36.

NOTE ON DETERMINANTS AND DUADIC DISYNTHEMES.

[*American Journal of Mathematics,* II. (1879), pp. 89—96, 214—222.]

A GENERAL algebraical determinant in its developed form (viewed in relation to any one arbitrarily selected term) may be likened to a mixture of liquids seemingly homogeneous, but which being of differing boiling points, admit of being separated by the process of fractional distillation. Thus, for example, suppose a general determinant of the 6th order. The 720 terms which make it up will fall, in relation to the leading diagonal product, into as many classes (most of which comprise several similarly constituted families) as there are unlimited partitions of 6. These, 11 in number, are

$$6;\ 5,1;\ 4,2;\ 4,1,1;\ 3,3;\ 3,2,1;\ 3,1,1,1;\ 2,2,2;\ 2,2,1,1;\ 2,1,1,1,1;\ 1,1,1,1,1,1.$$

Let the determinant be represented, in the umbral notation, by

$$\begin{matrix} a & b & c & d & e & f \\ a & b & c & d & e & f \end{matrix}\ *.$$

Let us, by way of illustration, consider the class corresponding to 6; this will consist of the 1 . 2 . 3 . 4 . 5 (120) terms obtained by forming the 120 distinct circular arrangements that belong to *a b c d e f*. Thus:

$$\begin{matrix} & \longrightarrow & \\ & a \quad c & \\ b & & e \\ & f \quad d & \\ & \longleftarrow & \end{matrix}$$

* The cyclical method of the text shows what was not previously apparent, that the umbral notation $\begin{matrix} ab \dots l \\ ab \dots l \end{matrix}$ possesses an essential advantage over $\begin{matrix} ab \dots l \\ \alpha\beta \dots \lambda \end{matrix}$ even for unsymmetrical determinants. This mode of notation of course implies some ground of preference for one diagonal group over all others and thus virtually regards a general determinant as related to a lineo-linear as a symmetrical one is to a quadratic form. For instance the general determinant of the second order is to be regarded as appurtenant to the lineo-linear form $aaxx' + abxy' + bayx' + bbyy'$.

will signify $ac \times ce \times ed \times df \times fb \times ba$, which will be one of the 120 in question. So, again, 3, 3 will denote, in the first place, the 10 sets of double triads of the general form $abc : def$, and, as each triad will give two cyclical orders, there will in all be 10×2^2, that is, 40, terms of the form $ab \, . \, bc \, . \, ca \, . \, de \, . \, ef \, . \, fd$. So, again, there will be $15 \, . \, 1^3$, that is, 15, corresponding to 2, 2, 2. So 3, 2, 1 will give 10 groupings of the form $abc : de : f$, and each of these will give rise to two terms, namely,

$$ab \, . \, bc \, . \, ca \, . \, de \, . \, ed \, . \, ff, \quad ac \, . \, cb \, . \, ba \, . \, de \, . \, ed \, . \, ff,$$

the number of cycles corresponding to two elements de being 1, and to one element f also 1.

This simple theory affords us a direct means of calculating the number of distinct terms in a symmetrical determinant, that is, one in which $i \, . \, j$ and $j \, . \, i$ are identical. It enables us to see at once that the coefficient of every term is unity or a power of 2; the rule being that plus or minus terms* of the class corresponding to $m_1, m_2, m_3, \ldots$ will take the coefficient 2^ν, ν being the number of the quantities m which are neither 1 nor 2, for, in every other case, the total number of cycles in each partial group will arrange themselves in pairs which give the same result, thus, for example,

$$\begin{array}{ccc} & a & \\ d & & b \\ & c & \end{array} \quad \text{and} \quad \begin{array}{ccc} & a & \\ b & & d \\ & c & \end{array}$$

will give the equal products $ab \, . \, bc \, . \, cd \, . \, da$ and $ad \, . \, dc \, . \, cb \, . \, ba$.

As an example of the direct method of computation, take a symmetrical determinant of the 5th order. Write

$$5 \quad 4 \, . \, 1 \quad 3 \, . \, 2 \quad 3 \, . \, 1 \, . \, 1 \quad 2 \, . \, 2 \, . \, 1 \quad 2 \, . \, 1 \, . \, 1 \, . \, 1 \quad 1 \, . \, 1 \, . \, 1 \, . \, 1 \, . \, 1.$$

To these 7 classes there will belong respectively

1 . 12	with the	coefficient	2
5 . 3	"	"	2
10 . 1	"	"	2
10 . 1	"	"	2
15	"	"	1
10	"	"	1
1	"	"	1.

Thus the number of distinct terms will be

$$12 + 15 + 10 + 10 + 15 + 10 + 1 = 73,$$

and the sum of the coefficients

$$24 + 30 + 20 + 20 + 15 + 10 + 1 = 120,$$

both of which are right.

* The complete value of the coefficient is $(-)^\mu \, 2^\nu$, ν being the number of elements in the partition other than 1 or 2, and μ the number of even elements.

Again, if we have a skew determinant of an even order, it will easily be seen that any partition embracing one or more odd numbers will give rise to pairs of terms that mutually cancel, but when all the parts into which the exponent of the order is divided are even, the coefficient will be given by the same rule as for symmetrical determinants, that is, its arithmetical value will be 2^ν, where ν is the number of parts exceeding 2. Thus, for example, for a skew determinant of the order 6 we have

$$6 \quad 4.2 \quad 2.2.2.$$

The number of terms corresponding to these partitions being 60 with coefficient 2, 15×3 also with coefficient 2, and 15 with coefficient 1, making 120 distinct terms in all, the sum of the coefficients will be

$$120 + 90 + 15 = (1.3.5)^2,$$

which is right, because the result is the square of the sum of 15 synthemes of the form $1.2 \times 3.4 \times 5.6$. It may be observed that 120 is $\frac{15.16}{2}$, as it ought to be, because, until we reach the order 8, the same *double duadic syntheme* can only be made up in one way of two simple ones, but this ceases to be the case from and after 8. Thus, for example, the pair of synthemes

$$1.2 \quad 3.4 \quad 5.6 \quad 7.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.7 \quad 6.8$$

combined will produce the same double syntheme as the pair

$$1.2 \quad 3.4 \quad 5.7 \quad 6.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.6 \quad 7.8,$$

and accordingly for 8 we have the partitions

$$8 \quad 6.2 \quad 4.4 \quad 4.2.2 \quad 2.2.2.2,$$

giving rise to

2520	with	coefficient	2
28.60	,,	,,	2
35.3^2	,,	,,	4
210.3	,,	,,	2
105	,,	,,	1,

making in all $2520 + 1680 + 315 + 630 + 105$, that is, 5250, distinct terms, whereas

$$\frac{(1.3.5.7)^2 + (1.3.5.7)}{2} = 5565,$$

the difference, 315, being due to the fact that there are that number of double synthemes which admit of a twofold resolution into two single synthemes.

I will not stop to prove, but any person conversant with the subject will see at once that this method gives an intuitive and direct proof of the theorem that a pure skew determinant for an even order is a perfect square*. Having

* That a skew determinant of an odd order vanishes is apparent from the fact that an odd number cannot be made up of a set of even ones. I use the term skew determinant in its strict sense as referring to a matrix for which $ij = -ji$ and $ii = 0$.

only a limited space at my command, I will pass on at once to forming the equation in differences for the case of a symmetrical, a skew, and one or two other special forms of determinants.

For a symmetrical determinant, taking as a diagram, to fix the ideas, the matrix of the 6th order

$$\begin{matrix} a & b & c & d & e & f \\ b & g & h & k & l & m \\ c & h & n & p & q & r \\ d & k & p & s & t & u \\ e & l & q & t & v & w \\ f & m & r & u & w & \omega \end{matrix},$$

calling u_m the number of distinct terms in a symmetrical matrix of the mth order, and, resolving the entire determinant into a sum of determinants of the order $(m-1)$ multiplied by the letters in the top line, we shall obviously get u_{m-1} together with $(m-1)$ quantities, positive or negative (and we know, by what precedes, that there can be no cancelling, so that the sign, for the object in view, may be entirely neglected) of the form

$$b \times \begin{matrix} b & h & k & l & m \\ c & n & p & q & r \\ d & p & s & t & u \\ e & q & t & v & w \\ f & r & u & w & \omega \end{matrix}.$$

Among these $(m-1)$ quantities all the terms containing bc, bd, be, bf will occur twice over, but those containing b^2 do not recur. Hence, to find the number of distinct terms we may reckon each of such distinct terms as contain bc, bd, be, bf worth only $\frac{1}{2}$, the others counting as 1. But if, instead of the column (which I write as a line) *bcdef*, we had the column *bhklm*, the rule for calculating the number of distinct terms might be calculated by this very same rule, except that the terms multiplied by hc, kd, le, mf ought to count as *units* instead of *halves*. Hence obviously

$$u_m + (m-1)(m-2)\,u_{m-3} \times \tfrac{1}{2} = u_{m-1} + (m-1)\,u_{m-1} = mu_{m-1},$$

or

$$u_m = mu_{m-1} - \tfrac{1}{2}(m-1)(m-2)\,u_{m-3},$$

which is Mr Cayley's equation, but obtained by a much more expeditious process (see Salmon's *Higher Algebra*, 3rd edition, pp. 40—42); writing $u_m = (1 \,.\, 2 \ldots m)\,v_m$ we obtain the equation in differences, linear in regard to the independent variable,

$$mv_m - mv_{m-1} + \tfrac{1}{2}v_{m-3} = 0,$$

and this, treated by the general method applicable to all such, gives rise to a linear differential equation in which, on account of the particular initial values of u_0, u_1, u_2, the third term is wanting, and finally v_m is found to be the coefficient of t^m in

$$\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}.$$

If we apply a similar method to the case of a symmetrical determinant in which the diagonal of symmetry is filled out with zeros (an invertebrate symmetrical or symmetrical bialar determinant, as we may call it) we shall easily obtain the equation in differences

$$u_m = (m-1)\,[u_{m-1} + u_{m-2}] - \tfrac{1}{2}\,(m-1)\,(m-2)\,u_{m-3},$$

and, making $u_m = 1 \,.\, 2 \ldots m v_m$,

$$m v_m - (m-1)\,v_{m-1} - v_{m-2} + \tfrac{1}{2} v_{m-3} = 0,$$

from which, calling $y = v_0 + v_1 t + v_2 t^2 + \ldots$ and having regard to the initial values v_0, v_1, v_2, we obtain

$$2\,\frac{dy}{y} = \frac{2t - t^2}{1-t}\,dt,$$

and

$$y = \frac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}.$$

By way of distinction, using u' and v' for this case, and u, v for the preceding one, the slightest consideration shows that

$$u_m = u'_m + m u'_{m-1} + \frac{m\,(m-1)}{2}\,u'_{m-2} + \frac{m\,(m-1)\,(m-2)}{2\,.\,3}\,u'_{m-3} + \ldots,$$

or

$$v_m = v'_m + v'_{m-1} + \frac{v'_{m-2}}{1\,.\,2} + \frac{v'_{m-3}}{1\,.\,2\,.\,3} + \ldots.$$

Hence the generating function for v_m ought to be that for u_m multiplied by e^t, as we see is the case.

So, in like manner, the generating function for v_m, that is, $\frac{u_m}{1\,.\,2\ldots m}$, in the case of a general determinant being $\frac{1}{1-t}$, that of v_m for an invertebrate or zero-axial but otherwise general determinant we see must be $\frac{e^{-t}}{1-t}$*, that is,

$$v_m = 1 - 1 + \frac{1}{1\,.\,2} - \frac{1}{1\,.\,2\,.\,3} + \ldots \pm \frac{1}{1\,.\,2\ldots m}.$$

* It may easily be proved that the difference between the numbers of positive and negative combinations in the development of an invertebrate determinant of the mth order is $(-)^{m-1}(m-1)$ in favour of the former. From this it is easy to prove that the generating function for $\frac{\text{number of positive terms in such determinant}}{1\,.\,2\,.\,3\ldots m}$ is

$$\frac{1}{2}\left\{\frac{e^{-t}}{1-t} - (1+t)\,e^{-t}\right\}, \text{ or } \frac{t^2 e^{-t}}{2\,(1-t)}.$$

the well known value $\left(\text{ultimately equal to } \frac{1}{e}\right)$, as it ought obviously to be, of the chance of two cards of the same name not coming together when one pack of m distinct cards is laid card for card under another precisely similar pack.

Returning to the case of the invertebrate symmetrical determinant, it will readily be seen, by virtue of the prolegomena, that the number of terms (the u_m) for such a determinant of the mth order is the same thing as the total number of duadic disynthemes that can be formed with m things, meaning by a duadic disyntheme any combination of duads with or without repetition, in which each element occurs twice and no oftener. Thus, when $m=6$, 1.2 2.3 1.3 4.5 4.6 5.6 and 1.2 2.3 3.4 5.6 6.1 and 1.2 2.3 3.4 1.4 5.6 5.6 are all three of them disynthemes. But the two latter ones are each resoluble into single synthemes, whereas the first one is not. It is clear that, when a disyntheme is formed by means of cycles all of an even order, it will be resoluble into a pair of single synthemes, and in no other case. The problem, then, of finding the number of distinct double synthemes with m elements is one and the same as that of finding the number of distinct terms in a *proper* (that is, invertebrate) skew determinant, which I proceed to consider.

Following a method (not identical with but) analogous to that adopted for the symmetrical cases, we shall find, by a process which the terms below written will sufficiently suggest

$$u_m + \frac{(m-1)(m-2)(m-3)}{2} u_{m-4} = (m-1) u_{m-2} + (m-1)(m-2) u_{m-2},$$

or
$$u_m = (m-1)^2 u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2} u_{m-4}.$$

Of course, when m is odd $u_m = 0$. From this it is readily seen that $\frac{u_{2m}}{1.3.5\ldots.2m-1}$, say ω_m, is an integer; for we shall have

$$\omega_m = (2m-1)\,\omega_{m-1} - (m-1)\,\omega_{m-2},$$

also,
$$\omega_1 = 1, \quad \omega_2 = 2,$$

Whence it follows that the number of positive terms in a general invertebrate determinant of the mth order is $m\frac{m-1}{2}$ times the total number of the terms in one of the $(m-2)$th order. The equation of differences for U_m, the total number, is of course

$$U_m = (m-1)(U_{m-1} + U_{m-2}),$$

and the successive values of

U_m for 1, 2, 3, 4, 5, 6, 7, 8, ...,

are 0, 1, 2, 9, 44, 265, 1854, 14833, ...

so that
$$\omega_3 = 5\,.\,2 \quad -2\,.\,1 \quad = \quad 8,$$
$$\omega_4 = 7\,.\,8 \quad -3\,.\,2 \quad = \quad 50,$$
$$\omega_5 = 9\,.\,50 \quad -4\,.\,8 \quad = 418,$$
$$\omega_6 = 11\,.\,418 - 5\,.\,50 = 4348,$$
and the conventional $\omega_0 = 3\omega_1 - \omega_2 = 1$.

By the above formula u_m can be calculated with prodigious rapidity. If, however, we wish to obtain a generating function for u_m, the differential equation obtained from the above equation in differences does not lead to a simple explicit integral, but if we make $u_{2m} = (1\,.\,2\,.\,3 \dots 2m)\,v_m$, as in the preceding cases, or, which is the same thing, $\omega_m = 2^m (1\,.\,2 \dots m)\,v_m$, we get
$$4mv_m - 4\,(m-1)\,v_{m-1} - 2v_{m-1} + v_{m-2} = 0,$$
and, writing as before $\quad y = v_0 + v_1 t + v_2 t^2 + \dots,$
$$4\,\frac{dy}{dt} - 4t\,\frac{dy}{dt} - 2y + ty$$
will be found to be equal to zero. [This vanishing of the 3rd term in the differential equation being a feature common to all the cases we have considered, and due to the initial values of the v series in each case.] We have thus
$$\frac{4y'}{y} = \frac{1}{1-t} + 1, \quad y = \frac{e^{\frac{t}{4}}}{(1-t)^{\frac{1}{4}}}.$$
By way of verification, we may observe that
$$v_0 = 1, \quad v_1 = \frac{1}{2}, \quad v_2 = \frac{1}{4}, \quad v_3 = \frac{1}{6}, \dots,$$
$$y = \left(1 + \frac{t}{4} + \frac{t^2}{32} + \frac{t^3}{384} + \dots\right)\left(1 + \frac{t}{4} + \frac{5t^2}{32} + \frac{45t^3}{384} + \dots\right),$$
and $$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \quad \frac{1}{32} + \frac{1}{16} + \frac{5}{32} = \frac{1}{4}, \quad \frac{45}{384} + \frac{5}{128} + \frac{1}{128} + \frac{1}{384} = \frac{1}{6}\text{*}.$$

We may now proceed to calculate the number of distinct terms in an improper or vertebrated skew-determinant, which is interesting on account of its connection with the theory of orthogonal transformations. Using v_{2m}, instead of v_m, the generating function for the case last considered becomes $\dfrac{e^{\frac{t^2}{4}}}{\sqrt[4]{(1-t^2)}}$. Let $(1\,.\,2\,.\,3 \dots m)\,V_m = U_m$ in general be used to denote the number of distinct terms in a vertebrate skew-determinant of the mth order. Then obviously
$$U_{2m} = u_{2m} + m\,.\,\frac{m-1}{2}\,u_{2m-2} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{m-3}{4}\,u_{2m-4} + \dots,$$

* The values of $v_1, v_2, v_3 \dots$ are $\frac{1}{2}, \frac{2}{2\,.\,4}, \frac{8}{2\,.\,4\,.\,6}, \frac{50}{2\,.\,4\,.\,6\,.\,8}, \dots$; that is, $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{25}{192}, \dots$

or $$V_{2m} = v_{2m} + \frac{v_{2m-2}}{1.2} + \frac{v_{2m-4}}{1.2.3.4} + \dots$$

Hence the generating function for V_{2m}

$$= \frac{e^{\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\left\{1 + \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} + \dots\right\} = \frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} + e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\},$$

and in like manner, since

$$U_{2m-1} = mu_{2m-2} + \frac{m(m-1)(m-2)}{1.2.3} u_{2m-4} + \dots,$$

the generating function for V_{2m-1} will be

$$\frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} - e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\}.$$

Hence the number of distinct cross-products in the development of an orthogonal transformation-matrix of the mth order is

$$(1.2.3 \dots m) \times \text{coefficient of } t^m \text{ in } \frac{e^{t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}.$$

Postscript.—Let us consider the case of $2m$ elements; call the number of ways in which any disyntheme composed with them may be resolved into a pair of single synthemes one in each hand* its weight; furthermore, call the aggregate of those which appertain to an odd number of cycles the first class, and the other the second class. The entire sum of the weights we know is $1^2 . 2^2 . 3^2 \dots (2m-1)^2$, but, furthermore, I find that the excess of the total weight of the first class over that of the second is

$$1^2 . 2^2 . 3^2 \dots (2m-3)^2 (2m-1);$$

or, in other words, the weights of the two classes are in the ratio of m to $m-1$.

The expressions for the sum and for the difference may, of course, by the *prolegomena* be translated into two theorems on the partition of numbers, neither of which, as far as I can see, is obvious upon the face of it†.

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* The two hands are introduced in order to double, by the effect of permutation, what the weight otherwise would be, except when the two component synthemes are identical, in which case the weight remains unity.

† Remark [by F. Franklin].—The equation in differences for the number of double duadic synthemes may be obtained without recourse to determinants, as follows: Single out any element, 1; it may be paired in each of the component synthemes with any one of the remaining elements 2, 3, 4, ..., and there are two cases to be distinguished, namely, 1 may be paired either with the same element (2) or with two different elements (2, 3), in the two synthemes. The former may be done in $(m-1)$ ways, and, after having made our choice, we have still the choice of all the double synthemes that can be formed from 3, 4, ... m; 3, 4, ... m. The choice of two *different* elements may be made in $\frac{(m-1)(m-2)}{2}$ ways, and having chosen, we have still the choice of all the double

The properties of the ω series 1, 1, 2, 8, 50, ... [see p. 269] present some features of interest. These are the numbers of distinct terms in pure skew determinants of the order $2n$ divided by the product of the odd integers inferior to $2n$. Such numbers themselves may be termed the denumerants, and the quotients, when they are so divided, the reduced denumerants of the corresponding determinants; or for greater brevity we may provisionally call these reduced denumerants *skew numbers*. We have found, in what precedes, that

$$\frac{e^{\frac{t}{4}}}{\sqrt[4]{(1-t)}} = \omega_0 + \omega_1 \frac{t}{2} + \omega_2 \frac{t^2}{2\,.\,4} + \omega_3 \frac{t^3}{2\,.\,4\,.\,6} + \dots.$$

From this we may easily obtain

$$\omega_x = \frac{Fx}{2^x},$$

where

$$Fx = 1 + 1\,.\,x + 1\,.\,5\,\frac{x(x-1)}{1\,.\,2} + 1\,.\,5\,.\,9\,\frac{x(x-1)(x-2)}{1\,.\,2\,.\,3} + \dots$$
$$+ \dots + 1\,.\,5\,.\,9 \dots (4x-3),$$

which shows that Fx, for all values of x, contains 2^x as a factor, and that if we take x greater than unity, 2^{x+1} will be a factor of Fx. In general, it follows from the fundamental equation $\omega_x = (2x-1)\,\omega_{x-1} - (x-1)\,\omega_{x-2}$ that if two consecutive skew numbers ω_c, ω_{c+1} have a common factor, all those of superior orders, and consequently $\frac{Fx}{2^x}$, for all values of x from c upwards, will contain such factor. It becomes then a matter of interest to assign, if possible, a general expression for the greatest common measure of ω_x, ω_{x+1}.

In the first place I say these can have no common odd factor other than unity.

Lemma. It is well known that, in the development of $(1+a)^x$, all the coefficients except the first and last will contain x when it is a prime number. More generally it may easily be shown (and the mode of proof* is too obvious

synthemes that can be formed from 3, 4, ... m; 3, 4, ... m. Now it is plain that the number of these can be obtained from the number of double synthemes that can be formed from 3, 4, ... m; 3, 4, ... m, by counting twice all except those in which 3 is paired twice with the same element; and is equal, therefore, from what precedes, to

$$2u_{m-2} - (m-3)\,u_{m-4}.$$

We have, therefore,

$$u_m = (m-1)\,u_{m-2} + \frac{(m-1)(m-2)}{2}\,[2u_{m-2} - (m-3)\,u_{m-4}]$$
$$= (m-1)^2\,u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2}\,u_{m-4}.$$

* Some of the prolixity of the more obvious mode of proof of this lemma may be avoided by the substitution of the following method :

Call $(1+t)^n = 1 + A_1 t + A_2 t^2 + A_3 t^3 + \dots$, so that

$$n\,(1+t)^{n-1} = A_1 + 2A_2 t + 3A_3 t^2 + \dots$$
$$= B_0 + B_1 t + B_2 t^2 + \dots = \phi t.$$

to need setting out) that whatever x may be, any prime number contained in it must either divide any number r, or else the coefficient of a^r in the binomial expression above referred to. Hence we may prove that ω_x and x cannot have a common odd factor other than unity. For if possible, let $x = qp$, where p is a prime number contained in ω_x. Let the qp terms in Fx subsequent to the first term be divided into q groups, each containing p terms. Each of the terms in any one group (except the last) contains a binomial coefficient, which, by virtue of the lemma, will contain p. Moreover, the last term in the kth group will contain the factor

$$1 . 5 . 9 \ldots (4kp - 1).$$

If p is of the form $4n - 3$, the nth term of the series 1, 5, 9, ... will be p, and if it is of the form $4n - 1$, the $(3n)$th term will be $3p$; and as $\frac{p+3}{4}$ and $3\frac{p+1}{4}$ are each not greater than p (and *à fortiori* not greater than kp) when p is greater than 1, it follows that the last coefficient, as well as all the others in any group, contains p. Hence $Fx = pP + 1$, and therefore ω_x, that is, $\frac{Fx}{2^x}$, cannot contain p. Hence the greatest common measure of ω_x and ω_{x+1} is a power of 2.

It will presently be shown by induction (waiting a strict proof)* that $\frac{\omega_{4x-2}}{2^x}$, $\frac{\omega_{4x-1}}{2^x}$, $\frac{\omega_{4x}}{2^x}$, $\frac{\omega_{4x+1}}{2^x}$ are all of them integers, and the first, third and fourth, odd integers; from this it will easily be seen that the greatest common measure of ω_x, ω_{x+1} is $2^{\theta\left(\frac{2x+1}{8}\right)}$, where, in general, $\theta(\mu)$ means the integer nearest to μ. Let us call the above fractions q_{4x-2}, q_{4x-1}, q_{4x}, q_{4x+1}, to which we may give the name of simplified skew numbers. In the subjoined table I have calculated the values of the residues of these numbers by a regular algorithm in respect to *moduli* beginning with 2^{23} and regularly decreasing according to the descending powers of 2. R stands for the words *residue of*.

Suppose $n = qp$: then designating the qth roots of unity by $\rho_1, \rho_2 \ldots \rho_q$, we have

$$\frac{1}{q}\Sigma\rho^{q-k}\phi(\rho t) = B_k t^k + B_{k+q}t^{k+q} + B_{k+2q}t^{k+2q} + \ldots + B_{k+(p-1)q}t^{k+(p-1)q},$$

and the left hand side of the equation is obviously a multiple of p. Hence, putting t successively equal to 0, 1, 2, 3, ... $(p-1)$, we obtain, by a well-known theorem of determinants,

$$\Delta B_{k+\lambda q} \equiv 0 \text{ [mod } p],$$

where Δ, being the product of the differences of 0, 1, 2, ... $(p-1)$, cannot contain p. Hence $B_{k+\lambda q} \equiv 0$ [mod p], and consequently giving k all values from 0 to $(q-1)$, and λ all values from 0 to $(p-1)$, we see that all the B's, from B_0 to B_{pq-1}, must contain p as a factor as was to be proved.

* Since the above was set up in print, I have found an easy proof, for which see *Postscript* [p. 279 below].

Modulus	x	Rq_{4x-2}	Rq_{4x-1}	Rq_{4x}	Rq_{4x+1}
8,388,608	0			1	1
4,194,304	1	1	4	25	209
2,097,152	2	1,087	13,504	194,951	1,088,983
1,048,576	3	929,451	442,068	992,179	576,715
524,288	4	287,913	118,168	393,089	71,201
262,144	5	201,913	14,228	126,417	179,945
131,072	6	51,071	56,656	46,407	127,767
65,536	7	56,531	24,452	15,131	46,739
32,768	8	12,521	29,928	22,753	29,729
16,384	9	14,289	5,412	15,209	14,305
8,192	10	1,119	2,784	4,063	4,751
4,096	11	3,283	3,156	2,331	3,059
2,048	12	1,721	1,632	425	1,801
1,024	13	913	84	1,001	385
512	14	215	240	479	239
256	15	91	132	99	219
128	16	81	8	9	9
64	17	41	36	1	57
32	18	23	0	31	15
16	19	3	4	11	3
8	20	1	0	1	1
4	21	1	0	1	1
2	22	1	0	1	1

From this table it appears that q_{8i-5} is 4 times an odd number, and that q_{8i-1} is 8 times a number which may be odd or even; thus we know the exact number of times that 2 will divide out all the skew numbers other than those whose orders are of the form $8i-1$, and an inferior limit to that number for that case.

It will further be noticed that, when x is of the form $4i$, or $4i+1$, the simplified skew numbers q_{4x-2}, q_{4x}, q_{4x+1} are all of the form $8\lambda+1$, that when x is of the form $4i+2$ the above named simplified skew numbers are of the form $8\lambda+7$, and when x is of the form $4i+3$, they are of the form $8\lambda+3$.

Before quitting this subject, I think it desirable briefly to refer to other series of integers closely connected with those which I have called *skew numbers*. To this end we may write, in general,

$$e^{\frac{t}{4}}(1-t)^{\frac{4\mu-1}{4}} = 1 + \omega_{1,\mu}\frac{t}{2} + \omega_{2,\mu}\frac{t^2}{2\,.\,4} + \omega_{3,\mu}\frac{t^3}{2\,.\,4\,.\,6} + \dots,$$

μ being any positive or negative integer, so that $\omega_{x,0}$ is the same as I have called hitherto ω_x. It may then easily be shown that $\omega_{x,\mu+1} = \frac{2\omega_{x+1,\mu} - \omega_{x,\mu}}{4\mu+1}$, that $\omega_{x,\mu-1} = \omega_{x,\mu} - 2x\omega_{x-1,\mu}$, and that the equation in differences for $\omega_{x,\mu}$, for μ constant, becomes

$$\omega_{x,\mu} = (2x + 2\mu - 1)\,\omega_{x-1,\mu} - (x-1)\,\omega_{x-2,\mu},$$

with the initial conditions $\omega_{0,\mu} = 1$, $\omega_{1,\mu} = 2\mu + 1$. Also, it is clear from the definition, that the explicit value of $\omega_{x,\mu}$ in a series becomes

$$\frac{1}{2^x}\Big\{1 + (4\mu+1)\,x + (4\mu+1)(4\mu+5)\,x\,\frac{x-1}{2} + (4\mu+1)(4\mu+5)(4\mu+9)\,x\,\frac{x-1}{2}\cdot\frac{x-2}{3} + \ldots\Big\},$$

which is easily seen to verify the equation

$$2\omega_{x,\mu} - \omega_{x-1,\mu} = (4\mu+1)\,\omega_{x-1,\mu+1}.*$$

We might call the $\omega_{x,\mu}$ series skew numbers of the μth degree, and, as for the case of $\mu = 0$, so it may be shown in general that two consecutive skew numbers of the same degree can have no common odd factor. Also, it remains true that the greatest common factor of any two consecutive skew numbers of the same degree and the orders x, $x+1$, is $2^{\theta\left(\frac{2x+1}{8}\right)}$; $\omega_{4x-2,\mu}$, $\omega_{4x-1,\mu}$, $\omega_{4x,\mu}$, $\omega_{4x+1,\mu}$ being all divisible by 2^x, and the resulting quotients being, the first, third and fourth of them, always odd integers, and the second divisible by 4 or some higher power of 2 when μ is even, but only by the first power of 2 when μ is odd. But it would carry me too far away from the original object of this note, and from other investigations of more pressing moment to myself, to pursue further the theory of general skew numbers, which, however, seems to me to be well worthy of the study of arithmeticians.

I will only stop to point out that the rule for the greatest common measure of ω_x and ω_{x+1}, serves to prove the rule for the general case of $\omega_{x,\mu}$ and $\omega_{x+1,\mu}$. Thus suppose μ to be positive. Then since $\omega_{k,1} = 2\omega_{k+1} - \omega_k$, and $\omega_{4k-2} = 2^k(2\lambda+1)$, $\omega_{4k-1} = 2^{k+1}\tau$, $\omega_{4k} = 2^k(2\nu+1)$, $\omega_{4k+1} = 2^k(2\pi+1)$, $\omega_{4k+2} = 2^{k+1}(2\rho+1)$; it follows that

$\omega_{4k-2,1} = 2^k(2\lambda'+1)$, $\omega_{4k-1,1} = 2^{k+1}\tau'$, $\omega_{4k,1} = 2^k(2\nu'+1)$, and $\omega_{4k+1,1} = 2^k(2\pi'+1)$.

It is obvious further that, τ being even, τ' is odd. So again from these results we may, in like manner, deduce

$\omega_{4k-2,2} = 2^k(2\lambda''+1)$, $\omega_{4k-1,2} = 2^{k+1}\tau''$, $\omega_{4k,2} = 2^k(2\nu''+1)$, $\omega_{4k+1,2} = 2^k(2\pi''+1)$,

* And of course, in general, the equation

$$\lambda u_{x,y} - u_{x-1,y} + \phi y u_{x-1,y+\delta} = 0,$$

with the condition that $u_{0,y}$ is constant, has for its integral

$$u_{x,y} = \frac{c}{\lambda^x}\left\{1 - \phi y x + \phi y \phi(y+\delta)\,x\,\frac{x-1}{2} - \phi y \phi(y+\delta)\,\phi(y+2\delta)\,x\,\frac{x-1}{2}\,\frac{x-2}{3} + \ldots\right\}.$$

subject also to the remark that, τ' being odd, τ'' is even, and so on continually, τ being alternately even and odd. Again if μ is negative, we may, in like manner, by means of the formula $\omega_{k,\mu-1} = \omega_{k,\mu} - 2k\omega_{k-1,\mu}$, pass successively from the case of ω_k to that of $\omega_{k,-1} : \omega_{k,-2} : \ldots \omega_{k,-\mu}$, and establish precisely the same conclusion in regard to powers of 2 as for the case of μ positive, and it will be remembered that I have already shown how to establish that $\omega_{k,\mu}$ and $\omega_{k+1,\mu}$ have no common *odd* factor.

In the first note on this subject (Vol. II, No. 1, of the *Journal**) I showed how a general determinant could be completely represented by means of systems of cycles and that accordingly the terms in the total development would split up into families, as many in number as there are indefinite partitions of the index of the order of the determinant—the particular mode of aggregation depending upon the term chosen to represent the product of the elements in the principal diagonal, so that for the order n there would be $1.2.3 \ldots n$ distinct modes of distribution into families. This gives rise to a theory of transformation of cycles, corresponding to a transposition of the rows or columns of the matrix. Thus, for example, suppose the *umbrae* to be $1, 2, 3, \ldots n$: r, s signifying the element in the rth row and sth column. Then if we interchange the mth and nth columns, this will have the effect of changing pm into pn and pn into pm.

Suppose now that a term of the developed determinant is expressed by a system of cycles such that m and n lie in two distinct cycles, say Xm and nY, where X, Y are each of them single elements, or aggregates of single elements; then the effect of the interchange will be to bring these cycles into the single cycle $XnYm$. If Xm, nY were both odd ordered or both even ordered cycles, their sum will be even ordered, and the number of *even* cycles will be increased or diminished by unity; so if one was of odd and the other of even order, their sum will be of odd order, and the number of even cycles will be diminished by unity. In either case, therefore, the sign, which depends on the *parity* of the number of even cycles, is reversed.

Again, suppose m and n to lie in the same cycle $mXnY$. Then the effect of the interchange will be to break this up into two cycles mX, nY, and for the same reason as above the sign will be reversed. Thus the sign of every term in the development will, we see, be reversed, as we know *à priori* ought to be the case.

[* p. 264 above.]

I shall conclude with applying the formula $\omega_x = \frac{Fx}{2^x}$ to determining the *asymptotic* mean value of the coefficients in a skew determinant of the order $2x$, that is, the function of x to which the mean value of the coefficients converges when x is taken indefinitely great. We know that all the coefficients, both in this case and in that of a symmetrical determinant, are different powers of 2; to find the mean of the indices of these powers would be seemingly an investigation of considerable difficulty, but there will be little or none in finding the ultimate expression for the mean of the coefficients themselves, or, which is the same thing, the first term in the function which expresses this mean in terms of descending powers of x. We shall find that, for symmetrical determinants, this is a certain multiple of the square root and, for skew determinants, of the fourth root of x, as I proceed to show.

From the equation

$$2^x \omega_x = 1 + x + 5x\frac{x-1}{2} + \ldots + \{1 . 5 \ldots (4x-3)\},$$

we have, when $x = \infty$,

$$2^x \omega_x = 1 . 5 . 9 \ldots (4x-3)\left\{1 + \frac{x}{4x-3} + \frac{1}{2}\frac{x(x-1)}{(4x-3)(4x-7)} + \ldots\right\}$$

$$= e^{\frac{1}{4}} . 1 . 5 . 9 \ldots (4x-3).$$

The number of terms in the Pfaffian (the square root of the determinant taken with suitable algebraical sign) being $1 . 3 . 5 \ldots (2x-1)$ and—as follows from what was shown in the first note—cancelling being out of the question, the sum of the coefficients all taken positively in the determinant itself will be $\{1 . 3 . 5 \ldots (2x-1)\}^2$. Hence the mean value required is $\{1 . 3 . 5 \ldots (2x-1)\}^2$ divided by $1 . 3 . 5 \ldots (2x-1)\,\omega_x$, to express which quotient in exact terms we may make use of the formula

$$\frac{a(a+\delta)(a+2\delta)\ldots(a+x\delta)}{b(b+\delta)(b+2\delta)\ldots(b+x\delta)} = \frac{\Gamma\frac{b}{\delta}}{\Gamma\frac{a}{\delta}}\, x^{\frac{a-b}{\delta}}.$$

For the mean value is

$$\frac{1}{e^{\frac{1}{4}}} . \frac{1 . 3 . 5 \ldots (2x-1)}{2 . 4 . 6 \ldots (2x)} . \frac{4 . 8 . 12 \ldots (4x)}{1 . 5 . 9 \ldots (4x-3)} = \frac{1}{e^{\frac{1}{4}}} . \frac{1}{\Gamma\frac{1}{2}}\, x^{-\frac{1}{2}} . \Gamma\tfrac{1}{4}\, x^{\frac{3}{4}} = \frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\sqrt{\pi}}\, x^{\frac{1}{4}}.$$

If we write this under the form $Qx^{\frac{1}{4}}$, we have

$$Q = \frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\,\Gamma\frac{1}{2}},$$

$$\log Q = \log \Gamma \tfrac{5}{4} + \log 2 - \log \Gamma \tfrac{3}{2} - \tfrac{1}{4} \log e$$
$$= 9{\cdot}9573211 + {\cdot}3010300 - 9{\cdot}9475449 - {\cdot}1085736$$
$$= {\cdot}2022326,$$

or $Q = 1{\cdot}59306.$

This result as may easily be seen remains unaffected when, instead of a pure skew determinant, one is taken in which the diagonal terms retain general values. The effect of this change will be to increase the numerator and denominator of the fraction which expresses the mean value, in the proportion of $\frac{e^2+1}{2e}$ to 1.

Finally, as regards the ultimate mean value of the coefficients of symmetrical determinants. This, for one of the order x, by virtue of Professor Cayley's formula previously given, will be the reciprocal of the coefficient of t^x in $\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$. It may readily be shown in general that, ϕt being any series of integer powers of t, the coefficient of t^x (when x becomes infinite) in $\frac{e^{\phi t}}{\sqrt{(1-t)}}$ is in a ratio of equality to the coefficient of t^x in $\frac{e^{(\phi 1)\,t}}{\sqrt{(1-t)}}$, so that in the present case this coefficient is the same as the coefficient of t^x in $\frac{e^{\frac{3}{4}t}}{\sqrt{(1-t)}}$, that is, in

$$\left\{1+\frac{1}{2}t+\frac{1\,.\,3}{2\,.\,4}t^2+\ldots+\frac{1\,.\,3\,.\,5\ldots(2x-1)}{2\,.\,4\,.\,6\ldots 2x}t^x+\ldots\right\}$$
$$\times\left\{1+\frac{3}{4}t+\left(\frac{3}{4}\right)^2\frac{t^2}{2}+\ldots+\left(\frac{3}{4}\right)^x\frac{t^x}{1\,.\,2\ldots x}+\ldots\right\},$$

which is obviously, when x is infinite, equal to $\frac{1\,.\,3\,.\,5\ldots(2x-1)}{2\,.\,4\,.\,6\ldots 2x}e^{\frac{3}{4}}$. Hence the ultimate mean value of the coefficients is $\frac{1}{e^{\frac{3}{4}}}\frac{2\,.\,4\,.\,6\ldots 2x}{1\,.\,3\,.\,5\ldots(2x-1)}$, or $\frac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$.

For a symmetrical determinant in which all the diagonal terms are wanting, the numerator of the fraction giving the mean value becomes $e^{-1}(1\,.\,2\,.\,3\ldots x)$ and the denominator is $(1\,.\,2\,.\,3\ldots x)$ into the coefficient of t^x in $\frac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$, which is the same as in $\frac{e^{-\frac{1}{4}t}}{\sqrt{(1-t)}}$. The result then is $\frac{\pi^{\frac{1}{2}}e^{\frac{1}{4}}}{e}\sqrt{x}$, or $\frac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$ as before. It may perhaps be just worth while to notice that the *skew numbers* (the ω's of the text) may be put under the form of a determinant, the nature of which is sufficiently indicated by the annexed diagram.

1	1	0	0	0	0	0
1	3	2	0	0	0	0
0	1	5	3	0	0	0
0	0	1	7	4	0	0
0	0	0	1	9	5	0
0	0	0	0	1	11	6
0	0	0	0	0	1	13

The successive principal minors in this matrix represent the successive skew numbers of all orders from 1 to 6 inclusive.

Postscript. [See p. 273 above, footnote.]

Since $\omega_{x+1} = (2x+1)\,\omega_x - x\omega_{x-1}$, we have

$$\omega_{x+2} = (4x^2 + 7x + 2)\,\omega_x - (2x^2 + 3x)\,\omega_{x-1},$$

$$\omega_{x+3} = (8x^3 + 32x^2 + 34x + 8)\,\omega_x - (4x^3 + 15x^2 + 13x)\,\omega_{x-1},$$

$$\omega_{x+4} = (16x^4 + 116x^3 + 273x^2 + 231x + 50)\,\omega_x - (8x^4 + 56x^3 + 122x^2 + 82x)\,\omega_{x-1}.$$

Suppose now that, for a given value of i, $q_{4i-2} = \frac{\omega_{4i-2}}{2^i} = 2\lambda + 1$, $q_{4i-1} = \frac{\omega_{4i-1}}{2^i} = 4\mu$, $q_{4i} = \frac{\omega_{4i}}{2^i} = 2\nu + 1$ and $q_{4i+1} = \frac{\omega_{4i+1}}{2^i} = 2\rho + 1$. Call $\omega_{x+4} = E_x\omega_x - F_x\omega_{x-1}$. Then when $x \equiv \pm 2$, $F_x \equiv 4$ [mod 8], and therefore, assuming that $q_{4i-3} = \frac{\omega_{4i-3}}{2^{i-1}}$ is odd, $\frac{F_{4i-2}\omega_{4i-3}}{2^{i+1}}$ is odd. Also, $E_{4i-2} \equiv 462 + 50 \equiv 0$ [mod 4], and consequently $\frac{E_{4i-2}\omega_{4i-2}}{2^{i+1}}$ is even; hence $q_{4i+2} = \frac{\omega_{4i+2}}{2^{i+1}}$ is integer and odd. Again when $x = 4i - 1$, $E_x \equiv 1 - 3 + 50 \equiv 0$ [mod 4], and $F_x \equiv 122 - 82 \equiv 0$ [mod 8]; hence $q_{4i+3} = \frac{\omega_{4i+3}}{2^{i+1}}$ is an integer divisible by 4. Again, when $x = 4i$, $E_{4i} \equiv 2$ and $F_{4i} \equiv 0$ [mod 4]; hence $q_{4i+4} = \frac{\omega_{4i+4}}{2^{i+1}}$ is integer and odd; and when $x = 4i + 1$, $E_{4i+1} \equiv 2$ and $F_{4i+1} \equiv 0$ [mod 4]; hence $q_{4i+5} = \frac{\omega_{4i+5}}{2^{i+1}}$ is integer and odd.

Thus it has been shown that if it be true up to $\lambda = i$ that $\frac{\omega_{4\lambda-2}}{2^\lambda}$, $\frac{\omega_{4\lambda-1}}{2^{\lambda+2}}$, $\frac{\omega_{4\lambda}}{2^\lambda}$, $\frac{\omega_{4\lambda+1}}{2^\lambda}$ are all integer, and the first, third and fourth odd integers, the same proposition can be affirmed for all superior values of i, and being true for ω_0, ω_1, ω_2, ω_3, the quotients corresponding to which are 1, 1, 1, 1, the theorem is true universally. It is inconceivable that it could have occurred to any human being to lay down so singular a train of induction as the one above employed, unless previously prompted to do so by an *à priori* perception of the law to be established, acquired through a preliminary study and direct inspection of the earlier terms in the series of numbers to which it applies. Here then we have a salient example (if any were needed) of the importance of the part played by the *faculty of observation* in the discovery and establishment of pure mathematical laws.