11.

ON A GENERALIZATION OF TAYLOR'S THEOREM.

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CONNECTED with the study of the Theory of the symmetrical functions of the differences of the roots of an Algebraical Equation, a theorem presents itself in Dr Salmon's *Lessons on Higher Algebra*, 3rd edition, p. 59, art. 63, only partially indicated and insufficiently demonstrated there, which on a closer inspection will be found to be well deserving of notice as containing a true generalization of Taylor's theorem, leading to a development of the same form, subject to a like law of convergence, and easily demonstrable by the same method as that theorem.

Let f be *any function whatever* of $a, b, c, \ldots$, and f_1 the same function of $a_1, b_1, c_1, \ldots$, where

$$a_1 = a, \quad b_1 = b + ah, \quad c_1 = c + 2bh + ah^2,$$
$$d_1 = d + 3ch + 3bh^2 + ah^3, \ldots$$

and let Ω represent the operator

$$a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots;$$

then the theorem in question affirms that

$$f_1 = f + \Omega\,.\,fh + (\Omega\,.)^2 f\frac{h^2}{1\,.\,2} + (\Omega\,.)^3 f\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

On making $a = 1,\ b = x,\ c = 0,\ d = 0 \ldots$, the theorem becomes Taylor's. To prove it in its general form, let

$$\phi x = ax^n + nbx^{n-1} + n\frac{n-1}{2}cx^{n-2} + \ldots;$$

then, on substituting $x+h$ for x, ϕx becomes

$$= a_1 x^n + nb_1 x^{n-1} + n\frac{n-1}{2}c_1 x^{n-2} + \dots.$$

Let h become $h+\delta h$, then obviously

$$\delta f_1 = \frac{d}{dh} f_1 \delta h.$$

But we may obtain the new values of a_1, b_1, c_1, ... corresponding to the change of h into $h+\delta h$, by substituting in ϕx first $x+\delta h$ and then $x+h$ for x.

The effect of the first substitution is to change a, b, c, ... into $a+\delta a$, $b+\delta b$, $c+\delta c$, ..., where

$$\delta a = 0, \quad \delta b = a\delta h, \quad \delta c = 2b\delta h, \quad \delta d = 3c\delta h, \dots.$$

Hence the increment

$$\delta f_1 = \left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d}\dots\right) f_1 \,.\, \delta h;$$

consequently

$$\frac{d}{dh} f_1 = \Omega \,.\, f_1.^*$$

Hence, if we write

$$f_1 = f + Bh + Ch^2 + Dh^3 + \dots,$$

we shall have

$$B + 2Ch + 3Dh^2 + \dots$$

$$= \Omega f + \Omega Bh + 2\Omega Ch^2 + \dots.$$

Hence $\quad B = \Omega\,.\,f,\ C = \tfrac{1}{2}(\Omega\,.)^2 f,\ D = \dfrac{1}{2\,.\,3}(\Omega\,.)^3 f \dots;$

and consequently

$$f_1 = f + \Omega\,.\,f + (\Omega\,.)^2 f\frac{h^2}{1\,.\,2} + (\Omega\,.)^3 f\frac{h^3}{1\,.\,2\,.\,3} + \dots,^\dagger$$

and the first part of the theorem is demonstrated. It will of course be understood that $(\Omega\,.)^i$ means not (Ω^i)., but $\Omega\,.\,\Omega\,.\,\Omega\,.$ (to i factors).

* Or without introducing ϕx, the equations between a_1, b_1, c_1 ... and a, b, c, ... show by direct inspection that the effect upon the former is the same, whether we augment h by δh or b, c, d ... respectively and simultaneously by $a\delta h$, $2b\delta h$, $3c\delta h$, ... so that $\frac{d}{dh} f_1 = \Omega\,.\,f_1$, as in the text.

† Consequently, if Ωf vanishes, since also $(\Omega\,.)^i f$ will also vanish for all values of i, we shall have $f_1 = f$. It is this fact of $(\Omega f = 0)$ being the complete solution of $(f_1 = f)$ which constitutes the importance of the theorem in the Calculus of Invariants.

Lagrange's or any other rule for the Remainder in the old Taylor's theorem may be extended to this generalization of it; that is to say, if in the development of f_1 we stop at the nth term, the *remainder* will be equal to

$$\frac{h^n}{\Pi n}(\Omega\,.)^n f(\alpha, \beta, \gamma \ldots),$$

where $\alpha, \beta, \gamma \ldots$ are what $a_1, b_1, c_1 \ldots$ become when we write θh for h, θ being some proper positive fraction. The demonstration proceeds *pari passu* for the generalized form and for Taylor's case of it. Thus, consider Bertrand's proof as given in Williamson's *Calculus*, second edition, p. 64.

The *lemma* upon which the proof depends takes the form, that if f_1 (supposed continuous between two values of h) has the same value (zero, as it happens in the matter in hand) for two values of h, Ωf must vanish for some intermediate value of h; which is obviously true, since $\delta f = \Omega f \delta h$. The rest of the demonstration remains essentially the same, *mutatis mutandis*, at each point as for Taylor's theorem properly so called.

The theorem above established easily admits of extension to the case of $a_1, b_1, c_1 \ldots$ being the values assumed by $a, b, c \ldots$, when in the quantic $(a, b, c \ldots \between x, y, z)^n$ we substitute $x + hy + kz + \ldots$ for x. We may thus obtain a theorem which will bear to Taylor's theorem for any number of variables the same relation as the theorem given in the text to Taylor's theorem for a single variable.

Since the effect of changing x into $x + h + \delta h$ may be obtained either by first substituting $x + h$ for x and then $x + \delta h$ for x in ϕx, or by a reversal of the order of these two processes, we obtain the interesting consequence that the two operators

$$a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots$$

and

$$a_1\frac{d}{db_1} + 2b_1\frac{d}{dc_1} + 3c_1\frac{d}{d\,.\,d_1} + \ldots$$

are absolutely identical,—a theorem which of course admits, but not without a somewhat complicated process, of an *à posteriori* direct proof; so that the operator Ω is to all intents and purposes what Professor Cayley calls a semi-invariant or pene-invariant, but to which I am accustomed to give the name of a *differentiant* to ϕx.

Finally, it may be observed that a development for f_1 may be obtained by the use of the ordinary Taylor's theorem for *several* variables. If we make use of this method, and write in addition to

$$\Omega = a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots,$$

$$\Omega_1 = a\frac{d}{dc} + 3b\frac{d}{d\,.\,d} + 6c\frac{d}{d\,.\,e} + \ldots,$$

$$\Omega_2 = a\frac{d}{d\,.\,d} + 4b\frac{d}{d\,.\,e} + 10c\frac{d}{d\,.\,f} + \ldots,$$

$$\&c. = \&c.,$$

we shall obtain the noteworthy symbolical and absolute identity

$$e^{h\Omega\,\cdot} = e^{h\Omega + h^2\Omega_1 + h^3\Omega_2 + \ldots}\ *,$$

which may be verified, but not without some little trouble, by direct expansion.

If we use $\Omega\,!$ to signify that Ω is to be used as a pure operator on the matter coming after it (operating that is to say solely on the symbols of quantity $a, b, c, \ldots$ and not on the operators $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc} \ldots$), we shall have

$$\Omega_1 = \frac{(\Omega\,!\,\Omega)}{1\,.\,2}, \quad \Omega_2 = \frac{(\Omega\,!\,\Omega\,!\,\Omega)}{1\,.\,2\,.\,3},$$

and so on. Hence the "noteworthy" symbolical equation above written may be put under the hypersymbolical form

$$e^{h\Omega\,\cdot} = e^{(e^{h\Omega\,!} - 1)\frac{\Omega}{\Omega\,!}}$$

a suggestive identity that may possibly call forth a sneer from the mathematical cynic, but not from the thoughtful mathematician, who, aware that algebra is in its essence a language which it is the proper business of his art to fathom and develop, is prompt to recognize every step in expression as a gain in power.

* If we write

$$\Delta = (-\Omega\,.\,+\Omega)\,h + \Omega_1 h^2 + \Omega_2 h^3 + \ldots,$$

we ought to have $e^{\Delta} - 1 = 0$, and the coefficients in the expansion of $e^{\Delta} - 1$ according to ascending powers of h ought all to vanish identically; and so they will be found to do, provided that in each such coefficient expressed as the sum of the product of powers of $\Omega_0, \Omega_1, \Omega_2, \ldots$ and of $\Omega\,.$ the power of the *dotted* Ω be taken last in order. As soon as that expansion is made (but of course not before) we may write $\Omega\,. - \Omega = 0$, and we may readily calculate *à priori* the value of each power of $(\Omega\,. - \Omega)$; thus we shall obtain

$$(\Omega\,. - \Omega)^2 = \Omega\,.^2 - 2\Omega\Omega\,. + \Omega^2 = \Omega\,.^2 - 2\Omega^2 + \Omega^2 = 2\Omega_1;$$

and so by a similar calculation, having first determined $\Omega\,.^2$, $\Omega\,.^3$, $\Omega\,.^4$, &c., we shall obtain

$$(\Omega\,. - \Omega)^3 = 6\Omega_2, \quad (\Omega\,. - \Omega)^4 = 24\Omega_3 + 12\Omega_1{}^2, \ \&c.;$$

on substituting these values in $\Delta + \frac{\Delta^2}{1\,.\,2} + \frac{\Delta^3}{1\,.\,2\,.\,3} + \ldots$ the coefficients of the several powers of h will be found to vanish.

The appearance in the above process of a zero whose powers are not zero is a phenomenon which will not shock those who are acquainted with Professor Peirce's discussions of possible algebras; but it is new to find it occur in working out a symbolical identity.

The theorem $f_1 = e^{h\Omega} \cdot f$ having, as far as I am aware, been first given by Dr Salmon in a form, if not quite complete, still sufficient for the immediate purpose to which it was to be applied, ought, I think, in justice to bear his name; and I see no reason why Salmon's Theorem in its totality should not be expected in the future to bear new fruit in algebraical expansions and other uses as important as have flowed from the one familiar and simplest case of it, known as Taylor's Theorem. Thus, *ex. gr.*, for the special case where f_1 becomes a function of one only of the quantities b_1, c_1, ... the Salmonian theorem reproduces Arbogast's celebrated one for expanding a rational integral function by the method of derivations, but under a greatly improved form of notation, and with the advantage of a test of convergency supplied by the *limit to the remainder* given in the text above. Who on a first casual reading could have imagined that Arbogast's problem in the differential calculus was virtually solved in an improved form in an article treating "on the symmetrical functions of the differences of the roots of an equation"? "*Que diable allait-il faire dans cette galère là?*" may rise to the lips of many a reader on being made acquainted with the fact*.

* Using Q to denote any rational integral function of x, Salmon's theorem is a theorem for expanding any function of Q, $\frac{dQ}{dx}$, $\frac{d^2Q}{dx^2}$, ... in terms of ascending powers of x.