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TABLES OF THE GENERATING FUNCTIONS AND GROUNDFORMS FOR THE BINARY QUANTICS OF THE FIRST TEN ORDERS.

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IN what follows, "G. F." stands for the words *Generating Function*. In the Generating Functions, the exponents of the letter a refer to degree in the coefficients, and the exponents of the letter x to order in the variables. The Generating Functions for differentiants take account only of degree in the coefficients, without regard to the order in the variables of the covariant of which the differentiant is the "*source*." In the *tabulated* numerators of the Generating Functions, the *minus* sign is placed *over* instead of *to the left of* the number which it affects.

QUADRIC.

G. F. for differentiants, $\dfrac{1}{(1-a)(1-a^2)}$.

G. F. for covariants, $\dfrac{1}{(1-a^2)(1-ax^2)}$.

Groundforms: 1 of deg. 1, ord. 2; 1 of deg. 2, ord. 0.

CUBIC.

G. F. for differentiants, $\dfrac{1+a^3}{(1-a)(1-a^2)(1-a^4)}$.

G. F. for covariants, reduced form, $\dfrac{1-ax+a^2x^2}{(1-a^4)(1-ax)(1-ax^3)}$.

G. F. for covariants, representative form, $\dfrac{1+a^3x^3}{(1-a^4)(1-a^2x^2)(1-ax^3)}$.

Groundforms: 1 of deg. 1, ord. 3; 1 of deg. 2, ord. 2; 1 of deg. 3, ord. 3; 1 of deg. 4, ord. 0.

QUARTIC.

G. F. for differentiants, $\dfrac{1+a^3}{(1-a)(1-a^2)^2(1-a^3)}$.

G. F. for covariants, reduced form, $\dfrac{1-ax^2+a^2x^4}{(1-a^2)(1-a^3)(1-ax^2)(1-ax^4)}$.

G. F. for covariants, representative form, $\dfrac{1+a^3x^6}{(1-a^2)(1-a^3)(1-a^2x^4)(1-ax^4)}$.

Groundforms: 1 of deg. 1, ord. 4; 1 of deg. 2, ord. 0; 1 of deg. 2, ord. 4; 1 of deg. 3, ord. 0; 1 of deg. 3, ord. 6.

QUINTIC.

G. F. for differentiants,

$$\frac{1+a^2+3a^3+3a^4+5a^5+4a^6+6a^7+6a^8+4a^9+5a^{10}+3a^{11}+3a^{12}+a^{13}+a^{15}}{(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-ax)(1-ax^3)(1-ax^5)$.

Numerator: $1+a(-x-x^3)+a^2(x^2+x^4+x^6)-a^3x^7+a^4x^4+a^5(x+x^3-x^5)$
$+a^6(-1-x^4)+a^7(2x+x^3+x^5)+a^8(-x^2-x^4-2x^6)$
$+a^9(x^3+x^7)+a^{10}(x^2-x^4-x^6)-a^{11}x^3+a^{12}+a^{13}(-x-x^3+x^5)$
$+a^{14}(x^4+x^6)-a^{15}x^7$.

G. F. for covariants, representative form,

Denominator: $(1-a^4)(1-a^8)(1-a^{12})(1-a^2x^2)(1-a^2x^6)(1-ax^5)$.

Numerator: $1+a^3(x^3+x^5+x^9)+a^4(x^4+x^6)+a^5(x+x^3+x^7-x^{11})$
$+a^6(x^2+x^4)+a^7(x+x^5-x^9)+a^8(x^2+x^4)+a^9(x^3+x^5-x^7)$
$+a^{10}(x^2+x^4-x^{10})+a^{11}(x+x^3-x^9)+a^{12}(x^2-x^8-x^{10})$
$+a^{13}(x-x^7-x^9)+a^{14}(x^4-x^6-x^8)+a^{15}(-x^7-x^9)$
$+a^{16}(x^2-x^6-x^{10})+a^{17}(-x^7-x^9)+a^{18}(1-x^4-x^8-x^{10})$
$+a^{19}(-x^5-x^7)+a^{20}(-x^2-x^6-x^8)-a^{23}x^{11}$.

Table of Groundforms.

Degree in the Coefficients	Order in the Variables: 0	1	2	3	4	5	6	7	9
1						1			
2			1				1		
3				1		1			1
4	1				1		1		
5		1		1				1	
6			1		1				
7		1				1			
8	1		1						
9				1					
11		1							
12	1								
13		1							
18	1								

Sextic.

G. F. for differentiants, $\dfrac{1 + a^2 + 3a^3 + 4a^4 + 4a^5 + 4a^6 + 3a^7 + a^8 + a^{10}}{(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)}$.

G. F. for covariants, reduced form,*

Denominator: $(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-ax^2)(1-ax^4)(1-ax^6)$.

Numerator: $1 + a(-x^2 - x^4) + a^2(-1 + x^4 + x^6 + x^8) + a^3(-1 + 2x^2 + x^4 - x^{10})$
$+ a^4(x^2 - x^6 - x^8) + a^5(-x^6 - x^8 + x^{10}) + a^6(1 - x^2 - x^8 + x^{10})$
$+ a^7(1 - x^2 - x^4) + a^8(-x^2 - x^4 + x^8) + a^9(-1 + x^6 + 2x^8 - x^{10})$
$+ a^{10}(x^2 + x^4 + x^6 - x^{10}) + a^{11}(-x^6 - x^8) + a^{12}x^{10}$.

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^4)(1-a^6)(1-a^{10})(1-a^2x^4)(1-a^2x^8)(1-ax^6)$.

Numerator: $1 + a^3(x^2 + x^6 + x^8 + x^{12}) + a^4(x^4 + x^6 + x^{10}) + a^5(x^2 + x^4 + x^8 - x^{16})$
$+ a^6(x^4 + 2x^6) + a^7(x^2 + x^4 + x^8 - x^{12}) + a^8(x^2 + x^4 + x^6 - x^{14})$
$+ a^9(x^4 + x^6 - x^{10} - x^{12}) + a^{10}(x^2 + x^4 - x^{12} - x^{14}) + a^{11}(x^4$
$+ x^6 - x^{10} - x^{12}) + a^{12}(x^2 - x^{10} - x^{12} - x^{14}) + a^{13}(x^4 - x^8 - x^{12} - x^{14})$
$+ a^{14}(-2x^{10} - x^{12}) + a^{15}(1 - x^8 - x^{12} - x^{14}) + a^{16}(-x^6 - x^{10} - x^{12})$
$+ a^{17}(-x^4 - x^8 - x^{10} - x^{14}) - a^{20}x^{16}$.

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12
1				1			
2	1		1		1		
3		1		1	1		1
4	1		1	1		1	
5		1	1		1		
6	1			2			
7		1	1				
8		1					
9			1				
10	1	1					
12		1					
15	1						

* This is not strictly the minimum form, its numerator and denominator being divisible by $1-a$; it is, however, the lowest form to which the fraction can be reduced when the factors of the denominator are all of the forms $1-a^r$, $1-a^r x^s$. The same remark applies to the "reduced form" in the case of the decimic.

Septimic.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})$.

Numerator: $1+2a^2+6a^3+10a^4+19a^5+28a^6+44a^7+61a^8+79a^9$
$+102a^{10}+129a^{11}+156a^{12}+173a^{13}+196a^{14}+215a^{15}$
$+230a^{16}+231a^{17}+231a^{18}+230a^{19}+215a^{20}+196a^{21}$
$+173a^{22}+156a^{23}+129a^{24}+102a^{25}+79a^{26}+61a^{27}+44a^{28}$
$+28a^{29}+19a^{30}+10a^{31}+6a^{32}+2a^{33}+a^{35}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-ax)(1-ax^3)$
$(1-ax^5)(1-ax^7)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
a^0	1														
a^1		$\bar{1}$		$\bar{1}$		$\bar{1}$									
a^2			1		1		2		1		1				
a^3								$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$	
a^4					2				1						1
a^5		1		2						$\bar{1}$		$\bar{1}$			
a^6	$\bar{1}$		2		$\bar{1}$				$\bar{1}$		$\bar{1}$		1		
a^7		4		1		3				$\bar{1}$		1			
a^8	2		$\bar{1}$				$\bar{3}$		$\bar{3}$		$\bar{1}$		$\bar{1}$		
a^9		1		3		1		$\bar{1}$		2				2	
a^{10}	$\bar{1}$		4				$\bar{1}$		$\bar{2}$		$\bar{2}$				$\bar{1}$
a^{11}		5		3		2		$\bar{1}$		$\bar{2}$		$\bar{1}$		1	
a^{12}	5		1				$\bar{4}$		$\bar{6}$		$\bar{4}$		$\bar{1}$		2
a^{13}		1				$\bar{4}$		$\bar{4}$		$\bar{1}$		1		4	
a^{14}	2		5		1		1		$\bar{2}$				3		$\bar{1}$
a^{15}		8		$\bar{1}$		$\bar{1}$		$\bar{7}$		$\bar{5}$		$\bar{1}$		$\bar{1}$	
a^{16}	6		3		3		$\bar{4}$		$\bar{8}$				$\bar{1}$		5
a^{17}		$\bar{1}$		$\bar{2}$		$\bar{9}$		$\bar{8}$		$\bar{4}$		$\bar{3}$		4	

Numerator—(*Continued.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
a^{18}	2		6		1		2		2		1		6		2
a^{19}		4		$\bar{3}$		$\bar{4}$		$\bar{8}$		$\bar{9}$		$\bar{2}$		$\bar{1}$	
a^{20}	5		$\bar{1}$				$\bar{3}$		$\bar{4}$		3		3		6
a^{21}		$\bar{1}$		$\bar{1}$		$\bar{5}$		$\bar{7}$		$\bar{1}$		$\bar{1}$		3	
a^{22}	$\bar{1}$		3				$\bar{2}$		1		1		5		2
a^{23}		4		1		$\bar{1}$		$\bar{4}$		$\bar{4}$				1	
a^{24}	2		$\bar{1}$		$\bar{4}$		$\bar{6}$		$\bar{4}$				1		5
a^{25}		1		$\bar{1}$		$\bar{2}$		$\bar{1}$		2		3		5	
a^{26}	$\bar{1}$				$\bar{2}$		$\bar{2}$		$\bar{1}$				4		$\bar{1}$
a^{27}		2				2		$\bar{1}$		1		3		1	
a^{28}			$\bar{1}$		$\bar{1}$		$\bar{3}$		$\bar{3}$				$\bar{1}$		2
a^{29}				1		$\bar{1}$				3		1		4	
a^{30}			1		$\bar{1}$		$\bar{1}$				$\bar{1}$		2		$\bar{1}$
a^{31}				$\bar{1}$		$\bar{1}$						2		1	
a^{32}	1						1				2				
a^{33}		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$							
a^{34}					1		1		2		1		1		
a^{35}										$\bar{1}$		$\bar{1}$		$\bar{1}$	
a^{36}															1

Owing to the non-existence of an irreducible invariant whose degree is 10, or any multiple of 10, no representative generating function with a *finite* numerator can be obtained for the septimic; the factor $1-a^{10}$ in the denominator has to be got rid of by dividing numerator and denominator by it, or, in other words, by striking it out of the denominator and multiplying the numerator by the infinite series $1+a^{10}+a^{20}+\ldots$. We thus obtain:

G. F. for covariants, representative form, (with infinite numerator),

Denominator: $(1-a^4)(1-a^8)(1-a^{12})^2(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})(1-ax^7)$.

Numerator: (Given to the terms containing the 45th power of a, inclusive; after which, each column can be continued by repeating *the last five coefficients* occurring in it, *ad inf.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1																							
a^3				1		1		1		1		1				1								
a^4					2		1		2		1				1									
a^5		1		2		2		2		2								$\bar{1}$				$\bar{1}$		
a^6			3		2		3		3				2		$\bar{1}$		$\bar{1}$							
a^7		3		2		4		4				1				$\bar{2}$				1				1
a^8	2		3		4		6		1		3		$\bar{1}$		$\bar{2}$				$\bar{1}$					
a^9		3		5		7		1		4				$\bar{2}$		$\bar{1}$		$\bar{2}$				1		
a^{10}			5		8		6		4		1		$\bar{4}$				$\bar{3}$		$\bar{1}$					
a^{11}		5		8		8		8		4		$\bar{4}$		$\bar{1}$		$\bar{5}$		$\bar{1}$						
a^{12}	4		9		9		12		4		$\bar{1}$		$\bar{3}$		$\bar{5}$		$\bar{6}$				$\bar{1}$		1	
a^{13}		9		9		12		6		$\bar{1}$		$\bar{3}$		$\bar{8}$		$\bar{9}$		$\bar{3}$		$\bar{1}$		1		
a^{14}	4		9		13		11		$\bar{1}$		$\bar{3}$		$\bar{9}$		$\bar{10}$		$\bar{7}$		$\bar{2}$				3	
a^{15}		9		12		16		3		2		$\bar{10}$		$\bar{11}$		$\bar{8}$		$\bar{3}$				3		2
a^{16}	5		14		15		12		1		$\bar{5}$		$\bar{16}$		$\bar{9}$		$\bar{9}$		$\bar{1}$		3		3	
a^{17}		12		15		16		6		$\bar{3}$		$\bar{17}$		$\bar{13}$		$\bar{15}$		$\bar{5}$		2		3		
a^{18}	9		14		15		14		$\bar{3}$		$\bar{13}$		$\bar{20}$		$\bar{15}$		$\bar{15}$		2		2		5	
a^{19}		15		16		18				$\bar{8}$		$\bar{18}$		$\bar{20}$		$\bar{19}$		$\bar{3}$		3		5		4
a^{20}	7		14		18		12		$\bar{10}$		$\bar{16}$		$\bar{25}$		$\bar{19}$		$\bar{12}$		2		5		9	
a^{21}		14		17		19		$\bar{1}$		$\bar{8}$		$\bar{27}$		$\bar{25}$		$\bar{16}$		$\bar{2}$		4		8		4
a^{22}	9		17		19		11		$\bar{8}$		$\bar{18}$		$\bar{31}$		$\bar{17}$		$\bar{15}$		-6		9		9	
a^{23}		17		19		18		$\bar{3}$		13		$\bar{31}$		$\bar{25}$		21		$\bar{4}$		9		9		5
a^{24}	8		17		17		10		$\bar{12}$		$\bar{27}$		$\bar{32}$		$\bar{22}$		$\bar{16}$		9		9		12	
a^{25}		18		17		19		$\bar{6}$		$\bar{17}$		$\bar{31}$		$\bar{28}$		$\bar{22}$		3		10		12		9
a^{26}	9		18		18		11		$\bar{17}$		$\bar{23}$		$\bar{34}$		$\bar{21}$		$\bar{10}$		10		14		15	

Numerator—(*Continued.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{27}		17		17		19		$\bar{9}$		$\overline{16}$		$\overline{36}$		$\overline{29}$		$\overline{19}$		3		13		14		7
a^{28}	8		17		18		9		$\overline{16}$		$\overline{26}$		$\overline{38}$		$\overline{18}$		$\overline{13}$		14		15		14	
a^{29}		18		19		17		$\bar{8}$		$\overline{16}$		$\overline{36}$		$\overline{25}$		$\overline{21}$		6		16		16		9
a^{30}	9		18		18		10		$\overline{18}$		$\overline{27}$		$\overline{35}$		$\overline{19}$		$\overline{11}$		16		15		17	
a^{31}		17		17		17		$\bar{8}$		$\overline{19}$		$\overline{36}$		$\overline{29}$		$\overline{19}$		8		15		17		8
a^{32}	9		18		18		8		$\overline{18}$		$\overline{26}$		$\overline{35}$		$\overline{19}$		$\overline{10}$		17		17		18	
a^{33}		18		18		18		$\bar{9}$		$\overline{18}$		$\overline{34}$		$\overline{26}$		$\overline{17}$		8		18		18		9
a^{34}	8		17		17		9		$\overline{17}$		$\overline{28}$		$\overline{36}$		$\overline{18}$		$\bar{8}$		17		17		17	
a^{35}		18		17		18		$\bar{9}$		$\overline{17}$		$\overline{35}$		$\overline{27}$		$\overline{18}$		9		18		17		8
a^{36}	9		19		18		9		$\overline{18}$		$\overline{25}$		$\overline{34}$		$\overline{17}$		$\bar{9}$		17		19		18	
a^{37}		17		17		18		$\bar{9}$		$\overline{18}$		$\overline{37}$		$\overline{26}$		$\overline{18}$		9		17		17		9
a^{38}	9		17		17		9		$\overline{18}$		$\overline{26}$		$\overline{37}$		$\overline{18}$		$\bar{9}$		18		17		17	
a^{39}		18		19		17		$\bar{9}$		$\overline{17}$		$\overline{34}$		$\overline{25}$		$\overline{18}$		9		18		19		9
a^{40}	9		17		18		9		$\overline{18}$		$\overline{27}$		$\overline{35}$		$\overline{17}$		$\bar{9}$		18		17		18	
a^{41}		17		17		17		$\bar{8}$		$\overline{18}$		$\overline{36}$		$\overline{28}$		$\overline{17}$		9		17		17		8
a^{42}	9		18		18		8		$\overline{17}$		$\overline{26}$		$\overline{34}$		$\overline{18}$		$\bar{9}$		18		18		18	
a^{43}		18		18		18		$\bar{9}$		$\overline{18}$		$\overline{34}$		$\overline{26}$		$\overline{17}$		8		18		18		9
a^{44}	8		17		17		9		$\overline{17}$		$\overline{28}$		$\overline{36}$		$\overline{18}$		$\bar{8}$		17		17		17	
a^{45}		18		17		18		$\bar{9}$		$\overline{17}$		$\overline{35}$		$\overline{27}$		$\overline{18}$		9		18		17		9

etc. etc. etc.

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	1	2	3	4	5	6	7	8	9	10	11	14	15
1								1						
2			1				1				1			
3				1		1		1		1		1		1
4	1				2		1		2		1		1	
5		1		2		2		2		2				
6			3		2		2		2					
7		3		2		4		2						
8	3		3		3		3							
9		3		5		2								
10			4		3									
11		5		3										
12	6		6											
13		7												
14	4													
15		3												
16	2													
17		2												
18	9													
22	1													

Octavic.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)^2(1-a^3)^2(1-a^4)(1-a^5)(1-a^7)$.

Numerator: $1+2a^2+6a^3+12a^4+19a^5+25a^6+31a^7+36a^8+38a^9+36a^{10}$
$+31a^{11}+25a^{12}+19a^{13}+12a^{14}+6a^{15}+2a^{16}+a^{18}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)$
$(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}
a^0	1									
a^1		$\bar{1}$	$\bar{1}$	$\bar{1}$						
a^2			1	1	2	1	1			
a^3			1			$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^4			2							1
a^5		1	2		$\bar{1}$		$\bar{1}$			
a^6		1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$	1		
a^7		2	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	1		
a^8	1	2			$\bar{2}$	$\bar{2}$	$\bar{2}$	1		
a^9	1	2	$\bar{2}$		$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	
a^{10}	1	1	$\bar{2}$		$\bar{2}$	$\bar{1}$			1	
a^{11}		1	$\bar{1}$		$\bar{1}$	$\bar{1}$		$\bar{1}$	1	
a^{12}		1			$\bar{1}$	$\bar{2}$		$\bar{2}$	1	1
a^{13}		1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$		$\bar{2}$	2	1
a^{14}			1	$\bar{2}$	$\bar{2}$	$\bar{2}$			2	1
a^{15}			1	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1	2	
a^{16}			1	$\bar{1}$	$\bar{1}$	$\bar{1}$		1	1	
a^{17}				$\bar{1}$		$\bar{1}$		2	1	
a^{18}	1							2		
a^{19}		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$			1		
a^{20}				1	1	2	1	1		
a^{21}							$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^{22}										1

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^2x^4)$
$(1-a^2x^8)(1-a^2x^{12})(1-ax^8)$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}
a^0	1															
a^3			1	1	1	1	1	1		1						
a^4			2	1	1	2	1	1		1						
a^5		1	2	2	1	3		1		$\bar{1}$		$\bar{1}$		$\bar{1}$		
a^6		1	2	3	2	2		1		$\bar{1}$		$\bar{1}$				
a^7		2	2	3	2	2		$\bar{1}$		$\bar{2}$		$\bar{1}$				1
a^8	1	2	2	3	3	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{1}$					
a^9	1	3	1	3	2	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{1}$		1		
a^{10}	1	2	1	2	1	$\bar{2}$	$\bar{2}$	$\bar{5}$	$\bar{4}$	$\bar{4}$	$\bar{2}$	$\bar{1}$		1		
a^{11}		2	1	1	1	$\bar{4}$	$\bar{2}$	$\bar{6}$	$\bar{4}$	$\bar{4}$	$\bar{3}$			2		
a^{12}		1	1	1	1	$\bar{4}$	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\bar{2}$	$\bar{3}$			2	1	
a^{13}		1	2			$\bar{3}$	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\bar{2}$	$\bar{4}$	1	1	1	1	
a^{14}			2			$\bar{3}$	$\bar{4}$	$\bar{4}$	$\bar{6}$	$\bar{2}$	$\bar{4}$	1	1	1	2	
a^{15}			1		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{5}$	$\bar{2}$	$\bar{2}$	1	2	1	2	1
a^{16}			1		$\bar{1}$	$\bar{1}$	$\bar{4}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	2	3	1	3	1
a^{17}						$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	3	3	2	2	1
a^{18}	$\bar{1}$				$\bar{1}$		$\bar{2}$		$\bar{1}$		2	2	3	2	2	
a^{19}					$\bar{1}$		$\bar{1}$		1		2	2	3	2	1	
a^{20}			$\bar{1}$		$\bar{1}$		$\bar{1}$		1		3	1	2	2	1	
a^{21}							1		1	1	2	1	1	2		
a^{22}							1		1	1	1	1	1	1		
a^{25}																1

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	18
1					1				
2	1		1		1		1		
3	1		1	1	1	1	1	1	1
4	1		2	1	1	2	1	1	1
5	1	1	2	2	1	3		1	
6	1	1	2	3	1	1			
7	1	2	2	3					
8	1	2	2	2					
9	1	3	1						
10	1	2							
11		2							
12		1							

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G. F. for differentiants,

Denominator: $(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})$
$(1-a^{14})(1-a^{16})$.

Numerator: $1+3a^2+10a^3+23a^4+49a^5+93a^6+172a^7+289a^8+457a^9$
$+701a^{10}+1036a^{11}+1477a^{12}+2023a^{13}+2720a^{14}+3568a^{15}$
$+4573a^{16}+5702a^{17}+7013a^{18}+8466a^{19}+10043a^{20}+11672a^{21}$
$+13400a^{22}+15155a^{23}+16880a^{24}+18487a^{25}+20013a^{26}$
$+21392a^{27}+22539a^{28}+23398a^{29}+24013a^{30}+24355a^{31}$
$+24355a^{32}+24013a^{33}+23398a^{34}+22539a^{35}+21392a^{36}$
$+20013a^{37}+18487a^{38}+16880a^{39}+15155a^{40}+13400a^{41}$
$+11672a^{42}+10043a^{43}+8466a^{44}+7013a^{45}+5702a^{46}+4573a^{47}$
$+3568a^{48}+2720a^{49}+2023a^{50}+1477a^{51}+1036a^{52}+701a^{53}$
$+457a^{54}+289a^{55}+172a^{56}+93a^{57}+49a^{58}+23a^{59}+10a^{60}$
$+3a^{61}+a^{63}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-a^{14})(1-a^{16})$
$(1-ax)(1-ax^3)(1-ax^5)(1-ax^7)(1-ax^9)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1																							
a^1		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$																
a^2			1		1		2		2		2		1		1									
a^3								$\bar{1}$		$\bar{1}$		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{1}$		$\bar{1}$				
a^4	1				2		1		2				1		1		1		1		1		1	
a^5				2		1				$\bar{2}$		$\bar{1}$		$\bar{2}$		$\bar{1}$		$\bar{1}$						$\bar{1}$
a^6	$\bar{1}$		4		1		3										1		1		1			
a^7		5		5		5		1		$\bar{1}$		$\bar{3}$		$\bar{2}$		$\bar{2}$						$\bar{1}$		
a^8	5		3		4		$\bar{2}$		$\bar{3}$		$\bar{7}$		$\bar{5}$		$\bar{3}$		1		2				1	
a^9		5		8		2		1		$\bar{4}$		$\bar{2}$		$\bar{1}$		3		3		2		1		$\bar{1}$
a^{10}	3		15		5		5		$\bar{3}$		$\bar{7}$		$\bar{7}$		$\bar{2}$		1		$\bar{1}$		1		$\bar{2}$	
a^{11}		17		11		9		$\bar{2}$		$\bar{10}$		$\bar{16}$		$\bar{5}$		$\bar{3}$		2		4		$\bar{1}$		1
a^{12}	18		14		15		$\bar{2}$		$\bar{11}$		$\bar{24}$		$\bar{14}$		$\bar{3}$		3		8		3		1	
a^{13}		17		17		2		$\bar{12}$		$\bar{27}$		$\bar{21}$		$\bar{6}$		8		11		9		3		$\bar{5}$
a^{14}	15		39		21		6		$\bar{13}$		$\bar{26}$		$\bar{13}$		2		13		10		8		$\bar{7}$	
a^{15}		42		24		10		$\bar{28}$		$\bar{45}$		$\bar{52}$		$\bar{17}$		$\bar{5}$		13		11		$\bar{5}$		$\bar{3}$
a^{16}	44		41		31		$\bar{15}$		$\bar{33}$		$\bar{59}$		$\bar{26}$		8		28		31		13		$\bar{2}$	
a^{17}		44		28		$\bar{14}$		$\bar{52}$		$\bar{78}$		$\bar{63}$		$\bar{9}$		15		34		18		1		$\bar{18}$
a^{18}	43		77		33		$\bar{5}$		$\bar{35}$		$\bar{63}$		$\bar{11}$		28		51		84		20		$\bar{20}$	
a^{19}		79		32		$\bar{6}$		$\bar{82}$		$\bar{113}$		$\bar{108}$		$\bar{20}$		3		36		19		$\bar{17}$		$\bar{15}$
a^{20}	82		76		43		$\bar{39}$		$\bar{70}$		$\bar{109}$		$\bar{22}$		48		80		69		29		$\bar{13}$	
a^{21}		76		37		$\bar{43}$		$\bar{121}$		$\bar{159}$		$\bar{117}$				36		70		29		$\bar{10}$		$\bar{44}$
a^{22}	76		122		41		$\bar{35}$		$\bar{75}$		$\bar{112}$		6		83		118		76		38		$\bar{45}$	
a^{23}		120		37		$\bar{41}$		$\bar{163}$		$\bar{201}$		$\bar{165}$		$\bar{5}$		31		75		33		$\bar{43}$		$\bar{43}$
a^{24}	122		112		37		$\bar{86}$		$\bar{121}$		$\bar{161}$		2		120		160		123		40		$\bar{40}$	
a^{25}		109		31		$\bar{92}$		$\bar{205}$		$\bar{242}$		$\bar{154}$		39		83		120		37		$\bar{40}$		$\bar{82}$
a^{26}	107		151		25		$\bar{82}$		$\bar{116}$		$\bar{147}$		52		166		203		117		39		$\bar{82}$	
a^{27}		148		25		$\bar{85}$		$\bar{239}$		$\bar{267}$		$\bar{190}$		44		79		113		33		$\bar{84}$		$\bar{76}$
a^{28}	147		125		13		$\bar{136}$		$\bar{161}$		$\bar{188}$		50		206		237		158		37		$\bar{74}$	
a^{29}		121		14		$\bar{137}$		$\bar{265}$		$\bar{286}$		$\bar{152}$		107		135		157		35		$\bar{77}$		$\bar{122}$
a^{30}	119		153		$\bar{1}$		$\bar{123}$		$\bar{141}$		$\bar{151}$		111		243		263		137		28		$\bar{124}$	
a^{31}		149		1		$\bar{123}$		$\bar{281}$		$\bar{286}$		$\bar{165}$		108		123		138		27		$\bar{127}$		$\bar{107}$
a^{32}	147		112		$\bar{15}$		$\bar{167}$		$\bar{169}$		$\bar{164}$		109		270		280		166		13		$\bar{108}$	

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{33}		108		$\overline{13}$		$\overline{166}$		$\overline{280}$		$\overline{270}$		$\overline{109}$		164		169		167		15		$\overline{112}$		$\overline{147}$
a^{34}	107		127		$\overline{27}$		$\overline{138}$		$\overline{123}$		$\overline{108}$		165		286		281		123		$\overline{1}$		$\overline{149}$	
a^{35}		124		$\overline{28}$		$\overline{137}$		$\overline{263}$		$\overline{243}$		$\overline{111}$		151		141		123		1		$\overline{153}$		$\overline{119}$
a^{36}	122		77		$\overline{35}$		$\overline{157}$		$\overline{135}$		$\overline{107}$		152		286		265		137		$\overline{14}$		$\overline{121}$	
a^{37}		74		$\overline{37}$		$\overline{158}$		$\overline{237}$		$\overline{206}$		$\overline{50}$		188		161		136		$\overline{13}$		$\overline{125}$		$\overline{147}$
a^{38}	76		84		$\overline{33}$		$\overline{113}$		$\overline{79}$		$\overline{44}$		190		267		239		85		$\overline{25}$		$\overline{148}$	
a^{39}		82		$\overline{39}$		$\overline{117}$		$\overline{203}$		$\overline{166}$		$\overline{52}$		147		116		82		$\overline{25}$		$\overline{151}$		$\overline{107}$
a^{40}	82		40		$\overline{37}$		$\overline{120}$		$\overline{83}$		$\overline{39}$		154		242		205		92		$\overline{31}$		$\overline{109}$	
a^{41}		40		$\overline{40}$		$\overline{123}$		$\overline{160}$		$\overline{120}$		$\overline{2}$		161		121		86		$\overline{37}$		$\overline{112}$		$\overline{122}$
a^{42}	43		43		$\overline{33}$		$\overline{75}$		$\overline{31}$		5		165		201		163		41		$\overline{37}$		$\overline{120}$	
a^{43}		45		$\overline{38}$		$\overline{76}$		$\overline{118}$		$\overline{83}$		$\overline{6}$		112		75		35		$\overline{41}$		$\overline{122}$		$\overline{76}$
a^{44}	44		10		$\overline{29}$		$\overline{70}$		$\overline{36}$				117		159		121		43		$\overline{37}$		$\overline{76}$	
a^{45}		13		$\overline{29}$		$\overline{69}$		$\overline{80}$		$\overline{48}$		22		109		70		39		$\overline{43}$		$\overline{76}$		$\overline{82}$
a^{46}	15		17		$\overline{19}$		$\overline{36}$		$\overline{3}$		20		108		113		82		6		$\overline{32}$		$\overline{79}$	
a^{47}		20		$\overline{20}$		$\overline{34}$		$\overline{51}$		$\overline{28}$		11		63		35		5		$\overline{33}$		$\overline{77}$		$\overline{43}$
a^{48}	18		$\overline{1}$		$\overline{18}$		$\overline{34}$		$\overline{15}$		9		63		78		52		14		$\overline{28}$		$\overline{44}$	
a^{49}		2		$\overline{13}$		$\overline{31}$		$\overline{28}$		$\overline{8}$		26		59		33		15		$\overline{31}$		$\overline{41}$		$\overline{44}$
a^{50}	3		5		$\overline{11}$		$\overline{13}$		5		17		52		45		28		$\overline{10}$		$\overline{24}$		$\overline{42}$	
a^{51}		7		$\overline{8}$		$\overline{10}$		$\overline{13}$		$\overline{2}$		13		26		13		$\overline{6}$		$\overline{21}$		$\overline{39}$		$\overline{15}$
a^{52}	5		$\overline{3}$		$\overline{9}$		$\overline{11}$		$\overline{3}$		6		21		27		12		$\overline{2}$		$\overline{17}$		$\overline{17}$	
a^{53}		$\overline{1}$		$\overline{3}$		$\overline{8}$		$\overline{3}$		3		14		24		11		2		$\overline{15}$		$\overline{14}$		$\overline{18}$
a^{54}	$\overline{1}$		1		$\overline{4}$		$\overline{2}$		3		5		16		10		2		$\overline{9}$		$\overline{11}$		$\overline{17}$	
a^{55}		2		$\overline{1}$		1		$\overline{1}$		2		7		7		3		$\overline{5}$		$\overline{5}$		$\overline{15}$		$\overline{3}$
a^{56}	1		$\overline{1}$		$\overline{2}$		$\overline{3}$		$\overline{3}$		1		2		4		$\overline{1}$		$\overline{2}$		$\overline{8}$		$\overline{5}$	
a^{57}		$\overline{1}$				$\overline{2}$		$\overline{1}$		3		5		7		3		2		$\overline{4}$		$\overline{3}$		$\overline{5}$
a^{58}			1						2		2		3		1		$\overline{1}$		$\overline{5}$		$\overline{5}$		$\overline{5}$	
a^{59}				$\overline{1}$		$\overline{1}$		$\overline{1}$										$\overline{3}$		$\overline{1}$		$\overline{4}$		1
a^{60}	1						1		1		2		1		2				$\overline{1}$		$\overline{2}$			
a^{61}		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$				$\overline{2}$		$\overline{1}$		$\overline{2}$				$\overline{1}$
a^{62}					1		1		2		2		2		1		1							
a^{63}										$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{1}$		$\overline{1}$		
a^{64}																	1		1		1		1	
a^{65}																								$\overline{1}$

G. F. for covariants, representative form,

Denominator: $(1-a^4)(1-a^8)(1-a^{10})(1-a^{12})^2(1-a^{14})(1-a^{16})(1-a^2x^6)$
$(1-a^2x^{10})(1-a^2x^{14})(1-ax^9)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^0	1																			
a^3				1		1		1		2		1		1		1		1		
a^4	1				2		2		3		2		2		2		1		1	
a^5		1		3		4		4		3		4		2		2				
a^6			4		4		7		7		5		6		1		2			
a^7		4		8		9		10		11		7		6		2				
a^8	5		8		13		16		16		14		7		6		1		$\overline{1}$	
a^9		10		17		20		22		19		15		7		1		$\overline{3}$		$\overline{7}$
a^{10}	4		20		25		30		33		20		13		2		$\overline{3}$		$\overline{10}$	
a^{11}		21		32		41		43		40		20		11		$\overline{4}$		$\overline{14}$		$\overline{13}$
a^{12}	17		35		50		60		57		37		16				$\overline{18}$		$\overline{25}$	
a^{13}		39		57		75		71		57		28		6		$\overline{29}$		$\overline{34}$		$\overline{41}$
a^{14}	20		64		86		90		92		44		13		$\overline{31}$		$\overline{46}$		$\overline{59}$	
a^{15}		67		94		121		108		96		23		$\overline{11}$		$\overline{63}$		$\overline{73}$		$\overline{79}$
a^{16}	47		103		135		143		135		57		$\overline{7}$		$\overline{65}$		$\overline{91}$		$\overline{117}$	
a^{17}		108		142		181		154		116		3		$\overline{45}$		$\overline{139}$		$\overline{136}$		$\overline{148}$
a^{18}	61		152		195		191		181		37		$\overline{43}$		$\overline{149}$		$\overline{176}$		$\overline{198}$	
a^{19}		157		201		257		199		149		$\overline{38}$		$\overline{104}$		$\overline{239}$		$\overline{221}$		$\overline{222}$
a^{20}	97		211		270		260		225		21		$\overline{107}$		$\overline{252}$		$\overline{271}$		$\overline{302}$	
a^{21}		215		273		339		239		157		$\overline{108}$		$\overline{200}$		$\overline{391}$		$\overline{330}$		$\overline{338}$
a^{22}	120		281		348		308		262		$\overline{42}$		$\overline{206}$		$\overline{412}$		$\overline{410}$		$\overline{434}$	
a^{23}		284		348		418		269		159		$\overline{215}$		$\overline{327}$		$\overline{562}$		$\overline{462}$		$\overline{440}$

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
																				a^0
	1																			a^3
		1																		a^4
			$\overline{2}$				$\overline{1}$				$\overline{1}$									a^5
		$\overline{1}$		$\overline{1}$				$\overline{1}$												a^6
	$\overline{3}$				$\overline{1}$								1				1			a^7
$\overline{3}$		$\overline{4}$		$\overline{1}$		$\overline{2}$				1										a^8
	$\overline{4}$		$\overline{6}$		$\overline{3}$		$\overline{1}$				1				1				$\overline{1}$	a^9
$\overline{11}$		$\overline{9}$		$\overline{7}$		$\overline{2}$						1		1						a^{10}
	$\overline{16}$		$\overline{11}$		$\overline{6}$				2		2		3							a^{11}
$\overline{23}$		$\overline{24}$		$\overline{9}$		$\overline{4}$		1		3		5		2						a^{12}
	$\overline{36}$		$\overline{29}$		$\overline{9}$				4		7		2		2				$\overline{1}$	a^{13}
$\overline{55}$		$\overline{46}$		$\overline{20}$		$\overline{4}$		7		9		11		4		1		$\overline{1}$		a^{14}
	$\overline{65}$		$\overline{40}$		$\overline{9}$		8		20		15		12		4					a^{15}
$\overline{89}$		$\overline{78}$		$\overline{20}$				27		24		23		9		1		$\overline{4}$		a^{16}
	$\overline{102}$		$\overline{74}$		$\overline{5}$		25		38		30		17		7		$\overline{4}$		$\overline{5}$	a^{17}
$\overline{147}$		$\overline{121}$		$\overline{23}$		19		57		41		45		13				$\overline{10}$		a^{18}
	$\overline{150}$		$\overline{87}$		25		57		83		55		39		6		$\overline{8}$		$\overline{4}$	a^{19}
$\overline{202}$		$\overline{164}$		9		50		112		83		74		16		$\overline{3}$		$\overline{21}$		a^{20}
	$\overline{194}$		$\overline{113}$		63		109		137		86		48		6		$\overline{19}$		$\overline{17}$	a^{21}
$\overline{276}$		$\overline{202}$		43		107		194		121		112		16		$\overline{11}$		$\overline{39}$		a^{22}
	$\overline{230}$		$\overline{102}$		149		194		232		126		81		$\overline{2}$		$\overline{34}$		$\overline{20}$	a^{23}

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^{24}	165		353		419		366		278		$\overline{122}$		$\overline{338}$		$\overline{586}$		$\overline{555}$		$\overline{569}$	
a^{25}		353		417		490		275		115		$\overline{356}$		$\overline{481}$		$\overline{777}$		$\overline{593}$		$\overline{551}$
a^{26}	189		415		484		386		269		$\overline{247}$		$\overline{496}$		$\overline{800}$		$\overline{716}$		$\overline{692}$	
a^{27}		413		478		544		254		68		$\overline{519}$		$\overline{652}$		$\overline{976}$		$\overline{708}$		$\overline{622}$
a^{28}	223		471		529		403		235		$\overline{374}$		$\overline{669}$		$\overline{996}$		$\overline{839}$		$\overline{794}$	
a^{29}		464		521		570		211		$\overline{22}$		$\overline{694}$		$\overline{821}$		$\overline{1181}$		$\overline{795}$		$\overline{671}$
a^{30}	241		506		551		375		171		$\overline{530}$		$\overline{840}$		$\overline{1186}$		$\overline{959}$		$\overline{844}$	
a^{31}		499		538		568		139		$\overline{120}$		$\overline{859}$		$\overline{978}$		$\overline{1326}$		$\overline{832}$		$\overline{649}$
a^{32}	254		521		541		332		87		$\overline{669}$		$\overline{988}$		$\overline{1327}$		$\overline{998}$		$\overline{839}$	
a^{33}		510		529		534		49		$\overline{224}$		$\overline{1007}$		$\overline{1088}$		$\overline{1420}$		$\overline{809}$		$\overline{584}$
a^{34}	254		508		508		260		$\overline{5}$		$\overline{792}$		$\overline{1098}$		$\overline{1401}$		$\overline{991}$		$\overline{773}$	
a^{35}		499		492		474		$\overline{42}$		$\overline{322}$		$\overline{1104}$		$\overline{1144}$		$\overline{1432}$		$\overline{729}$		$\overline{459}$
a^{36}	241		475		449		183		$\overline{101}$		$\overline{877}$		$\overline{1143}$		$\overline{1406}$		$\overline{915}$		$\overline{650}$	
a^{37}		464		435		399		$\overline{132}$		$\overline{398}$		$\overline{1144}$		$\overline{1137}$		$\overline{1376}$		$\overline{593}$		$\overline{297}$
a^{38}	223		419		380		97		$\overline{184}$		$\overline{905}$		$\overline{1133}$		$\overline{1335}$		$\overline{788}$		$\overline{483}$	
a^{39}		413		367		311		$\overline{205}$		$\overline{446}$		$\overline{1122}$		$\overline{1076}$		$\overline{1240}$		$\overline{423}$		$\overline{128}$
a^{40}	189		357		297		16		$\overline{240}$		$\overline{891}$		$\overline{1062}$		$\overline{1203}$		$\overline{619}$		$\overline{306}$	
a^{41}		353		288		222		$\overline{251}$		$\overline{456}$		$\overline{1049}$		$\overline{956}$		$\overline{1051}$		$\overline{250}$		47
a^{42}	165		284		217		$\overline{40}$		$\overline{272}$		$\overline{825}$		$\overline{940}$		$\overline{1011}$		$\overline{441}$		$\overline{121}$	
a^{43}		284		210		147		$\overline{274}$		$\overline{446}$		$\overline{923}$		$\overline{801}$		$\overline{844}$		$\overline{80}$		191
a^{44}	120		213		147		$\overline{88}$		$\overline{278}$		$\overline{728}$		$\overline{780}$		$\overline{818}$		$\overline{264}$		34	
a^{45}		215		146		85		$\overline{270}$		$\overline{386}$		$\overline{769}$		$\overline{630}$		$\overline{619}$		65		297
a^{46}	97		152		91		$\overline{101}$		$\overline{256}$		$\overline{588}$		$\overline{615}$		$\overline{599}$		$\overline{107}$		145	
a^{47}		157		94		36		$\overline{242}$		$\overline{333}$		$\overline{604}$		$\overline{465}$		$\overline{427}$		158		338
a^{48}	61		102		46		$\overline{112}$		$\overline{219}$		$\overline{468}$		$\overline{452}$		$\overline{422}$		$\overline{7}$		215	
a^{49}		108		52		5		$\overline{203}$		$\overline{255}$		$\overline{446}$		$\overline{817}$		$\overline{253}$		209		359
a^{50}	47		62		17		$\overline{91}$		$\overline{175}$		$\overline{333}$		$\overline{309}$		$\overline{258}$		76		243	
a^{51}		67		25		$\overline{10}$		$\overline{158}$		$\overline{192}$		$\overline{307}$		$\overline{196}$		$\overline{136}$		224		321

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
$\overline{321}$		$\overline{224}$		136		196		307		192		158		10		$\overline{25}$		$\overline{67}$		a^{24}
	$\overline{243}$		$\overline{76}$		258		309		333		175		91		$\overline{17}$		$\overline{62}$		$\overline{47}$	a^{25}
$\overline{359}$		$\overline{209}$		253		317		446		255		203		$\overline{5}$		$\overline{52}$		$\overline{108}$		a^{26}
	$\overline{215}$		7		422		452		468		219		112		$\overline{46}$		$\overline{102}$		$\overline{61}$	a^{27}
$\overline{338}$		$\overline{158}$		427		465		604		333		242		$\overline{36}$		$\overline{94}$		$\overline{157}$		a^{28}
	$\overline{145}$		107		599		615		588		256		101		$\overline{91}$		$\overline{152}$		$\overline{97}$	a^{29}
$\overline{297}$		$\overline{65}$		619		630		769		386		270		$\overline{85}$		$\overline{146}$		$\overline{215}$		a^{30}
	$\overline{34}$		264		818		780		728		278		88		$\overline{147}$		$\overline{213}$		$\overline{120}$	a^{31}
$\overline{191}$		80		844		801		923		446		274		$\overline{147}$		$\overline{210}$		$\overline{284}$		a^{32}
	121		441		1011		940		825		272		40		$\overline{217}$		$\overline{284}$		$\overline{165}$	a^{33}
$\overline{47}$		250		1051		956		1049		456		251		$\overline{222}$		$\overline{288}$		$\overline{353}$		a^{34}
	306		619		1203		1062		891		240		$\overline{16}$		$\overline{297}$		$\overline{357}$		$\overline{189}$	a^{35}
128		423		1240		1076		1122		446		205		$\overline{311}$		$\overline{367}$		$\overline{413}$		a^{36}
	483		788		1335		1133		905		184		$\overline{97}$		$\overline{380}$		$\overline{419}$		$\overline{223}$	a^{37}
297		593		1376		1137		1144		398		132		$\overline{399}$		$\overline{435}$		$\overline{464}$		a^{38}
	650		915		1406		1143		877		101		$\overline{183}$		$\overline{449}$		$\overline{475}$		$\overline{241}$	a^{39}
459		729		1432		1144		1104		322		42		$\overline{474}$		$\overline{492}$		$\overline{499}$		a^{40}
	773		991		1401		1098		792		5		$\overline{260}$		$\overline{508}$		$\overline{508}$		$\overline{254}$	a^{41}
584		809		1420		1088		1007		224		$\overline{49}$		$\overline{534}$		$\overline{529}$		$\overline{510}$		a^{42}
	839		998		1327		988		669		$\overline{87}$		$\overline{332}$		$\overline{541}$		$\overline{521}$		$\overline{254}$	a^{43}
649		832		1326		978		859		120		$\overline{139}$		$\overline{568}$		$\overline{538}$		$\overline{499}$		a^{44}
	844		959		1186		840		530		$\overline{171}$		$\overline{375}$		$\overline{551}$		$\overline{506}$		$\overline{241}$	a^{45}
671		795		1181		821		694		22		$\overline{211}$		$\overline{570}$		$\overline{521}$		$\overline{464}$		a^{46}
	794		839		996		669		374		$\overline{235}$		$\overline{403}$		$\overline{529}$		$\overline{471}$		$\overline{223}$	a^{47}
622		708		976		652		519		$\overline{68}$		$\overline{254}$		$\overline{544}$		$\overline{478}$		$\overline{413}$		a^{48}
	692		716		800		496		247		$\overline{269}$		$\overline{386}$		$\overline{484}$		$\overline{415}$		$\overline{189}$	a^{49}
551		593		777		481		356		$\overline{115}$		$\overline{275}$		$\overline{490}$		$\overline{417}$		$\overline{353}$		a^{50}
	569		555		586		338		122		$\overline{278}$		$\overline{366}$		$\overline{419}$		$\overline{353}$		$\overline{165}$	a^{51}

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^{52}	20		34		2		$\overline{81}$		$\overline{126}$		$\overline{232}$		$\overline{194}$		$\overline{149}$		102		230	
a^{53}		39		11		$\overline{16}$		$\overline{112}$		$\overline{121}$		$\overline{194}$		$\overline{107}$		$\overline{43}$		202		276
a^{54}	17		19		$\overline{6}$		$\overline{48}$		$\overline{86}$		$\overline{137}$		$\overline{109}$		$\overline{63}$		113		194	
a^{55}		21		3		$\overline{16}$		$\overline{74}$		$\overline{83}$		$\overline{112}$		$\overline{50}$		$\overline{9}$		164		202
a^{56}	4		8		$\overline{6}$		$\overline{39}$		$\overline{55}$		$\overline{83}$		$\overline{57}$		$\overline{25}$		87		150	
a^{57}		10				$\overline{13}$		$\overline{45}$		$\overline{41}$		$\overline{57}$		$\overline{19}$		23		121		147
a^{58}	5		4		$\overline{7}$		$\overline{17}$		$\overline{30}$		$\overline{38}$		$\overline{25}$		5		74		102	
a^{59}		4		$\overline{1}$		$\overline{9}$		$\overline{23}$		$\overline{24}$		$\overline{27}$				20		78		89
a^{60}					$\overline{4}$		$\overline{12}$		$\overline{15}$		$\overline{20}$		$\overline{8}$		9		40		65	
a^{61}		1		$\overline{1}$		$\overline{4}$		$\overline{11}$		$\overline{9}$		$\overline{7}$		4		20		46		55
a^{62}	1				$\overline{2}$		$\overline{2}$		$\overline{7}$		$\overline{4}$				9		29		36	
a^{63}						$\overline{2}$		$\overline{5}$		$\overline{3}$		$\overline{1}$		4		9		24		23
a^{64}							$\overline{3}$		$\overline{2}$		$\overline{2}$				6		11		16	
a^{65}						$\overline{1}$		$\overline{1}$						2		7		9		11
a^{66}	1				$\overline{1}$				$\overline{1}$				1		3		6		4	
a^{67}										$\overline{1}$				2		1		4		3
a^{68}			$\overline{1}$				$\overline{1}$								1				3	
a^{69}												1				1		1		
a^{70}									1				1				2			
a^{71}																		$\overline{1}$		
a^{72}																			$\overline{1}$	
a^{75}																				

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
440		462		562		327		215		$\overline{159}$		$\overline{269}$		$\overline{418}$		$\overline{348}$		$\overline{284}$		a^{52}
	434		410		412		206		42		$\overline{262}$		$\overline{308}$		$\overline{348}$		$\overline{281}$		$\overline{120}$	a^{53}
338		330		391		200		108		$\overline{157}$		$\overline{239}$		$\overline{339}$		$\overline{273}$		$\overline{215}$		a^{54}
	302		271		252		107		21		$\overline{225}$		$\overline{260}$		$\overline{270}$		$\overline{211}$		$\overline{97}$	a^{55}
222		221		239		104		38		$\overline{149}$		$\overline{199}$		$\overline{257}$		$\overline{201}$		$\overline{157}$		a^{56}
	198		176		149		43		$\overline{37}$		$\overline{181}$		$\overline{191}$		$\overline{195}$		$\overline{152}$		$\overline{61}$	a^{57}
148		136		139		45		$\overline{3}$		$\overline{116}$		$\overline{154}$		$\overline{181}$		$\overline{142}$		$\overline{108}$		a^{58}
	117		91		65		7		$\overline{57}$		$\overline{135}$		$\overline{143}$		$\overline{135}$		$\overline{103}$		$\overline{47}$	a^{59}
79		73		63		11		$\overline{23}$		$\overline{96}$		$\overline{108}$		$\overline{121}$		$\overline{94}$		$\overline{67}$		a^{60}
	59		46		31		$\overline{13}$		$\overline{44}$		$\overline{92}$		$\overline{90}$		$\overline{86}$		$\overline{64}$		$\overline{20}$	a^{61}
41		34		29		$\overline{6}$		$\overline{28}$		$\overline{57}$		$\overline{71}$		$\overline{75}$		$\overline{57}$		$\overline{39}$		a^{62}
	25		18				$\overline{16}$		$\overline{37}$		$\overline{57}$		$\overline{60}$		$\overline{50}$		$\overline{35}$		$\overline{17}$	a^{63}
13		14		4		$\overline{11}$		$\overline{20}$		$\overline{40}$		$\overline{43}$		$\overline{41}$		$\overline{32}$		$\overline{21}$		a^{64}
	10		3		$\overline{2}$		$\overline{13}$		$\overline{20}$		$\overline{33}$		$\overline{30}$		$\overline{25}$		$\overline{20}$		$\overline{4}$	a^{65}
7		3		$\overline{1}$		$\overline{7}$		$\overline{15}$		$\overline{19}$		$\overline{22}$		$\overline{20}$		$\overline{17}$		$\overline{10}$		a^{66}
	1		$\overline{1}$		$\overline{6}$		$\overline{7}$		$\overline{14}$		$\overline{16}$		$\overline{16}$		$\overline{13}$		$\overline{8}$		$\overline{5}$	a^{67}
				$\overline{2}$		$\overline{6}$		$\overline{7}$		$\overline{11}$		$\overline{10}$		$\overline{9}$		$\overline{8}$		$\overline{4}$		a^{68}
			$\overline{2}$		$\overline{1}$		$\overline{6}$		$\overline{5}$		$\overline{7}$		$\overline{7}$		$\overline{4}$		$\overline{4}$			a^{69}
				$\overline{2}$		$\overline{2}$		$\overline{4}$		$\overline{3}$		$\overline{4}$		$\overline{4}$		$\overline{3}$		$\overline{1}$		a^{70}
	$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{3}$		$\overline{2}$		$\overline{2}$				$\overline{1}$	a^{71}
		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{1}$		$\overline{1}$		$\overline{1}$				a^{72}
																			$\overline{1}$	a^{75}

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	21	22
1										1											
2			1				1				1				1						
3				1		1		1		2		1		1		1		1		1	
4	2				2		2		3		2		2		2		1		1		1
5		1		3		4		4		3		4		2		2					
6			4		4		6		6		3		3								
7		4		7		8		7		5											
8	5		8		10		10		2												
9		9		14		10		2													
10	5		15		14																
11		17		16																	
12	14		23																		
13		25																			
14	17		9																		
15		26																			
16	21																				
17		5																			
18	25																				

Decimic.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)$
$(1-a^8)(1-a^9)$.

Numerator: $1 + 3a^2 + 11a^3 + 27a^4 + 58a^5 + 112a^6 + 193a^7 + 318a^8 + 485a^9$
$+ 699a^{10} + 951a^{11} + 1245a^{12} + 1541a^{13} + 1842a^{14} + 2108a^{15}$
$+ 2321a^{16} + 2451a^{17} + 2506a^{18} + 2451a^{19} + 2321a^{20} + 2108a^{21}$
$+ 1842a^{22} + 1541a^{23} + 1245a^{24} + 951a^{25} + 699a^{26} + 485a^{27}$
$+ 318a^{28} + 193a^{29} + 112a^{30} + 58a^{31} + 27a^{32} + 11a^{33} + 3a^{34}$
$+ a^{36}$.

G. F. for covariants, reduced form,*

Denominator: $(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)$
$(1-a^9)(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)(1-ax^{10})$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}
a^0	1														
a^1		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$										
a^2	$\bar{1}$		1	1	2	2	2	1	1						
a^3	$\bar{1}$	2	1	2	1	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$			
a^4		1	2			$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	1	1	1	1	1	
a^5		2	2			$\bar{1}$	$\bar{2}$		1	1	1	1			$\bar{1}$
a^6	3	1	1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$			1		1	$\bar{1}$	$\bar{1}$	
a^7		1		1	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{1}$		1	1		$\bar{1}$	$\bar{1}$	1
a^8	2	3	4	2	1	2	$\bar{2}$	$\bar{1}$		2				$\bar{1}$	1
a^9	2	5		1	$\bar{2}$	$\bar{6}$	$\bar{7}$	$\bar{7}$	$\bar{3}$	$\bar{2}$					
a^{10}	4	3	3		$\bar{4}$	$\bar{6}$	$\bar{6}$	$\bar{3}$		4	5	4	2	1	
a^{11}		4	2	$\bar{3}$	$\bar{6}$	$\bar{7}$	$\bar{7}$	$\bar{4}$	2	2	5	1	1	$\bar{1}$	$\bar{3}$
a^{12}	6	5	4	1	$\bar{2}$	$\bar{5}$	$\bar{7}$	$\bar{2}$	2	5	4	3	1	$\bar{2}$	
a^{13}	$\bar{1}$	3	$\bar{1}$	$\bar{5}$	$\bar{11}$	$\bar{17}$	$\bar{12}$	$\bar{9}$		2	6	3	1	$\bar{1}$	$\bar{2}$
a^{14}	1	4	7	$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{5}$	5	10	14	11	7	4	$\bar{3}$	$\bar{2}$
a^{15}		5	1	$\bar{4}$	$\bar{9}$	$\bar{17}$	$\bar{12}$	$\bar{6}$	3	3	5	1	$\bar{4}$	$\bar{5}$	$\bar{4}$
a^{16}	3	1	2	$\bar{5}$	$\bar{11}$	$\bar{11}$	$\bar{6}$	3	10	17	13	8		$\bar{2}$	
a^{17}	$\bar{4}$	1	$\bar{3}$	$\bar{9}$	$\bar{10}$	$\bar{10}$	$\bar{2}$	4	15	13	8	1	$\bar{4}$	$\bar{6}$	$\bar{6}$
a^{18}	4		$\bar{1}$	$\bar{1}$	$\bar{1}$	2	3	13	13	14	4	1	$\bar{6}$	$\bar{8}$	1
a^{19}	$\bar{3}$	$\bar{5}$	$\bar{8}$	$\bar{8}$	$\bar{8}$	$\bar{7}$	1	2	4	4	1	$\bar{3}$	$\bar{9}$	$\bar{3}$	$\bar{1}$
a^{20}		$\bar{3}$	$\bar{1}$		4	14	13	16	13	14	4		$\bar{1}$	$\bar{3}$	

* Numerator and denominator divisible by $1-a$; see foot-note to reduced form for sextic.

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}
a^{21}	$\bar{1}$	$\bar{3}$	$\bar{9}$	$\bar{3}$	1	4	4	2	1	$\bar{7}$	$\bar{8}$	$\bar{8}$	$\bar{8}$	$\bar{5}$	$\bar{3}$
a^{22}	1	$\bar{8}$	$\bar{6}$	1	4	14	13	13	3	2	$\bar{1}$	$\bar{1}$	$\bar{1}$		4
a^{23}	$\bar{6}$	$\bar{6}$	$\bar{4}$	1	8	13	15	4	$\bar{2}$	$\bar{10}$	$\bar{10}$	$\bar{9}$	$\bar{3}$	1	$\bar{4}$
a^{24}		$\bar{2}$		8	13	17	10	3	$\bar{6}$	$\bar{11}$	$\bar{11}$	$\bar{5}$	2	1	3
a^{25}	$\bar{4}$	$\bar{5}$	$\bar{4}$	1	5	3	3	$\bar{6}$	$\bar{12}$	$\bar{17}$	$\bar{9}$	$\bar{4}$	1	5	
a^{26}	$\bar{2}$	$\bar{3}$	4	7	11	14	10	5	$\bar{5}$	$\bar{6}$	$\bar{3}$	$\bar{1}$	7	4	1
a^{27}	$\bar{2}$	$\bar{1}$	1	3	6	2		$\bar{9}$	$\bar{12}$	$\bar{17}$	$\bar{11}$	$\bar{5}$	$\bar{1}$	3	$\bar{1}$
a^{28}		$\bar{2}$	1	3	4	5	2	$\bar{2}$	$\bar{7}$	$\bar{5}$	$\bar{2}$	1	4	5	6
a^{29}	$\bar{3}$	$\bar{1}$	1	1	5	2	2	$\bar{4}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{3}$	2	4	
a^{30}		1	2	4	5	4		$\bar{3}$	$\bar{6}$	$\bar{6}$	$\bar{4}$		3	3	4
a^{31}						$\bar{2}$	$\bar{3}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{2}$	1		5	2
a^{32}	1	$\bar{1}$				2		$\bar{1}$	$\bar{2}$	2	1	2	4	3	2
a^{33}	1	$\bar{1}$	$\bar{1}$		1	1		$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	1		1	
a^{34}		$\bar{1}$	$\bar{1}$	1		1			$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	1	3
a^{35}	$\bar{1}$			1	1	1	1		$\bar{2}$	$\bar{1}$			2	2	
a^{36}		1	1	1	1	1	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$			2	1	
a^{37}				$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	1	2	1	2	$\bar{1}$
a^{38}							1	1	2	2	2	1	1		$\bar{1}$
a^{39}											$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^{40}															1

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^4)(1-a^6)^2(1-a^8)(1-a^9)(1-a^{10})(1-a^{14})$
$(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})(1-a^2x^{16})(1-ax^{10})$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}
a^0	1																								
a^3		1		2	1	1	2	1	1	1	1		1												
a^4			3	1	3	3	2	3	1	2	1	1		1											
a^5		3	3	4	5	4	5	2	4		1		$\overline{1}$		$\overline{2}$		$\overline{1}$		$\overline{1}$						
a^6	2	2	6	8	8	9	6	7	2	4				$\overline{2}$		$\overline{1}$		$\overline{1}$							
a^7		7	10	11	13	11	11	7	6	1			$\overline{4}$		$\overline{2}$				1		1		1		
a^8	4	8	14	18	20	22	12	11	4	2	$\overline{2}$	$\overline{4}$	$\overline{3}$	$\overline{6}$	$\overline{1}$	$\overline{3}$				1					
a^9	4	15	21	27	30	24	23	12	7	$\overline{1}$	$\overline{8}$	$\overline{6}$	$\overline{9}$	$\overline{5}$	$\overline{5}$	$\overline{1}$	$\overline{1}$		2		1				$\overline{1}$
a^{10}	7	20	31	37	39	39	22	15	2	$\overline{8}$	$\overline{11}$	$\overline{18}$	$\overline{14}$	$\overline{15}$	$\overline{5}$	$\overline{4}$		2	1	2		1			
a^{11}	8	28	41	50	56	46	31	12	$\overline{2}$	$\overline{17}$	$\overline{28}$	$\overline{25}$	$\overline{26}$	$\overline{18}$	$\overline{13}$	$\overline{5}$	1	2	5	1	2		$\overline{1}$		
a^{12}	15	38	54	67	69	60	33	11	$\overline{12}$	$\overline{33}$	$\overline{41}$	$\overline{45}$	$\overline{36}$	$\overline{31}$	$\overline{12}$	$\overline{3}$	2	7	6	7	2	1		$\overline{1}$	
a^{13}	15	49	72	84	90	70	37	3	$\overline{26}$	$\overline{54}$	$\overline{66}$	$\overline{62}$	$\overline{56}$	$\overline{39}$	$\overline{21}$	$\overline{2}$	8	12	14	7	4	1	$\overline{1}$		
a^{14}	20	61	87	104	106	82	32	$\overline{9}$	$\overline{48}$	$\overline{86}$	$\overline{95}$	$\overline{93}$	$\overline{73}$	$\overline{55}$	$\overline{20}$	2	14	20	18	16	5	2	$\overline{3}$	$\overline{3}$	
a^{15}	27	75	108	127	128	92	32	$\overline{26}$	$\overline{76}$	$\overline{120}$	$\overline{134}$	$\overline{119}$	$\overline{100}$	$\overline{66}$	$\overline{25}$	11	27	32	32	18	9		$\overline{2}$	$\overline{2}$	$\overline{2}$
a^{16}	29	90	129	147	146	100	22	$\overline{49}$	$\overline{110}$	$\overline{165}$	$\overline{172}$	$\overline{157}$	$\overline{120}$	$\overline{77}$	$\overline{18}$	26	41	52	44	32	9	1	$\overline{6}$	$\overline{7}$	
a^{17}	35	105	148	168	164	103	5	$\overline{81}$	$\overline{153}$	$\overline{218}$	$\overline{227}$	$\overline{195}$	$\overline{150}$	$\overline{88}$	$\overline{15}$	44	67	70	63	37	14		$\overline{9}$	$\overline{8}$	$\overline{4}$
a^{18}	40	119	168	191	179	105	$\overline{11}$	$\overline{115}$	$\overline{201}$	$\overline{272}$	$\overline{274}$	$\overline{232}$	$\overline{169}$	$\overline{94}$	1	71	93	101	82	51	13	$\overline{6}$	$\overline{13}$	$\overline{15}$	$\overline{4}$
a^{19}	44	132	189	204	192	101	$\overline{36}$	$\overline{154}$	$\overline{254}$	$\overline{330}$	$\overline{330}$	$\overline{267}$	$\overline{190}$	$\overline{88}$	24	108	132	133	112	62	17	$\overline{7}$	$\overline{20}$	$\overline{20}$	$\overline{7}$
a^{20}	47	147	202	221	200	94	$\overline{64}$	$\overline{202}$	$\overline{305}$	$\overline{395}$	$\overline{379}$	$\overline{303}$	$\overline{203}$	$\overline{85}$	48	150	172	171	133	74	14	$\overline{16}$	$\overline{30}$	$\overline{28}$	$\overline{8}$
a^{21}	55	154	216	232	203	83	$\overline{98}$	$\overline{241}$	$\overline{365}$	$\overline{447}$	$\overline{431}$	$\overline{327}$	$\overline{208}$	$\overline{70}$	92	196	222	208	166	85	11	$\overline{26}$	$\overline{41}$	$\overline{38}$	$\overline{15}$
a^{22}	52	164	226	236	202	63	$\overline{127}$	$\overline{292}$	$\overline{413}$	$\overline{506}$	$\overline{470}$	$\overline{346}$	$\overline{210}$	$\overline{42}$	130	257	272	255	194	93	7	$\overline{37}$	$\overline{53}$	$\overline{49}$	$\overline{15}$
a^{23}	57	166	229	237	194	50	$\overline{168}$	$\overline{333}$	$\overline{465}$	$\overline{550}$	$\overline{502}$	$\overline{359}$	$\overline{193}$	$\overline{17}$	186	310	327	296	220	103	$\overline{4}$	$\overline{52}$	$\overline{73}$	$\overline{61}$	$\overline{20}$
a^{24}	56	172	228	236	187	22	$\overline{191}$	$\overline{372}$	$\overline{499}$	$\overline{585}$	$\overline{527}$	$\overline{353}$	$\overline{176}$	28	238	375	380	336	247	104	$\overline{13}$	$\overline{71}$	$\overline{88}$	$\overline{75}$	$\overline{27}$
a^{25}	57	166	227	225	168	7	$\overline{229}$	$\overline{401}$	$\overline{536}$	$\overline{610}$	$\overline{529}$	$\overline{347}$	$\overline{143}$	66	298	433	430	376	266	105	$\overline{31}$	$\overline{89}$	$\overline{109}$	$\overline{90}$	$\overline{29}$
a^{26}	52	164	217	211	155	$\overline{24}$	$\overline{249}$	$\overline{431}$	$\overline{551}$	$\overline{624}$	$\overline{536}$	$\overline{323}$	$\overline{114}$	119	346	487	474	403	281	98	$\overline{46}$	$\overline{111}$	$\overline{131}$	$\overline{105}$	$\overline{35}$
a^{27}	55	154	203	198	130	$\overline{38}$	$\overline{273}$	$\overline{442}$	$\overline{562}$	$\overline{620}$	$\overline{512}$	$\overline{296}$	$\overline{65}$	160	407	537	512	430	286	93	$\overline{68}$	$\overline{134}$	$\overline{150}$	$\overline{119}$	$\overline{40}$
a^{28}	47	147	190	176	112	$\overline{64}$	$\overline{281}$	$\overline{448}$	$\overline{556}$	$\overline{603}$	$\overline{490}$	$\overline{252}$	$\overline{26}$	216	448	578	541	443	296	76	$\overline{87}$	$\overline{155}$	$\overline{169}$	$\overline{132}$	$\overline{44}$

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}
a^{29}	44	132	169	155	87	$\overline{76}$	$\overline{296}$	$\overline{443}$	$\overline{541}$	$\overline{578}$	$\overline{448}$	$\overline{216}$	26	252	490	603	556	448	281	64	$\overline{112}$	$\overline{176}$	$\overline{190}$	$\overline{147}$	$\overline{47}$
a^{30}	40	119	150	134	68	$\overline{93}$	$\overline{286}$	$\overline{430}$	$\overline{512}$	$\overline{537}$	$\overline{407}$	$\overline{160}$	65	296	512	620	562	442	273	38	$\overline{130}$	$\overline{198}$	$\overline{203}$	$\overline{154}$	$\overline{55}$
a^{31}	35	105	131	111	46	$\overline{98}$	$\overline{281}$	$\overline{403}$	$\overline{474}$	$\overline{487}$	$\overline{346}$	$\overline{119}$	114	323	536	624	551	431	249	24	$\overline{155}$	$\overline{211}$	$\overline{217}$	$\overline{164}$	$\overline{52}$
a^{32}	29	90	109	89	31	$\overline{105}$	$\overline{266}$	$\overline{376}$	$\overline{430}$	$\overline{453}$	$\overline{298}$	$\overline{66}$	143	347	529	610	536	401	229	$\overline{7}$	$\overline{168}$	$\overline{225}$	$\overline{227}$	$\overline{166}$	$\overline{57}$
a^{33}	27	75	88	71	13	$\overline{104}$	$\overline{247}$	$\overline{336}$	$\overline{380}$	$\overline{375}$	$\overline{238}$	$\overline{28}$	176	353	527	585	499	372	191	$\overline{22}$	$\overline{187}$	$\overline{236}$	$\overline{228}$	$\overline{172}$	$\overline{56}$
a^{34}	20	61	73	52	4	$\overline{103}$	$\overline{220}$	$\overline{296}$	$\overline{327}$	$\overline{310}$	$\overline{186}$	17	193	359	502	550	465	333	168	$\overline{50}$	$\overline{194}$	$\overline{237}$	$\overline{229}$	$\overline{166}$	$\overline{57}$
a^{35}	15	49	53	37	$\overline{7}$	$\overline{93}$	$\overline{194}$	$\overline{255}$	$\overline{272}$	$\overline{257}$	$\overline{130}$	42	210	346	470	506	413	292	127	$\overline{63}$	$\overline{202}$	$\overline{236}$	$\overline{226}$	$\overline{164}$	$\overline{52}$
a^{36}	15	38	41	26	$\overline{11}$	$\overline{85}$	$\overline{166}$	$\overline{208}$	$\overline{222}$	$\overline{196}$	$\overline{92}$	70	208	327	431	447	365	241	98	$\overline{83}$	$\overline{203}$	$\overline{232}$	$\overline{216}$	$\overline{154}$	$\overline{55}$
a^{37}	8	28	30	16	$\overline{14}$	$\overline{74}$	$\overline{133}$	$\overline{171}$	$\overline{172}$	$\overline{150}$	$\overline{48}$	85	203	303	379	395	305	202	64	$\overline{94}$	$\overline{200}$	$\overline{221}$	$\overline{202}$	$\overline{147}$	$\overline{47}$
a^{38}	7	20	20	7	$\overline{17}$	$\overline{62}$	$\overline{112}$	$\overline{133}$	$\overline{132}$	$\overline{108}$	$\overline{24}$	88	190	267	330	330	254	154	36	$\overline{101}$	$\overline{192}$	$\overline{204}$	$\overline{189}$	$\overline{132}$	$\overline{44}$
a^{39}	4	15	13	6	$\overline{13}$	$\overline{51}$	$\overline{82}$	$\overline{101}$	$\overline{93}$	$\overline{71}$	$\overline{1}$	94	169	232	274	272	201	115	11	$\overline{105}$	$\overline{179}$	$\overline{191}$	$\overline{168}$	$\overline{119}$	$\overline{40}$
a^{40}	4	8	9		$\overline{14}$	$\overline{37}$	$\overline{63}$	$\overline{70}$	$\overline{67}$	$\overline{44}$	15	88	150	195	227	218	153	81	$\overline{5}$	$\overline{103}$	$\overline{164}$	$\overline{168}$	$\overline{148}$	$\overline{105}$	$\overline{35}$
a^{41}		7	6	$\overline{1}$	$\overline{9}$	$\overline{32}$	$\overline{44}$	$\overline{52}$	$\overline{41}$	$\overline{26}$	18	77	120	157	172	165	110	49	$\overline{22}$	$\overline{100}$	$\overline{146}$	$\overline{147}$	$\overline{129}$	$\overline{90}$	$\overline{29}$
a^{42}	2	2	2		$\overline{9}$	$\overline{18}$	$\overline{32}$	$\overline{32}$	$\overline{27}$	$\overline{11}$	25	66	100	119	134	120	76	26	$\overline{32}$	$\overline{92}$	$\overline{128}$	$\overline{127}$	$\overline{108}$	$\overline{75}$	$\overline{27}$
a^{43}		3	3	$\overline{2}$	$\overline{5}$	$\overline{16}$	$\overline{18}$	$\overline{20}$	$\overline{14}$	$\overline{2}$	20	55	73	93	95	86	48	9	$\overline{32}$	$\overline{82}$	$\overline{106}$	$\overline{104}$	$\overline{87}$	$\overline{61}$	$\overline{20}$
a^{44}			1	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{14}$	$\overline{12}$	$\overline{8}$	2	21	39	56	62	66	54	26	$\overline{3}$	$\overline{37}$	$\overline{70}$	$\overline{90}$	$\overline{84}$	$\overline{72}$	$\overline{49}$	$\overline{15}$
a^{45}		1		$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{7}$	$\overline{2}$	3	12	31	36	45	41	33	12	$\overline{11}$	$\overline{33}$	$\overline{60}$	$\overline{69}$	$\overline{67}$	$\overline{54}$	$\overline{38}$	$\overline{15}$
a^{46}			1		$\overline{2}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{1}$	5	13	18	26	25	28	17	2	$\overline{12}$	$\overline{31}$	$\overline{46}$	$\overline{56}$	$\overline{50}$	$\overline{41}$	$\overline{28}$	$\overline{8}$
a^{47}				$\overline{1}$		$\overline{2}$	$\overline{1}$	$\overline{2}$		4	5	15	14	18	11	8	$\overline{2}$	$\overline{15}$	$\overline{22}$	$\overline{39}$	$\overline{39}$	$\overline{37}$	$\overline{31}$	$\overline{20}$	$\overline{7}$
a^{48}	1				$\overline{1}$		$\overline{2}$		1	1	5	5	9	6	8	1	$\overline{7}$	$\overline{12}$	$\overline{23}$	$\overline{24}$	$\overline{30}$	$\overline{27}$	$\overline{21}$	$\overline{15}$	$\overline{4}$
a^{49}						$\overline{1}$				3	1	6	3	4	2	$\overline{2}$	$\overline{4}$	$\overline{11}$	$\overline{12}$	$\overline{22}$	$\overline{20}$	$\overline{18}$	$\overline{14}$	$\overline{8}$	$\overline{4}$
a^{50}			$\overline{1}$		$\overline{1}$		$\overline{1}$				2		4			$\overline{1}$	$\overline{6}$	$\overline{7}$	$\overline{11}$	$\overline{11}$	$\overline{13}$	$\overline{11}$	$\overline{10}$	$\overline{7}$	
a^{51}								1		1		2				$\overline{4}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{9}$	$\overline{8}$	$\overline{8}$	$\overline{6}$	$\overline{2}$	$\overline{2}$
a^{52}							1		1		2		1		$\overline{1}$		$\overline{4}$	$\overline{2}$	$\overline{5}$	$\overline{4}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{3}$	
a^{53}												$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{1}$	$\overline{3}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{3}$		
a^{54}													$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	$\overline{2}$		$\overline{1}$	
a^{57}																									$\overline{1}$

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	16	18	20	22	24	26
1						1								
2	1		1		1		1		1					
3		1		2	1	1	2	1	1	1	1		1	
4	1		3	1	3	3	2	3	1	2	1	1		1
5		3	3	4	5	4	5	2	4		1			
6	4	2	5	8	6	8	2	3						
7		7	10	8	12	2	3							
8	5	8	11	15	4	5								
9	5	13	19	8	4									
10	8	20	12	10										
11	8	18	21											
12	12	30												
13	15	16												
14	13	17												
15	19													
16	5													
17	3													

The total number of irreducible invariants and covariants for the first 10 orders (counting in the absolute constant and the quantic itself), it appears from what precedes, is as follows:

Order of Quantic: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Number of Groundforms: 1, 2, 3, 5, 6, 24, 27, 125, 70, 416, 476.

For the benefit of those new to the subject, it may be well to recall the immediate algebraical meaning of either form of the generating function to a binary quantic $(x, y)^n$.

Suppose n an odd number, say 5, then if

$$\frac{1-x^{-2}}{(1-ax^{-5})(1-ax^{-3})(1-ax^{-1})(1-ax)(1-ax^{3})(1-ax^{5})}$$

is expanded in a *bivergent* series, (that is, one going, as regards the powers of x, in two directions towards infinity), either generating function of the tables for the quintic is the sum of the terms which contain no negative powers of x. So if n be an even number, say 6,

$$\frac{1-x^{-2}}{(1-ax^{-6})(1-ax^{-4})(1-ax^{-2})(1-a)(1-ax^{2})(1-ax^{4})(1-ax^{6})}$$

being similarly expanded, either generating function of the tables for the sextic is, as before, the sum of the terms which contain only positive or zero powers of x. And so in general, for $(x, y)^n$, the numerator of the so-called *crude* generating function, being always $1-x^{-2}$ and its denominator a product of factors of the form $1-ax^{n-2i}$ (where i takes all values from nought up to n inclusive). Either generating function of the tables for the n^{ic} is the algebraic equivalent of the *positive* branch of the corresponding bivergent series, (that in which only positive powers of x appear), *plus* the *neutral* branch or term, namely, that which contains neither positive nor negative powers of x, or, which is the same thing, is a function only of a.

I subjoin a few reflexions which appear to me to be desirable on the foregoing tables.

It is scarcely necessary to state, that, in the development of the generating function, whether reduced or representative, the coefficient of $a^m x^\mu$ is the total number of linearly independent covariants of the degree m in the coefficients and the order μ in the variables.

Mr Franklin will probably, in a future number of the *Journal*, draw up a statement of the mode in which the tables have been calculated and the precautions taken to insure accuracy *; as regards the reduced form, three methods have been employed in calculating it, namely, Mr Sylvester's first method, Professor Cayley's method, fully explained in a preceding number of the *Journal* by its eminent author, and Mr Sylvester's second method, much briefer than his other, but, in general, not so brief as Professor Cayley's, which last, however, involves a delicate point in the expansion of series, the assumed principle of which, although its validity on moral grounds of evidence is unquestionable, cannot be regarded as *a priori* self-evident †.

The theory of the generating function, alike for single and simultaneous forms, depends on the law for determining the number of linearly indepen-

* In especial I wish to single out an ingenious device of Mr Franklin to check the operation of tamisage by introducing a common superfluous factor into the numerator and denominator of the representative generating function so selected as that the augmented denominator shall not cease to be representative; the effect of this will be to cause the groundforms obtained by tamisage of the augmented numerator to be the same as before, except that the groundform represented by the additional factor will not be found among them.

† In Prof. Cayley's method the crude generating function is regarded as a function of a; in my two methods as a function of x.

dent in- and co-variants of given order and degree or degrees belonging to a given quantic or system of quantics, a proof of which will be found at the end of a memoir by Mr Sylvester in *Borchardt's Journal**, and also in the *London and Edinburgh Philosophical Magazine*†, that leaves nothing to be desired as regards rigour of demonstration. The law itself for the case of a single quantic was first stated by Professor Cayley whilst the theory was still in its infancy.

But besides this fundamental theorem, in order to deduce the tables of groundforms, a *fundamental postulate* still awaiting demonstration is necessary, which is, that no more linear relations between in- or co-variants are to be supposed to exist than are necessary in order to satisfy the *fundamental theorem*. The application of this principle in such a mode as to substitute a finite for an infinite process, leads to the use of representative generating functions and the simplified method of *tamisage*. The validity of the fundamental-postulate which is in accord with the law of parcimony is verified by its conducting to results which have been proved to be accurate for single binary quantics up to the sixth order inclusive, for pairs of binary quantics up to the fourth order inclusive, and also for systems of an indefinite number of linear and quadratic binary forms‡.

The application of this principle discloses the remarkable singularity that for the quantic of the seventh order, there exists no finite representative generating function as shown in what follows.

The invariantive part of the numerator of the reduced form for the seventhic is

$$1 - a^6 + 2a^8 - a^{10} + 5a^{12} + 2a^{14} + 6a^{16} + 2a^{18} + 5a^{20} - a^{22} + 2a^{24} - a^{26} + a^{32},$$

and the invariantive part of the denominator is $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})$. Multiplying numerator and denominator by $(1+a^6)$, their invariantive portions§ become, respectively,

$$1 + 2a^8 - a^{10} + 4a^{12} + 4a^{14} + 5a^{16} + 7a^{18} + 7a^{20} + 5a^{22} + 4a^{24} + 4a^{26} - a^{28} + 2a^{30} + a^{38},$$

and $$(1-a^4)(1-a^8)(1-a^{10})(1-a^{12}).$$

[* p. 232 above.] [† p. 117 above.]

‡ If the *fundamental postulate* were called into question, this (it may be proved) would not affect the fact of the existence of the groundforms obtained by its aid, but only the possibility of the existence of other groundforms over and above those so obtained. Thus my tables of groundforms could only err (were that possible, which I do not believe it to be) in defect; and as those found by the German method can only err in excess, it follows that, whenever the tables coincide, both must be correct. The tables of groundforms here given, up to the sixth order, inclusive, and all those that follow, coincide exactly with those obtained by Clebsch, Gordan and Gundelfinger, when these latter are rectified by the omission of certain supposed groundforms which, in the *Comptes Rendus*, I have conclusively proved to be composite.

§ The factors in the denominator which involve x never offer any difficulty, as they represent the given quantic along with the complete system of covariants of the second degree, the several orders of which follow a well known rule.

The factors of the denominator are now, with the exception of $1-a^{10}$, representative factors; $1-a^{10}$ is not such, as a^{10} occurs in the numerator with the coefficient -1. If we multiply numerator and denominator by $1+a^{10}$, the factor $1-a^{20}$ will take the place of $1-a^{10}$ in the denominator, and the numerator will become

$$1+2a^8+4a^{12}+4a^{14}+5a^{16}+9a^{18}+6a^{20}+\ldots.$$

Here the coefficient of a^{20} is not negative, but it is less than the number (8) obtained by composition from the terms $2a^8$ and $4a^{12}$; hence, by the fundamental postulate there is no irreducible invariant of the degree 20. If, instead of multiplying numerator and denominator by $1+a^{10}$, we multiply them by the infinite series $1+a^{10}+a^{20}+\ldots$, the denominator becomes representative and the invariantive part of the numerator becomes the *recurrent* series given in the table (p. [288]), in which the coefficient of a^{30}, a^{40} and, in general, all powers of a whose exponents are multiples of and greater than 20, is 9; but 9 is less than the number obtained in the composition of a^{30}, a^{40} (and *a fortiori* of a^{50}, a^{60}, ...) out of the preceding terms; therefore, by the fundamental postulate, there is no irreducible invariant whose degree is any multiple of 10. It is a remarkable and significant fact that in this case the erroneous assumption of $1-a^{10}$ being a representative factor in the denominator of the complete generating function will be found to lead to no subsequent further error in the determination of the other groundforms of the seventhic.

A chorographical law obtains in the numerical tables of the numerators of the representative forms, which plays a considerable part in the complete theory of tamisage, and is too important to be passed over without notice, namely, it will be seen that all these tables consist of a small number of irregular but continuous bands or blocks of alternately positive and negative coefficients which can be drawn asunder without tearing or leaving any hole in the paper*. For the first four orders there is but one such block, for the

* In the operation of tamisage on the numerator of the representative groundforms the terms of the negative blocks are disregarded. In every case treated in these tables, and those to follow in the next number of the *Journal*, the only surviving terms will be found to be comprised in the first block. Had it turned out otherwise it would have been necessary to ascertain whether the surviving terms belonging to the other odd-numbered blocks would survive the operation of tamisage performed on the infinite aggregate of terms obtained by the development of the generating function; if not, they would have to be rejected. This is what I have found actually happens in a system of quadratic or linear forms when a sufficient number of such forms is employed. In that case, terms not confined to the first block emerge from the tamisage of the numerator of the representative groundforms, but disappear when the tamisage is performed on the infinite aggregate of terms of which the groundform is the sum. Such aggregate, it may be noticed, (I have proved elsewhere), consists exclusively of positive terms, the coefficients corresponding to non-existing types being always zero and never negative. It is very likely to be found true hereafter that in no case need any, except the first block of terms in the numerator of the representative groundforms, be submitted to tamisage in order to obtain the groundforms not represented in the denominator, and so in like manner that, in order to obtain the ground-syzygies of the first kind, that is, those that concern the groundforms, only the first

quintic and the sextic two, for the seventhic five, for the octavic three, and for the 9^{ic} and 10^{ic} four. A similar law obtains for systems of quantics, as for instance in the case of two simultaneous quantics, the corresponding tables consist of detachable solid blocks, alternately positive and negative, and small in number in comparison with the number of terms which they contain, as will be seen in the tables to appear in the next number of the *Journal* which will contain a complete set of them for all the systems that can be formed of two binary quantics of orders, m, n where neither m nor n exceeds 4.

It is my duty to state that the expense of calculating the tables for quantics of the 7th, 8th, 9th and 10th orders, has been defrayed out of a grant made by the British Association for the Advancement of Science, and I have pleasure in returning my thanks to that distinguished body for this act of aid in enabling me to bring to a successful issue an undertaking of such unusual magnitude and of such pith and moment to the progress of Algebraical Theory.

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positive and the first negative block need be considered, and so on for syzygies of the higher orders, each time a new block being taken into account until all are exhausted, it being quite conceivable that the number of blocks may designate the highest order of syzygy that occurs in any case, subject in the case of a linear or quadratic form (for which the block reduces to a single term, namely, unity) to the obvious exception that, for them, the syzygies become abortive.

To explain what is meant by syzygies of successive orders, suppose Z to be a rational and integral function of groundforms which, regarded as a function of the coefficients, is identically zero, then $Z=0$ is a syzygy and Z may be termed a syzygant of the first order and, if incapable of being resolved into a sum of products of syzygants multiplied respectively by rational algebraic functions of the groundforms, will be an irreducible or ground-syzygy of the first order. In like manner, if Z' is a function of ground-syzygants which, regarded as a function of the ground-forms, vanishes identically $Z'=0$ is a syzygy and Z' is a counter-syzygant or a syzygant of the second order, and, if incapable of representation as a sum of products of other syzygants of the second order multiplied respectively by rational integral functions of syzygants of the first order, is a ground-syzygant of the second order; and so on indefinitely.