral}=\frac{\mu}{n+3}\int_0^{\frac{\pi}{2}}(2a\cos\theta)^{n+3}\sin\theta\,d\theta$$

$$=\frac{\mu}{n+3}(2a)^{n+3}\left[\frac{-\cos^{n+4}\theta}{n+4}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\mu}{(n+3)(n+4)}(2a)^{n+3}.$$

$$\left.\begin{aligned}\text{Hence}\qquad \bar{y}&=\frac{n+2}{(n+3)(n+4)}\cdot 2a\,.\,\frac{n+2}{n+1}\,.\,\frac{n}{n-1}\ldots\frac{3}{2},\qquad n\text{ odd},\\ \text{or}\qquad &\frac{n+2}{(n+3)(n+4)}\cdot 2a\,.\,\frac{n+2}{n+1}\,.\,\frac{n}{n-1}\ldots\frac{2}{1}\,.\,\frac{2}{\pi},\qquad n\text{ even}.\end{aligned}\right\}$$

Ex. 2. Find the centroid of a lamina in the form of the cardioide

$$r=a(1+\cos\theta)$$

in the case of uniform surface density.

As the initial line is an axis of symmetry, $\bar{y}$ is evidently $=0$ (see Fig. 82).

To find the abscissa we have

$$\bar{x}=\iint r\cos\theta\,.\,r\,d\theta\,dr\Big/\iint r\,d\theta\,dr,$$

the limits for r being

from $r=0$ to $r=a(1+\cos\theta)$,

and for θ, from $\theta=0$ to $\theta=\pi$

(and double to include the lower half).

$$2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\cos\theta\,.\,r\,d\theta\,dr=2\int_0^{\pi}\cos\theta\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}d\theta$$

$$=\frac{2}{3}a^3\int_0^{\pi}(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta$$

$$=\frac{4}{3}a^3\int_0^{\frac{\pi}{2}}(3\cos^2\theta+\cos^4\theta)\,d\theta$$

$$=\frac{4}{3}a^3\left[3\frac{1}{2}\frac{\pi}{2}+\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right]$$

$$=\frac{4}{3}a^3\,\frac{3\pi}{4}\,.\,\frac{5}{4}=\frac{5}{4}\pi a^3.$$

$$\text{The denominator} = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}}(1+\cos^2\theta)\,d\theta$$

$$= 2a^2\left[\frac{\pi}{2}+\frac{1}{2}\,\frac{\pi}{2}\right] = \frac{3\pi a^2}{2}.$$

Hence $$\bar{x} = \frac{5}{4}\pi a^3\Big/\frac{3\pi a^2}{2} = \frac{5a}{6}.$$

Ex. 3. Calculate the surface integral of μr^{2n} taken over *one loop* of a Bernoulli's Lemniscate.

The curve is $r^2 = a^2\cos 2\theta$ (*Diff. Calc.*, Art. 458).

The surface integral is plainly

$$S \equiv 2\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} \mu r^{2n}\,.\,r\,d\theta\,dr$$

$$= \frac{2\mu}{2n+2}\int_0^{\frac{\pi}{4}}(a^2\cos 2\theta)^{n+1}\,d\theta$$

$$= \frac{1}{2}\,\frac{\mu}{n+1}\,a^{2n+2}\int_0^{\frac{\pi}{2}}\cos^{n+1}\phi\,.\,d\phi, \quad \text{where } \phi = 2\theta,$$

$$= \frac{\mu}{2}\,\frac{a^{2n+2}}{n+1}\,.\,\frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+3}{2}\right)} = \frac{\mu}{4}\,\frac{a^{2n+2}\sqrt{\pi}}{n+1}\,\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)} = \text{etc.} \quad \text{.........(1)}$$

If the moment of inertia be required about an axis perpendicular to the plane through the pole,

$$\text{Mom. In.} = 2\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r^2\,.\,\mu r^{2n} r\,d\theta\,dr$$

$$= \frac{2\mu}{2n+4}\int_0^{\frac{\pi}{4}}(a^2\cos 2\theta)^{n+2}\,d\theta = \frac{\mu}{4}\,\frac{a^{2n+4}\sqrt{\pi}}{n+2}\,\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}$$

$$= Ma^2\,\frac{n+1}{n+2}\,\frac{\left[\Gamma\left(\frac{n+3}{2}\right)\right]^2}{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+4}{2}\right)} = 2Ma^2\,\frac{n+1}{(n+2)^2}\left\{\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\right\}^2, \quad \text{.........(2)}$$

where M is the mass.

If we put $n=0$ in (1), we get the mass M of the loop for uniform surface density μ, viz.

$$M = \frac{\mu}{4}\,a^2\sqrt{\pi}\,\frac{1}{\Gamma\left(\frac{3}{2}\right)} = \mu\,\frac{a^2}{2},$$

and $\mu = 1$ gives the area, viz. $A = \dfrac{a^2}{2}$.

Putting $n=1$ in (1), we have the moment of inertia for a uniform lamina about a perpendicular through the pole to the plane (or the mass for a superficial distribution μr^2), viz.

$$\text{Mom. In.} = \frac{\mu}{4}\frac{a^4}{2}\sqrt{\pi}\frac{\Gamma(\frac{3}{2})}{\Gamma(2)} = \frac{\mu a^4 \pi}{16} = \frac{M\pi a^2}{8}.$$

Similarly $n=2$ in (1) gives the moment of inertia for a superficial distribution μr^2 or the mass for a superficial distribution μr^4, etc.

Examples.

1. Find the centroid of a sector of a circle

 (α) when the surface density is uniform ;

 (β) when the surface density varies as the n^{th} power of the direct distance from the centre.

2. Find the centroid of a circular lamina whose surface density varies as the n^{th} power of the distance from a point O on the circumference.

Find also its moment of inertia

 (1) about the tangent at O;

 (2) about the diameter through O;

 (3) about a perpendicular to the plane through O.

3. (a) Show that the moment of inertia of the triangle of uniform surface density, bounded by the y-axis and the lines

$$y=m_1x+c_1, \quad y=m_2x+c_2$$

about the y-axis, is

$$\frac{M}{6}\left(\frac{c_1-c_2}{m_1-m_2}\right)^2,$$

where M is the mass of the triangle.

(b) Find the moments of inertia of the triangle of uniform surface density, bounded by the lines

$$y=m_1x+c_1, \quad y=m_2x+c_2, \quad y=m_3x+c_3,$$

about the coordinate axes; and show that if M be the mass of the triangle, they are the same as those of equal masses $\frac{M}{3}$ placed at the mid-points of the sides.

4. Find the centre of gravity and the moments of inertia about the coordinate axes of the rectangle $x=a_1$, $x=a_2$, $y=b_1$, $y=b_2$, the surface density being $\sigma=\mu x^p y^q$.

5. If A, B be the moments of inertia of any plane area about a pair of perpendicular axes Ox, Oy in its plane, and C the moment of inertia about an axis through O at right angles to the plane, prove that

$$C=A+B$$

for any law of surface density.

6. Show that the moments of inertia of a uniform ellipse bounded by $x^2/a^2+y^2/b^2=1$ about the major and minor axes are respectively $\frac{Mb^2}{4}$ and $\frac{Ma^2}{4}$, and about a line through the centre and perpendicular to its plane, $M\frac{a^2+b^2}{4}$, M being the mass of the ellipse.

7. Find the area remote from the pole between the circles

$$r=a, \quad r=2a\cos\theta;$$

and assuming a surface density varying inversely as the distance from the pole, find

(1) the centroid;

(2) the moment of inertia about a line through the pole perpendicular to the plane.

8. Find for the area included between the curves

$$y^2=4ax, \quad x^2=4ay,$$

(i) the moment of inertia about the x-axis;

(ii) the moment of inertia about an axis through the origin and at right angles to the plane of the area.

9. Find the coordinates of the centroid of the area bounded by the catenary $y=c\cosh\frac{x}{c}$, an ordinate, and the coordinate axes.

10. If the density at any point of a circular disc whose radius is a vary directly as the distance from the centre and a circle described on a radius as diameter be cut out, prove that the centroid of the remainder will be at a distance $\frac{6a}{5(3\pi-2)}$ from the centre. [MATH. TRIP., 1875.]

460. Trilinears and Areals.

These coordinates are not well adapted for metrical purposes. Their special rôle is the discussion of descriptive properties of curves.

With the usual notation of the trilinear system [Smith's *Conics*, Chapter XIII.], we have

$$a\alpha+b\beta+c\gamma=2\Delta,$$

as an identical relation between the three coordinates α, β, γ of a point, and in the areal system this is replaced by

$$x+y+z=1.$$

The transformation formulae from the one system to the other are

$$x=\frac{a\alpha}{2\Delta}, \quad y=\frac{b\beta}{2\Delta}, \quad z=\frac{c\gamma}{2\Delta}.$$

Variations $d\alpha$, $d\beta$, $d\gamma$ or dx, dy, dz of the coordinates are therefore connected by the equations

$$\left.\begin{aligned} a\,d\alpha+b\,d\beta+c\,d\gamma&=0,\\ dx+\ \ dy+\ \ dz&=0,\end{aligned}\right\}\text{ respectively.}$$

The evaluation of an area for such coordinates is best done by throwing back the homogeneous equation given into a Cartesian form, taking two sides of the triangle of reference as

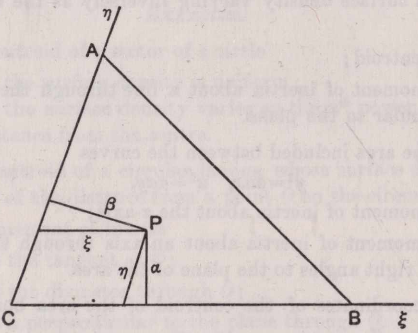

Fig. 87.

coordinate axes. Thus taking CB and CA, sides of the reference triangle, as axes of ξ and η, if ξ, η be the Cartesian coordinates of the point α, β, γ, we obviously have

$$\alpha=\eta\sin C,\quad \beta=\xi\sin C$$

and

$$\begin{aligned}\gamma&=(2\Delta-a\eta\sin C-b\xi\sin C)/c\\ &=\frac{2\Delta}{c}\left(1-\frac{\xi}{a}-\frac{\eta}{b}\right),\end{aligned}$$

and then the evaluation of the area will be obtained by

$$A=-\sin C\int\eta\,d\xi\quad\text{or}\quad\sin C\int\xi\,d\eta\quad\text{or}\quad\sin C\iint d\xi\,d\eta$$

or any of the methods customary for Cartesians.

461. Formulae can, however, be exhibited expressing the area directly in terms of areal or trilinear coordinates for use if necessary.

In the Case of Areals, since x, y, z, the areal coordinates of a point, are linear functions of ξ, η, the Cartesian coordinates with reference to any chosen rectangular axes and $x+y+z=1$, we have

$$\iint d\xi\, d\eta=\lambda\iint dx\, dy \quad \text{or} \quad \mu\iint dy\, dz \quad \text{or} \quad \nu\iint dz\, dx,$$

where λ, μ, ν are determinate constants depending upon the triangle of reference alone. To determine λ we shall apply the first of these formulae to the triangle of reference itself.

If Δ be the area of the triangle of reference,

$$\iint d\xi\, d\eta=\Delta,$$

where the integration is conducted over the triangle.

Now let us evaluate $\iint dx\, dy$ for the triangle.

The limits of y, keeping x constant, are from $y=0$ to $z=0$, *i.e.* to $y=1-x$, and for x from $x=0$ to $x=1$.

Thus $\iint dx\, dy$ for the triangle $=\int_0^1\int_0^{1-x} dx\, dy$

$$=\int_0^1 (1-x)dx=\left[x-\frac{x^2}{2}\right]_0^1=\frac{1}{2};$$

$$\therefore\ \lambda=2\Delta.$$

Hence if $f(x, y, z)=0$ be the equation of a closed curve in areals, its area is

$$2\Delta\iint dx\, dy,$$

the limits of integration being obtained from

$$f(x, y, 1-x-y)=0.$$

The corresponding result for trilinears will be

$$\frac{1}{\sin C}\iint d\alpha\, d\beta,$$

where the limits are to be found from

$$f\left(\alpha,\ \beta,\ \frac{2\Delta-a\alpha-b\beta}{c}\right)=0,$$

$f(\alpha, \beta, \gamma)=0$ being the curve to be considered.

462. Illustrative Cases.

Ex. 1. As a test let us apply this method to find the area of the circum-circle of the triangle of reference, viz. $a^2yz+b^2zx+c^2xy=0$ (in areals).

The result, from elementary considerations, should be

$$\pi R^2 = \pi\left(\frac{abc}{4\Delta}\right)^2, \quad R \text{ being the radius of the circle.}$$

Substituting $1-x-y$ for z, we have

$$(a^2y+b^2x)(1-x-y)+c^2xy=0,$$

$$a^2y+b^2x-a^2y^2-b^2x^2-2ab\cos C\,xy=0,$$

$$a^2y^2+(2ab\cos C\,x-a^2)y=b^2x-b^2x^2,$$

$$y^2+\left(2\frac{b}{a}\cos C\,x-1\right)y+\frac{1}{4}\left(2\frac{b}{a}\cos C\,x-1\right)^2$$

$$=\frac{1}{4}-\frac{b}{a}\cos C\,x+\frac{b^2}{a^2}\cos^2 C\,x^2+\frac{b^2}{a^2}x-\frac{b^2}{a^2}x^2$$

$$=\frac{1}{4}+\frac{bc}{a^2}\cos A\,x-\frac{b^2}{a^2}\sin^2 C\,x^2$$

$$=\frac{1}{4}+\frac{1}{4}\frac{c^2}{a^2}\frac{\cos^2 A}{\sin^2 C}-\frac{b^2}{a^2}\sin^2 C\left(x-\frac{1}{2}\frac{c}{b}\frac{\cos A}{\sin^2 C}\right)^2$$

$$=\frac{1}{4}\operatorname{cosec}^2 A\left[1-4\sin^2 B\sin^2 C\left(x-\frac{1}{2}\frac{\cos A}{\sin B\sin C}\right)^2\right]$$

$$=p^2-q^2(x-r)^2, \text{ say.}$$

The limits for y are therefore

$$-\frac{1}{2}\left(\frac{2b\cos C}{a}x-1\right)\pm\sqrt{p^2-q^2(x-r)^2},$$

and for x, $\qquad r\pm\frac{p}{q}.$

The area $=2\Delta\iint dx\,dy=4\Delta\int\sqrt{p^2-q^2(x-r)^2}\,dx$

$$=4\Delta\cdot\frac{1}{2q}\left[q(x-r)\sqrt{p^2-q^2(x-r)^2}+p^2\sin^{-1}\frac{q(x-r)}{p}\right]_{r-\frac{p}{q}}^{r+\frac{p}{q}}$$

$$=\frac{2\Delta}{q}p^2\left[\sin^{-1}1-\sin^{-1}(-1)\right]=2\pi\Delta\frac{p^2}{q}$$

$$=2\pi\Delta\cdot\frac{1}{4}\frac{\operatorname{cosec}^2 A}{\dfrac{\sin B}{\sin A}\sin C}=\frac{\pi\Delta}{2}\frac{1}{\sin A\sin B\sin C}$$

$$=\frac{\pi\Delta}{2}\cdot\frac{a^2b^2c^2}{(2\Delta)^3}=\pi\left(\frac{abc}{4\Delta}\right)^2,$$

the result to be expected.

Ex. 2. More generally consider the areal equation of an ellipse

$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0.$$

To obtain the integration limits put $z=1-x-y$.

We obtain $\qquad ax^2+2hxy+by^2+2gx+2fy+c=0,$

where

$$a=w+u-2v', \qquad g=-w+v',$$
$$h=w+w'-u'-v', \qquad f=-w+u',$$
$$b=w+v-2u', \qquad c=w.$$

Solving for y,

$$by = -(hx+f) \pm \sqrt{-Cx^2+2Gx-A} = -(hx+f) \pm \sqrt{\frac{G^2-AC}{C} - C\left(x-\frac{G}{C}\right)^2},$$

where $$A = \frac{\partial H}{\partial a},\quad 2G = \frac{\partial H}{\partial g},\quad C = \frac{\partial H}{\partial c},$$

and $H =$ the Hessian, viz. $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}$

The limits for y are

$$\left\{-(hx+f) \pm \sqrt{\frac{G^2-AC}{C} - C\left(x-\frac{G}{C}\right)^2}\right\} \Big/ b,$$

and for x, $$\frac{G}{C} \pm \frac{\sqrt{G^2-AC}}{C}.$$

Writing the radical

$$\sqrt{\frac{G^2-AC}{C} - C\left(x-\frac{G}{C}\right)^2} \text{ as } \sqrt{p^2-q^2(x-r)^2},$$

$$\text{area} = 2\Delta\iint dx\,dy = \pm\frac{4\Delta}{b}\int\sqrt{p^2-q^2(x-r)^2}\,dx$$

$$= \pm\frac{2\Delta}{bq}\left[q(x-r)\sqrt{p^2-q^2(x-r)^2} + p^2\sin^{-1}\frac{q(x-r)}{p}\right]_{r-\frac{p}{q}}^{r+\frac{p}{q}}$$

$$= \pm\frac{2\Delta}{bq}p^2[\sin^{-1}1 - \sin^{-1}(-1)] = \pm 2\pi\Delta\frac{p^2}{bq}.$$

Now $q = \sqrt{C} = \sqrt{ab-h^2} = \sqrt{\Sigma(vw-u'^2) + 2\Sigma(v'w'-uu')} = \sqrt{-K}$,

where $K = \begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$, the "bordered Hessian," and $G^2-AC = -bH$.

Hence $$\frac{p^2}{bq} = \frac{G^2-AC}{bC^{\frac{3}{2}}} = \frac{-H}{(-K)^{\frac{3}{2}}}.$$

Therefore the area sought is $\pm 2\pi\Delta\dfrac{H}{(-K)^{\frac{3}{2}}}$, the positive value to be taken, where

$\Delta \equiv$ area of triangle of reference,

$H \equiv$ the Hessian, viz. $\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}$,

$K \equiv$ the bordered Hessian, viz. $\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$

463. Corresponding Points and Areas.

Let $f(x, y)$ be any closed curve.

Its area (A_1) is expressed by taking the line-integral $-\int y\,dx$ or the line-integral $\int x\,dy$ round the complete contour.

If the coordinates of the current point x, y be connected with those of a second point (ξ, η) by the relations

$$x = m\xi, \quad y = n\eta,$$

this second point will trace out the curve

$$f(m\xi, n\eta) = 0,$$

whose area (A_2) is expressed by the line-integral $-\int \eta\,d\xi$ or the line-integral $\int \xi\,d\eta$ taken round the contour.

And we have

$$A_1 = -\int y\,dx = -\int n\eta\,m\,d\xi = -mn\int \eta\,d\xi = mnA_2,$$

or $$A_1 = \int x\,dy = \int m\xi\,n\,d\eta = mn\int \xi\,d\eta = mnA_2,$$

or, if we use surface integrals,

$$A_1 = \iint dx\,dy = \iint mn\,d\xi\,d\eta = mn\iint d\xi\,d\eta \equiv mnA_2,$$

whence it appears that the area of any closed curve $f(x, y) = 0$ is mn times that of the closed curve $f(mx, ny) = 0$.

464. Ex. 1. Thus, in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ put } \frac{x}{a} = \frac{\xi}{r}, \ \frac{y}{b} = \frac{\eta}{r}.$$

The corresponding point traces out the circle $\xi^2 + \eta^2 = r^2$, and area of the ellipse $= \dfrac{ab}{r^2} \times$ area of circle $= \dfrac{ab}{r^2}\pi r^2 = \pi ab$.

Ex. 2. Find the area of the curve $(m^2x^2 + n^2y^2)^2 = a^2x^2 + b^2y^2$.

Put $mx = \xi$, $ny = \eta$. Then the corresponding curve is

$$(\xi^2 + \eta^2)^2 = \frac{a^2}{m^2}\xi^2 + \frac{b^2}{n^2}\eta^2,$$

or in polars $$r^2 = \frac{a^2}{m^2}\cos^2\theta + \frac{b^2}{n^2}\sin^2\theta,$$

the central pedal of an ellipse, symmetrical about both coordinate axes.

Hence the area of the given curve

$$=\frac{1}{mn}\times \text{area of derived curve}$$

$$=\frac{1}{mn}\,2\int_0^{\frac{\pi}{2}}\left(\frac{a^2}{m^2}\cos^2\theta+\frac{b^2}{n^2}\sin^2\theta\right)d\theta$$

$$=\frac{\pi}{2mn}\left(\frac{a^2}{m^2}+\frac{b^2}{n^2}\right).$$

It will be noted that it is often possible by a selection of such a change of the variables to arrange that the derived curve is of a much more convenient form, and its area readily obtainable when expressed in polars.

Ex. 3. Find the area of the curve

$$\left(c^2+\frac{a^2y^2}{b^2}+\frac{b^2x^2}{a^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\left(\frac{a^2y^2}{b^2}+\frac{b^2x^2}{a^2}\right),$$

where c is less than both a and b.

Let $\frac{ay}{b}=\eta,\ \frac{bx}{a}=\xi.$

Then the derived curve is

$$(c^2+\xi^2+\eta^2)\left(\frac{\xi^2}{b^2}+\frac{\eta^2}{a^2}\right)=\xi^2+\eta^2,$$

or in polars, $$(c^2+r^2)\left(\frac{\cos^2\theta}{b^2}+\frac{\sin^2\theta}{a^2}\right)=1,$$

i.e. $$r^2=\frac{a^2b^2}{a^2\cos^2\theta+b^2\sin^2\theta}-c^2.$$

There is obviously symmetry about both axes, and though there is a conjugate point in the original curve at the origin, the curve does not pass through the origin, and the derived curve is one which could be obtained from an ellipse by writing r^2+c^2 for r^2.

Let $r^2+c^2=r'^2$. Then $r'^2=\frac{a^2b^2}{a^2\cos^2\theta+b^2\sin^2\theta}$, and the area of this ellipse is πab. The area of our first derived curve is therefore

$$4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}r^2\,d\theta=4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}(r'^2-c^2)\,d\theta=4\cdot\frac{1}{2}\left(\frac{\pi ab}{2}-\frac{\pi}{2}c^2\right)=\pi(ab-c^2);$$

$\therefore$ the area of the original curve is

$$\frac{b}{a}\cdot\frac{a}{b}\pi(ab-c^2),$$

which also $=\pi(ab-c^2)$.

465. In connection with the last example, it is worth noting that in any curve $r=f(\theta)$ if the area if any portion from $\vartheta=\alpha$ to $\theta=\beta$ be found as

$$\frac{1}{2}\int_\alpha^\beta[f(\theta)]^2\,d\theta\quad\text{and}\quad=A,$$

then the sectorial area of the curve $r^2=[f(\theta)]^2\pm c^2$ between the same limits is

$$\frac{1}{2}\int_\alpha^\beta \{[f(\theta)]^2\pm c^2\}\,d\theta = A\pm\frac{c^2}{2}(\beta-\alpha);$$

and if both be closed and the origin within both, then the area of the new curve differs from the area of the original curve by the area of a circle of radius c, supposing c to be such that r is real throughout the range of integration in each case.

EXAMPLES.

1. Find the whole area of a loop of each of the curves

$$\text{(i)}\qquad x(x^2+y^2)=a(x^2-y^2).$$
$$\text{(ii)}\quad (m^2x^2+n^2y^2)^2=a^2x^2-b^2y^2.$$

[St. John's, 1887.]

2. Trace the shape of the following curves, and find their areas:

$$\text{(i)}\quad (x^2+\ y^2)^3=axy^4.$$
$$\text{(ii)}\quad (x^2+2y^2)^3=axy^4.$$

[Barnes Scholarships, 1887.]

3. Prove that the area of

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 \quad\text{is}\quad \frac{\pi c^2}{2ab}(a^2+b^2).$$

4. Prove that the area in the positive quadrant of the curve

$$(a^2x^2+b^2y^2)^{\frac{5}{2}}=mx^3+ny^3 \quad\text{is}\quad \frac{1}{3ab}\left(\frac{m}{a^3}+\frac{n}{b^3}\right).$$

[α, 1890.]

5. Prove that the area of the curve

$$(a^2x^2+b^2y^2)^2=c^6(x^2-y^2) \quad\text{is}\quad \frac{c^6}{a^3b^3}\left\{ab+(b^2-a^2)\tan^{-1}\frac{b}{a}\right\}.$$

[St. John's, 1883.]

6. Show that the area of the loop of the curve

$$\frac{x^5}{a^5}+\frac{y^5}{b^5}=5\frac{x^2y^2}{a^2b^2} \quad\text{is}\quad \frac{5}{2}ab.$$

7. Find the area of the curve

$$\frac{l}{\sqrt{r^2-c^2}}=1+e\cos\theta \quad (e<1).$$

8. Show that the area bounded by

$$(x^2+y^2-c^2)(x^2+y^2)=4a^2x^2 \quad\text{is}\quad (2a^2+c^2)\pi.$$

9. Find the area included within the curve whose equation is

$$\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1.$$

[COLLEGES, 1885.]

10. Trace the curve

$$\left(\frac{x}{a}+\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{x}{a}-\frac{y}{b}\right)^{\frac{2}{3}}=2,$$

and show that its area is half as great again as that of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

[MATH. TRIPOS, 1884.]

11. Prove that the area of the curve

$$(x^2+y^2)^5=a^2y(x^7+y^7) \quad \text{is} \quad \tfrac{105}{512}\pi a^2+\tfrac{125}{384}a^2.$$

[ST. JOHN'S, 1889.]

12. Prove that the area of the curve

$$(a^2x^2+b^2y^2)^5=8a^4b^4xy(a^4x^6+b^4y^6) \quad \text{is} \quad a^2+b^2.$$

[ST. JOHN'S, 1889.]

13. Show that the area in the first quadrant of the curve

$$c^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^5=\left(\frac{x^3}{a}+\frac{y^3}{b}\right)^2 \quad \text{is} \quad \frac{ab(a^2+b^2)}{3c^2}.$$

14. Trace the curve $4(x^2+2y^2-2ay)^2=x^2(x^2+2y^2)$, proving that the area of a loop is $4\pi(2-\sqrt{3})a^2/\sqrt{3}$, and that the area included between the loops is

$$8a^2(2\pi-3\sqrt{3})/3\sqrt{3}.$$

[TRINITY, 1896.]

15. Find the whole area of the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{2xy}{ab},$$

[OXFORD I. P., 1890.]

and of a loop of the curve

$$\frac{x^4}{a^4}+\frac{y^4}{b^4}=\frac{2xy}{ab}.$$

[OXFORD II. P., 1900.]

16. Show that the area of either oval of

$$x^2\{x^2/a^2+y^2/b^2-1\}+c^2=0 \quad \text{is} \quad \tfrac{1}{2}\pi b(a-2c).$$

[ST. JOHN'S, 1890.]

17. If $f(x, y)=0$ be a closed curve, show that its area is mn times the area of the closed curve $f(mx, ny)=0$. Trace the curve $(4x^2+9y^2)^4=axy^6$, and find its area. [OXFORD II. P., 1890.]

18. Trace the curve $\dfrac{x^3}{a^3}+\dfrac{y^3}{b^3}=\dfrac{3xy}{ab}$, and show that the area of its loop is $\dfrac{3}{2}ab$.

19. A curve is defined by the equations

$$x = 6a\sin^2\phi, \quad y = 6a\sin^2\phi\tan\phi,$$

where ϕ is a variable parameter. Show that the centroid of the portion enclosed between the infinite branches and the asymptote is situated on the x-axis at a distance $5a$ from the origin.

[OXFORD II. P., 1889.]

20. (i) In an involute of a circle, show that the area swept out by the radius vector drawn from the centre of the circle to a point on the curve varies as the cube of the central perpendicular upon the tangent, the initial line being the radius to the point where the involute meets the circle.

(ii) In the Conchoid of Nicomedes $r = a\sec\theta - b$ in the case when $a < b$, show that the area of the loop is

$$a^2(\alpha\sec^2\alpha - 2\sec\alpha\cosh^{-1}\sec\alpha + \tan\alpha),$$

and that the distance of the centroid of the loop from the node is

$$\frac{2}{3}a\frac{3\alpha\sec\alpha - 3\cosh^{-1}\sec\alpha - \sin\alpha\tan^2\alpha}{\alpha\sec\alpha - 2\cosh^{-1}\sec\alpha + \sin\alpha},$$

where $$\alpha = \cos^{-1}a/b.$$

21. Prove that the area contained by the curve

$$x^4 + 2x^2y^2 + 4ax^2y + 2a^2(y^2 - x^2 - 2ay) + a^4 = 0 \quad \text{is} \quad \pi a^2(4 - 5/\sqrt{2}).$$

Find also the distance from the axis of y of the centre of gravity of that portion of the area which lies in the first quadrant.

[COLLEGES β, 1890.]

22. Show that the area included between the curve

$$s = a\tan\psi, \quad \text{its tangent at } \psi = 0 \quad \text{and its tangent at } \psi = \phi$$

is $$\tfrac{1}{2}a^2\tan\phi + a^2\tan\tfrac{1}{2}\phi - a^2\log(\sec\phi + \tan\phi).$$

[TRINITY, 1892.]

23. Show that an expression for the element of area in trilinear coordinates is

$$\operatorname{cosec} C\, d\alpha\, d\beta.$$

Show that the area of the conic whose trilinear equation is

$$a^{-1}\beta\gamma + b^{-1}\gamma\alpha + c^{-1}\alpha\beta = 0$$

is to that of the triangle of reference as

$$4\pi : 3\sqrt{3}.$$

[OXFORD II. P., 1890.]

24. Show that the coordinates of the centroid of the area bounded by half the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, the line of cusps and the y-axis are given by

$$\frac{9\pi\bar{x}}{(3\pi - 4)(3\pi + 4)} = \frac{3\bar{y}}{7} = \frac{a}{2}.$$

[WALLIS.]

25. OB and OC are any two semi-diameters of an ellipse conjugate to each other; find the locus of the intersection of the normals at B and C, and show that the area of the curve is

$$\frac{\pi(a^2-b^2)^2}{4ab}$$ [R. P.]

26. Tangents to a system of similar and similarly situated concentric ellipses are drawn such that the distance of each from the centre is the same. Find the area of the curve formed by the points of contact. [TRINITY, 1885.]

27. Show that the moment of inertia of the portion of a uniform parabolic lamina cut off by the latus rectum about the tangent at an extremity of the latus rectum, is equal to $\frac{12Ma^2}{7}$, $4a$ being the latus rectum and M the mass of the lamina. [OXF. I. P., 1914.]

28. Prove by integration that the moment of inertia of a uniform triangular lamina ABC of mass M about a perpendicular axis at A is $\frac{1}{12}M(3b^2+3c^2-a^2)$. [OX. I. P., 1915.]