26.

A SYNOPTICAL TABLE OF THE IRREDUCIBLE INVARIANTS AND COVARIANTS TO A BINARY QUINTIC, WITH A SCHOLIUM ON A THEOREM IN CONDITIONAL HYPERDETERMINANTS.

[*American Journal of Mathematics*, I. (1878), pp. 370—378.]

It is well known that every binary quintic can be expressed, and in only one way, as the sum of three fifth powers of linear functions of its variables, or which is the same thing, as the sum of the fifth powers of three variables connected by a linear equation, or finally, under the form

$$ax^5 + by^5 + cz^5,$$

subject to the equation

$$x + y + z = 0.$$

If ϕ, ψ be any two covariants of a binary quintic in x, y, the most general expression of the covariant produced by their operation on each other through the variables is

$$\left(\dot{x}\frac{d}{dy} - \dot{y}\frac{d}{dx}\right)^i \phi\psi,$$

where i is any positive integer and $\dot{x}, \dot{y}$ $\left(\text{abbreviations for } \frac{\delta}{\delta x}, \frac{\delta}{\delta y}\right)$ operate on ϕ only whilst $\frac{d}{dx}, \frac{d}{dy}$ operate on ψ.

Suppose now that ϕ, ψ are expressed as functions, say Φ, Ψ, of x, y, z, between which there exists the linear relation $lx + my + nz = 0$; it may be shown that the preceding expression becomes identical with

$$\begin{vmatrix} l, & m, & n \\ \dot{x}, & \dot{y}, & \dot{z} \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \end{vmatrix}^i \Phi\Psi,$$

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where x, y, z are to be treated as independent variables. In the present case, therefore, writing

$$(\dot{y}-\dot{z})\frac{d}{dx}+(\dot{z}-\dot{x})\frac{d}{dy}+(\dot{x}-\dot{y})\frac{d}{dz}=\Lambda,$$

$\Lambda^i\Phi\Psi$, or (which will be more convenient for writing) $\Psi\Lambda^i\Phi$ will represent the covariant derived from the alliance of Φ and Ψ.

The twenty-three irreducibles of the quintic may be arranged in the following partially symmetrical order, which is that which I shall adopt as the order of their successive deduction: the first figure denotes the degree in the coefficients, the second the order in the variables*.

			2·2		4·0					
	1·5		3·3		5·1					
	2·6	3·5	4·4	5·3	6·2	7·1	8·0			
		4·6		6·4		8·2				
3·9		5·7		7·5		9·3		11·1	12·0	
									13·1	
										18·0

* Comparing this arrangement to the distribution of stars in a firmament, it will be observed that there is a tendency to concentration, or the formation of a sort of milky-way, in the zone situated towards the centre, consisting of three bands which comprise between them 15 out of 23, the total number of forms. This phenomenon becomes very much more distinctly marked in the distribution of the 124 irreducible forms appertaining to the septic, the *corrected* table of which I anticipate will have appeared, about simultaneously with the publication of this, in the *Comptes Rendus*†. The table previously given in that journal for the seventhic is affected with some inaccuracies chiefly arising from arithmetical errors of calculation, as I made the computation hurriedly and on the point of leaving England for this continent, and also, in part, from the existence of some errors in the table of the reduced generating function, which I accepted, without sufficient examination, as the basis of my work. It may perhaps be worthy of notice that, if we add a unit to the ordinarily received number of irreducible forms in each case (which it is proper to do, since an absolute number is an invariant of the order zero), the numbers of the irreducibles for the 1st, 3rd, 5th and 7th orders become 2, 5, 24, 125 respectively. As I am about to compute the irreducibles for the 9th order, we shall soon be in a position to ascertain whether the law indicated in this progression has any foundation in nature: if so, the number for that case should be 626, or thereabouts, but it is not unlikely that the fact of 9 being a composite number may have a tendency to affect the result, probably in the direction of decrease. For binary quantics of the even orders 0, 2, 4, 6, 8 the number of irreducible covariants is 1, 3, 6, 27, 70 respectively (for the last see *Comptes Rendus*‡, June 24, 1878), which appear to indicate a geometrical progression with the common ratio 3, subject to diminution for higher powers of 2 entering into the order of the quantic.

[† p. 146 above.] [‡ p. 114 above.]

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There will be two sources of indeterminateness in the expressions obtained for these forms, one universal, arising from the arbitrary addition

$$(x+y+z)\,M,$$

the other special to those forms (such as 13·1) which can be obtained by the multiplication of lower forms (as 8·0, 5·1). Our object must be to seek in all cases the simplest expressions that can be obtained.

$2{\cdot}2 = 1{\cdot}5\Lambda^4 1{\cdot}5 \equiv \Sigma\,(abxy + acxz) \equiv \Sigma abxy.$

I use the sign of equivalence to signify that numerical common multipliers are to be rejected.

$$4{\cdot}0 = 2{\cdot}2\Lambda^2 2{\cdot}2$$
$$= \Sigma\,(\dot{y}-\dot{z})\,(\dot{z}-\dot{x})\,(abxy+acxz+bcyz)\,\frac{d}{dx}\cdot\frac{d}{dy}\,(abxy+acxz+bcyz)$$
$$= \Sigma\,(-ab+ac+bc)\,ab \equiv a^2b^2+b^2c^2+c^2a^2-2abc\,(a+b+c)$$

$$1{\cdot}5 = ax^5+by^5+cz^5$$
$$3{\cdot}3 = 2{\cdot}2\Lambda^2 1{\cdot}5 \equiv \Sigma ax^3\,(\dot{y}-\dot{z})^2\,(abxy+bcyz+cazx) \equiv abc\Sigma x^3.$$

Since $x^3+y^3+z^3 = 3xyz+(x+y+z)\,(x^2+y^2+z^2-xy-yz-zx)$ we have

(*bis*) $$3{\cdot}3 = abcxyz$$

$$5{\cdot}1 = 3{\cdot}3\Lambda^2 2{\cdot}2 \equiv abc\Sigma x\,(\dot{y}-\dot{z})^2\,(abxy+bcyz+cazx) \equiv abc\Sigma bcx$$
$$2{\cdot}6 = 1{\cdot}5\Lambda^2 1{\cdot}5 \equiv \Sigma ax^3\,(\dot{y}-\dot{z})^2\,(ax^5+by^5+cz^5) \equiv \Sigma ax^3\,(by^3+cz^3) \equiv \Sigma abx^3y^3$$
$$3{\cdot}5 = 2{\cdot}2\Lambda\ 1{\cdot}5 = \Sigma\,(aby+acz)\,(\dot{y}-\dot{z})\,(ax^5+by^5+cz^5)$$
$$\equiv \Sigma\,(aby+acz)\,(by^4-cz^4) = \Sigma a\,(b^2y^5-c^2z^5)+abc\Sigma\,(zy^4-yz^4)$$
$$4{\cdot}4 = 3{\cdot}3\Lambda^2 1{\cdot}5 \equiv abc\Sigma x\,(\dot{y}-\dot{z})^2\,(ax^5+by^5+cz^5) \equiv abc\Sigma\,(bxy^3+cxz^3)$$
$$= abc\Sigma\,[(ax^3+by^3+cz^3)\,x-ax^4] \equiv abc\Sigma ax^4$$
$$5{\cdot}3 = 2{\cdot}2\Lambda\ 3{\cdot}3 = \Sigma\,(aby+acz)\,(\dot{y}-\dot{z})\,abc\,(x^3+y^3+z^3)$$
$$\equiv abc\Sigma\,(y^2-z^2)\,(aby+acz)^* \equiv abc\Sigma ax\,(by^2-cz^2)$$
$$6{\cdot}2 = 3{\cdot}3\Lambda^2 3{\cdot}3 \equiv a^2b^2c^2\Sigma x\,(\dot{y}-\dot{z})^2\,(x^3+y^3+z^3) \equiv a^2b^2c^2\Sigma\,(xy+xz)$$
$$\equiv a^2b^2c^2\,(xy+yz+zx) \equiv a^2b^2c^2\,(x^2+y^2+z^2)$$
$$7{\cdot}1 = 4{\cdot}4\Lambda^4 3{\cdot}5 \equiv abc\Sigma\,a\,(\dot{y}-\dot{z})^4\Sigma\,\{(ab^2y^5-ac^2z^5)+abc\,(zy^4-yz^4)\}$$
$$\equiv a^2b^2c^2\Sigma\,a\,(\dot{y}-\dot{z})^4\Sigma\,(zy^4-yz^4) = a^2b^2c^2\Sigma a\,(y-z)$$
$$8{\cdot}0 = 4{\cdot}4\Lambda^4 4{\cdot}4 \equiv a^2b^2c^2\Sigma a\,(\dot{y}-\dot{z})^4\,(ax^4+by^4+cz^4) \equiv a^2b^2c^2\,(ab+ac+bc)$$
$$4{\cdot}6 = 3{\cdot}3\Lambda\ 1{\cdot}5 \equiv abc\Sigma x^2\,(\dot{y}-\dot{z})\,(ax^5+by^5+cz^5) \equiv abc\Sigma a\,(y^2-z^2)\,x^4$$
$$6{\cdot}4 = 2{\cdot}2\Lambda\ 4{\cdot}4 = \Sigma\,(aby+acz)\,(\dot{y}-\dot{z})\,abc\,(ax^4+by^4+cz^4)$$
$$= abc\,(aby+acz)\,(by^3-cz^3)$$
$$= abc\Sigma\,(ab^2y^4+ac^2z^4)+a^2b^2c^2\Sigma\,(zy^3-yz^3),$$

which, since $\Sigma\,(zy^3-yz^3)$ contains $x+y+z$,

$$= abc\Sigma\,(c-b)\,a^2x^4$$

$$8{\cdot}2 = 4{\cdot}4\Lambda^4 4{\cdot}6 = a^2b^2c^2\Sigma a\,(\dot{y}-\dot{z})^4\,\{\Sigma a\,(y^2-z^2)\,x^4\} = a^2b^2c^2\Sigma ab\,(x^2-y^2)$$
$$3{\cdot}9 = 2{\cdot}6\Lambda\ 1{\cdot}5 \equiv \Sigma\,(abx^2y^3+acx^2z^3)\,(\dot{y}-\dot{z})\,(ax^5+by^5+cz^5)$$
$$\equiv \Sigma\,(abx^2y^3+acx^2z^3)\,(by^4-cz^4)$$
$$= \Sigma ax^2\,(b^2y^7-c^2z^7)+abcx^2y^2z^2\Sigma\,(zy^2-yz^2)\,\dagger$$

* For y^2-z^2 I substitute $xz-xy$.

† Possibly this expression may be simplifiable by the addition of a suitable multiple of $x+y+z$.

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$$5{\cdot}7 = 4{\cdot}4\Lambda\ 1{\cdot}5 \equiv abc\Sigma ax^3(\dot{y}-\dot{z})(ax^5+by^5+cz^5) \equiv abc\Sigma ab(x-y)x^3y^3$$

$$7{\cdot}5 = 4{\cdot}4\Lambda\ 3{\cdot}3 \equiv a^2b^2c^2\Sigma ax^3(\dot{y}-\dot{z})(x^3+y^3+z^3) \equiv a^2b^2c^2\Sigma ax^2(y^3-z^3)$$

$$11{\cdot}1 = 5{\cdot}1\Lambda\ 6{\cdot}2 \equiv a^3b^3c^3\Sigma bc(\dot{y}-\dot{z})(x^2+y^2+z^2) \equiv a^3b^3c^3\Sigma bc(y-z)$$

$$9{\cdot}3 = 6{\cdot}2\Lambda\ 3{\cdot}3 \equiv a^3b^3c^3\Sigma x(\dot{y}-\dot{z})(x^3+y^3+z^3) \equiv a^3b^3c^3(x-y)(y-z)(z-x)$$

$$12{\cdot}0 = 6{\cdot}2\Lambda^2 6{\cdot}2 \equiv a^4b^4c^4\Sigma(\dot{y}-\dot{z})^2(x^2+y^2+z^2) \equiv a^4b^4c^4$$

$$\begin{aligned}13{\cdot}1 = 7{\cdot}1\Lambda\ 6{\cdot}2 &= a^4b^4c^4\Sigma(b-c)(\dot{y}-\dot{z})(x^2+y^2+z^2) \equiv a^4b^4c^4\Sigma(b-c)(y-z)\\ &= a^4b^4c^4\{\Sigma(by+cz)-\Sigma(bz+cy)\}\\ &= a^4b^4c^4\{2(ax+by+cz)+(bx+cz+ay)\} \equiv a^4b^4c^4\Sigma ax\end{aligned}$$

$$\begin{aligned}18{\cdot}0 = 13{\cdot}1\Lambda 5{\cdot}1 &= a^4b^4c^4\Sigma a(\dot{y}-\dot{z})(bcx+cay+abz) = a^4b^4c^4\Sigma a(c-b)\\ &= a^5b^5c^5(a-b)(b-c)(c-a).\end{aligned}$$

18·0 may also be obtained by the operation of 11·1 on 7·1, or instantaneously as the resultant of 1·5, $abcxyz$ and $x+y+z$. In the following table the preceding results are collected; for greater brevity instead of the sign of summation I employ the sign + or − to signify respectively the symmetrical or semi-symmetrical completion of the terms to which it is affixed; m is used to signify abc.

1—2	$abxy+:(a^2b^2-2abc^2)+$
3—5	$ax^5+:mx^3+$, or $mxyz:mbcx+$
6—12	$\begin{cases}abx^3y^3+:a^2bx^5+myz^4-:max^4+:\\ mabxy^2-:m^2x^2+:m^2bx-:m^2ab+\end{cases}$
13—15	$max^4y^2-:ma^2cx^4-:m^2abx^2-$
16—21	$\begin{cases}ab^2x^2y^7+mx^2y^4z^3-:mabx^4y^3-:m^2ax^2y^3-:\\ m^3bcy-:m^3x^2y-:m^4\end{cases}$
22	m^4ax+
23	m^5a^2b-

I propose, at some future time, to apply a similar method to obtain an explicit representation of the irreducible forms appertinent to the binary seventhic, an arduous undertaking, but one that seems likely to lead to the apperception of new forms of complex symmetry. The primitive may, for that case, be represented by $x^7+y^7+z^7+t^7$, connected by the linear equations $(l, m, n, p \between x, y, z, t) = 0$, $(\lambda, \mu, \nu, \pi \between x, y, z, t) = 0$, and Λ, the symbol of alliance, will be represented by

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, & \frac{d}{dt} \\ \dot{x}, & \dot{y}, & \dot{z}, & \dot{t} \\ l, & m, & n, & p \\ \lambda, & \mu, & \nu, & \pi \end{vmatrix}.$$

Every in- and co-variant will then be a rational integer function of x, y, z, t and the six minor determinants, which are the parameters of the line represented by the above two linear equations.

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It may be worth while to notice the representations of the irreducible derivatives of the quartic when put under the indeterminate form

$$ax^4 + by^4 + cz^4,$$

subject to the relation $x + y + z = 0$. We get

$$2{\cdot}0 = 1{\cdot}4\Lambda^4 1{\cdot}4 = \Sigma a\,(\dot{y} - \dot{z})^4\,(ax^4 + by^4 + cz^4) \equiv ab + bc + ca$$
$$2{\cdot}4 = 1{\cdot}4\Lambda^2 1{\cdot}4 = \Sigma ax^2\,(\dot{y} - \dot{z})^2\,(ax^4 + by^4 + cz^4) = abx^2y^2 + acx^2z^2 + bcy^2z^2$$
$$3{\cdot}0 = 1{\cdot}4\Lambda^4 2{\cdot}4 = \Sigma a\,(\dot{y} - \dot{z})^4\,(abx^2y^2 + acx^2z^2 + bcy^2z^2) \equiv abc$$
$$3{\cdot}6 = 1{\cdot}4\Lambda\ 2{\cdot}4 = \Sigma ax^3\,(\dot{y} - \dot{z})\,(2{\cdot}4)$$
$$= \Sigma\,(a^2bx^5y - a^2cx^5z) + abcxyz\Sigma\,(yz^2 - y^2z).$$

As regards the sextic form, the first idea would be to regard it as the resultant, in respect to one of the variables (say z), of the canonical system discovered by me so long ago,

$$\left.\begin{matrix} ax^6 + by^6 + cz^6 + mxyz\,(x - y)\,(y - z)\,(z - x) \\ x + y + z \end{matrix}\right\},$$

but this will be found to give rise to expressions for the invariants and covariants of extreme complexity. The representations will, I think, be simplified by adopting the new canonical system

$$\left.\begin{matrix} x^3 + y^3 + z^3 + 3mxyz \\ ayz + bxz + cyx \end{matrix}\right\}\ \begin{matrix}(1)\\(2)\end{matrix}$$

and considering the sextic as the resultant of (1) and (2). It will then be found that every covariant proper (calling its order, which is always an even number, 2ϵ) will still be a resultant of (2) and of some new form in x, y, z of order ϵ*. The fact of the lowering, by one-half, the order of the form in x, y, z, corresponding to a covariant of any given order in x, y, gives a great (though it may be not an unbalanced) advantage to the new canonical system over the old. On setting out the equation connecting the four completely symmetrical invariants with the square of the skew one of the sextic, and then making this latter equal to zero, we obtain an equation between three absolute invariants of the sextic which may be regarded as the equation to a surface, the analogue of my Bicorn, the *Nomen Triviale* for the bicuspidal unicursal quartic curve. This surface will divide space into two parts, one corresponding to equations of the sixth order with real, the other with conjugate coefficients, or by real linear substitutions transformable into such, the surface itself being the locus of equations of the recurrent form. The facultative part of space, that is, the part corresponding to the case of real coefficients will then separate into two pairs of regions, one pair belonging to the case of 0 and 4, the other to that of 2 and 6 imaginary roots. By this method, however laborious, the solution of the problem of determining the invariantive criteria of the quality of the roots of the sextic (to borrow a term

* For every quantic of an even order in x, y is a ternary quantic in x^2+xy, y^2+yx, $-xy$, which quantities are proportional to x, y, z connected by the equation $xy+xz+yz=0$.

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from the chess table) becomes *forced*, and no other mode of attacking the question appears to me to be practicable; nor can it fail to bring into view a surface possessed of remarkable properties*.

SCHOLIUM.

The mode of representing the covariants to a sextic above employed made it imperative, or at least expedient, to discover a method by aid of which the process of alliance, or hyperdetermination, could be performed upon the representative forms themselves, without eliminating one of the variables by means of the equation of condition, and I have obtained the following very general theorem, which, it will presently be seen, contains a solution of the problem in question, and which, as the first example of conditional alliance, or hyperdetermination, it seems to me desirable to put on record.

Let $\phi, \psi, \ldots, \theta$ be i homogeneous functions of the orders $\alpha, \beta, \ldots, \lambda$ in $i+j$ variables, $x, y, \ldots, t$ being i of them and $u, v, \ldots, z$ the j others, and let the variables be connected by the j homogeneous equations

$$L=0,\quad M=0,\quad \ldots,\quad N=0.$$

Call the Jacobian $\dfrac{d(L, M, \ldots, N)}{d(u, v, \ldots, z)}=\Omega$.

Let $\Phi, \Psi, \ldots, \Theta$ be the values of $\phi, \psi, \ldots, \theta$ expressed in terms of $x, y, \ldots, t$ alone, and let

$$\begin{vmatrix} \delta_{x_1}, & \delta_{y_1}, & \ldots, & \delta_{t_1} \\ \delta_{x_2}, & \delta_{y_2}, & \ldots, & \delta_{t_2} \\ \ldots & \ldots & \ldots & \ldots \\ \delta_{x_i}, & \delta_{y_i}, & \ldots, & \delta_{t_i} \end{vmatrix}^q (\Omega_1{}^\alpha\Omega_2{}^\beta\ldots\Omega_i{}^\lambda\Phi_1\Psi_2\ldots\Theta_i)$$

be called D, it being understood that the meaning of any subscript, say μ, is to cause the letters $x, y, \ldots, t$ to be changed into $x_\mu, y_\mu, \ldots, t_\mu$. Again let the operative determinant of the $(i+j)$th order written below

$$\begin{vmatrix} \delta_{x_1}, & \delta_{y_1}, & \ldots\ldots, & \delta_{z_1} \\ \delta_{x_2}, & \delta_{y_2}, & \ldots\ldots, & \delta_{z_2} \\ \ldots & \ldots & \ldots & \ldots \\ \delta_{x_i}, & \delta_{y_i}, & \ldots\ldots, & \delta_{z_i} \\ (\delta_x L)_\epsilon, & (\delta_y L)_\epsilon, & \ldots\ldots, & (\delta_z L)_\epsilon \\ (\delta_x M)_\epsilon, & (\delta_y M)_\epsilon, & \ldots\ldots, & (\delta_z M)_\epsilon \\ \ldots & \ldots & \ldots & \ldots \\ (\delta_x N)_\epsilon, & (\delta_y N)_\epsilon, & \ldots\ldots, & (\delta_z N)_\epsilon \end{vmatrix}$$

* One may see at a glance that this surface cannot be of a higher order than 7, the integer part of 30 : 4. Possibly however, it may not be so high; there will be no difficulty in finding the actual order by means of the known expression for R^2 (Clebsch, *Binäre Formen*, p. 299), in terms of the invariants of even degrees.

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be called J_ϵ (ϵ being any of the suffixes 1, 2, 3, ..., i) then it will be found that to a numerical factor près

$$(\Omega_1^{\alpha-q}\Omega_2^{\beta-q}\ldots\Omega_i^{\lambda-q})(J_1+J_2+\ldots+J_i)^q(\phi_1\psi_2\ldots,\theta_i)=D.$$

As a corollary, if the functions $L, M, \ldots, N$ are all linear in respect to $u, v, \ldots, z$, and if in respect to $1, u, v, \ldots, z$ the resultants of $\phi, L, M, \ldots, N$; $\psi, L, M, \ldots, N$; ... are $[\Phi], [\Psi], \ldots, [\Theta]$ (which is what we mean by saying that $\phi, \psi, \ldots, \theta$ represent $[\Phi], [\Psi], \ldots, [\Theta]$), it will be easily seen to follow from the above theorem that the qth alliance of these quantics will be itself *represented by*

$$(J_1+J_2+\ldots+J_i)^q(\phi_1\psi_2\ldots,\theta_i)^*.$$

Thus in the particular case where $x, y, \ldots, t$ becomes x, y and $u, \ldots, z$ becomes z and $L, M, \ldots, N$ becomes the single function $xy+yz+zx$, we see that the qth alliance of the quantics represented by ϕ, ψ will be itself represented by

$$\left\{\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_1+z_1, & z_1+x_1, & x_1+y_1\end{vmatrix}+\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_2+z_2, & z_2+x_2, & x_2+y_2\end{vmatrix}\right\}^q(\phi_1\psi_2)$$

on replacing x_1, y_1, z_1; x_2, y_2, z_2 by x, y, z after the differentiations have been executed. It will, of course, be understood that the factors in each cross product of the determinants above are to be taken in *their natural order*, that is,

$$\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_1+z_1, & z_1+x_1, & x_1+y_1\end{vmatrix}^\mu$$

is to be understood to mean, not

$$[\Sigma(x_1+y_1)(\delta_{x_1}\delta_{y_2}-\delta_{y_1}\delta_{x_2})]^\mu,$$

but $$[\Sigma(\delta_{x_1}\delta_{y_2}-\delta_{y_1}\delta_{x_2})(x_1+y_1)]^\mu,$$

and so in general.

* This expression may be put under the more compact form J^q, J being a matrix in which the first i lines are the same as those common to $J_1, J_2, \ldots, J_i$, and the last j lines are the sums of the corresponding ones in $J_1, J_2, \ldots, J_i$. Although I had submitted it to a mental process of demonstration (or what seemed such) before sending it to the press, I am not without some little misgiving as to the exactitude of the theorem so far as it regards the higher alliances; for those of the first order it is easily verifiable, and, in that case, it should be noticed that each of the i terms in the expression given by it will reproduce separately (but under quite a distinct form) the value of the Jacobian of $\phi, \psi, \ldots, \theta$; $L, \ldots, N$. Some corresponding simplification in practice, it is not improbable, will apply in the general case, supposing my doubts as to the validity of the theorem to prove unfounded. It is important, and greatly enlarges the horizon of the subject, to remark that, inasmuch as any ternary quadric is linearly transformable into the form $xy+yz+zx$, it will follow that any binary quantic of an even order, with its train of covariants, may be represented by corresponding ternary forms of half their respective orders, combined with a perfectly general final conic, so that, for example, instead of the form $xy+yz+zx$, useful though it be as an intermediate step in the evolution of the theory, we may substitute the handier and more advantageous one $x^2+y^2+z^2$ as the auxiliary quadric.

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The result of this investigation has been to open my eyes to the unquestionable fact that, as we know that the first "Ueberschiebung," or "transvectant," or "alliance," of two or more quantics (names significant and useful enough to indicate the particular modes under which they are considered to be generated) is the ordinary Jacobian, so the right general name for the Ueberschiebung or alliance of any order viewed *per se* (as a *Ding an sich*) and without reference to its mode of origination, which ought to supersede all others, is the *Jacobian of the corresponding order*; or, in other words, the theory of invariants falls into the theory of compound differentiation, and just as $\left(\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}\right)$ is called a Jacobian and designated by $\frac{d(u,v)}{d(x,y)}$, so $\frac{d^2u}{dx^2}\frac{d^2v}{dy^2}-2\frac{d^2u}{dxdy}\frac{d^2v}{dxdy}+\frac{d^2u}{dy^2}\frac{d^2v}{dx^2}$ is entitled to be called the second Jacobian and to be designated by $\frac{d^2(u,v)}{d(x,y)^2}$, and more generally every hyperdeterminant may be designated as a compound differential coefficient (or derivative) of the type $\frac{d^\alpha d^\beta \ldots}{d(\quad)^\alpha d(\quad)^\beta \ldots}$, where the vacant spaces are to be filled up by the insertion of a certain number of letters, with liberty for any number of them in each parenthesis to be identical with the like number in any other. Since we are now in possession of a definite analogue to ordinary differential coefficients of all orders, I do not know whether I shall be considered too bold or fanciful in suggesting that there ought to exist, in the nature of things, some theorem of development for several sets of variables analogous to Taylor's for a single set: what such theorem is or could be I have at present no conception, but as little, be it remembered, could anyone, even Jacobi himself, before the creation of hyperdeterminants, have had the remotest conception in regard to a function of several variables bearing to $\left(\frac{d}{dx}\right)^i\phi$ the same relation of analogy as the ordinary functional determinant to $\frac{d\phi}{dx}$, whether such function could exist, and, if so, what it would be. I have always thought and felt that beyond all others the algebraist, in his researches, needs to be guided by the principle of faith, so well and philosophically defined as "the substance of things hoped for, the evidence of things not seen."

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