42.

ON THE THEOREM CONNECTED WITH NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS OF EQUATIONS.

[*Messenger of Mathematics*, IX. (1880), pp. 71—84.]

To save needless repetition in what follows I beg to refer the reader to Mr Todhunter's section 26, p. 236, in the third edition of his *Treatise on the Theory of Equations*. It will there be seen that in order to provide against any loss of double permanences consequent upon any of the f's changing sign $\gamma_1, \gamma_2, \gamma_3, \dots \gamma_{n-1}$ must all be positive; and in order to provide against the same thing happening consequent upon any of the G's changing sign we must have, from $i=2$ to $i=n-1$ inclusive, $2-\gamma_i=\dfrac{1}{\gamma_{i+1}}$; and, moreover, $2-\gamma_{n-1}$ [denoted by $\dfrac{1}{\gamma_n}$, although strictly there is no γ_n, since G_n is simply a positive absolute], as well as $\gamma_1, \gamma_2, \dots \gamma_{n-1}$, must be positive.

The solution of the equation $2-\gamma_i=\dfrac{1}{\gamma_{i+1}}$ is $\gamma_i=\dfrac{C+i-1}{C+i}$; and, in order that $\gamma_1, \gamma_2, \dots \gamma_{n-1}, \gamma_n$ may all be *positive*, it is necessary that C shall be either positive or, if negative, of greater absolute value than n.

If we put $C=0$, $\gamma_1=0$; if we put $C=-n$, $\gamma_n=\infty$, so that, the condition of γ_i being *positive*, from 1 to n, will not in either case be complied with, the signs of zero and of infinity being ambiguous. It is well known, however, that we may put $C=-n$; in fact, $-n$ is the value ordinarily attributed to C, for the corresponding value of γ_i, namely, $\dfrac{n-i+1}{n-i}$, it is which leads to that form of the theorem in which, when we put $\mu=\infty$ and $\lambda=0$, or $\mu=0$ and $\lambda=-\infty$ in the equation $pP(\mu)-pP(\lambda)=$ (the number of roots between λ and μ) $+2i$, gives Newton's rule as stated by Newton himself. Equally, we shall find it is lawful to put $C=0$, but each of these two suppositions requires to be subjected to a special examination before its validity can be admitted. Take the much more important case first, that where $C=-n$, we have then $\gamma_{n-1} = 2$, and the only object of $2 - \gamma_{n-1}$ being positive is to prevent mischief in the event of G_{n-1}, that is, $(f_{n-1}x)^2 - 2f_{n-2}xf_nx$, changing its sign. But in this case $\frac{dG_{n-1}}{dx} = 0$ by simple differentiation from $\frac{df_nx}{dx} = 0$: in other words, G_{n-1} *is a constant* and never can change its sign. Thus, then, all necessity for $2 - \gamma_{n-1}$ being positive is abolished by the very fact of its being zero.

It is worth noticing that this critical value of C, which makes $\gamma_i = \frac{n-i+1}{n-i}$, has the effect of lowering the degree of each G by two units; for if $\lambda = n - i + 1$, we may write $f_{i-1} = px^\lambda + qx^{\lambda-1} + \ldots$, and then

$$G_i = f_i^2 - \frac{\lambda}{\lambda-1} f_{i-1}f_{i+1} = \{p\lambda x^{\lambda-1} + q(\lambda-1)x^{\lambda-2} + \ldots\}^2$$

$$+ \frac{\lambda}{\lambda-1}(px^\lambda + qx^{\lambda-1} + \ldots)\{p\lambda(\lambda-1)x^{\lambda-2} + q(\lambda-1)(\lambda-2)x^{\lambda-3} + \ldots\};$$

so that the coefficient of $x^{2\lambda-2}$ becomes

$$p^2\left\{\lambda^2 - \frac{\lambda}{\lambda-1}(\lambda^2 - \lambda)\right\} = 0,$$

and that of $x^{2\lambda-3}$ becomes

$$pq\left\{2\lambda(\lambda-1) + \frac{\lambda}{\lambda-1}\lambda(\lambda-1) + (\lambda-1)(\lambda-2)\right\} = 0.$$

So again it will be found that C may be taken at the other extremity of the chasm or gap, which it is not permitted to enter; for if $C = 0$ so that $g_1 = 0$, $G_1 = (f'x)^2$.

Consider now the first three terms of the double series

$$\begin{matrix} fx, & f'x, & f''x, \\ I, & I, & G_2x, \end{matrix}$$

where the two I's denote absolute positive quantities; at the moment of $f'x$ becoming zero, G_2x becomes positive, so that the succession of double permanences of sign for this double series is the same as of single permanences for $fx, f'x, f''x$, and consequently no double permanences can be lost by $f'x$ changing its sign. Since, then, we have shown that values of C giving rise to no negative but to an ambiguous sign, either of γ_1 or of γ_n, are not prohibited, it might for a moment be imagined that any negative integer value of C, say $-\omega$, lying in the gap between 0 and $-n$ might also be admissible, seeing that such value would also *not introduce any negative value of* γ, but only two values of ambiguous signs, namely, for γ_ω and $\gamma_{\omega+1}$, ∞ and 0 respectively; all the other γ's will be positive. But it will be seen that this is inadmissible, for the course of the demonstration shows that every γ_i and $2 - \gamma_i$ must both be positive, which conditions cannot be fulfilled for γ_ω, whether we consider it equal to plus or minus infinity.

As I have referred to Mr Todhunter's treatise, I may notice the omission therein of the equation

$$\nu P\lambda - \nu P\mu = (\mu, \lambda) + 2i',$$

where i' is any positive integer and (μ, λ) the number of real roots between λ and μ. This may be deduced *pari passu*, and in precisely the same way as the parallel equation

$$pP\mu - pP\lambda = (\mu, \lambda) + 2i,$$

or either of these may be deduced from the other as follows. Let $fx = \phi(-x)$, and using the same parameter γ_i for the G's belonging to f and for those belonging to ϕ, let f_i, G_i for f become ϕ_i, T_i for ϕ. Then obviously

$$T_i(-c) = G_i c \text{ and } \phi_i(-c) = (-)^{n-1} f_i(c).$$

Hence, using π, Π in regard to ϕ in the same sense as p, P in regard to f, $\pi\Pi(-c) = \nu Pc$; also $(-\lambda, -\mu)$ in regard to ϕ is the same as (μ, λ) in regard to f. But remembering that if μ is greater than λ, then $-\lambda$ is greater than $-\mu$, the second equation above written applied to ϕ becomes

$$\pi\Pi(-\lambda) - \pi\Pi(-\mu) = (-\lambda, -\mu) \text{ in regard to } \phi + 2i.$$

Hence $$\nu P(\lambda) - \nu P(\mu) = (\mu, \lambda) \text{ in regard to } f + 2i,$$

as was to be shown*.

One other point deserves mentioning. If any G, say G_i, becomes incapable of changing its sign (of which G_1 becoming f_1^2 when $C = 0$, offers a particular example), the necessity for the equation $2 - \gamma_i = \frac{1}{\gamma_{i+1}}$ is done away with for that value of i, so that γ_{i+1} becomes arbitrary (within limits), and we may start with a new definition of the values of the γ's lying beyond γ_i, namely, $\gamma_{i+i'} = \frac{C' - 1 + i}{C' + i'}$ and so on, *toties quoties*, whenever in passing from G_1 to G_{n-1}, any of the G's becomes incapable of changing its sign†.

* This equation is stated in the original memoir in the *Proceedings of the Mathematical Society of London*‡. Dr Julius Petersen, of Copenhagen, in his treatise on Algebraical Equations, not having had the opportunity, as he has since informed me, of consulting this, and taking Mr Todhunter's chapter on the subject as his authority, was led to lay the fault of the omission at my door.

† Thus we see that in the expression $\gamma_r = \frac{C-1+r}{C+r}$, C is not absolutely prohibited from entering the gap comprised between 0 and $-n$, but that C may be $-i$ where i is an integer, or any quantity between $-i$ and $-\infty$, provided that G_{i-1}, that is, $f^2_{i-1} - \gamma_{i-1} f_{i-2} f_i$ is incapable of changing its sign. If $C = -i$, $\gamma_{i-1} = 2$.

As an application of the same principle we may make the γ series begin with G_2, that is, make G a positive absolute so as to have two positive absolutes instead of one positive absolute at the beginning of the series of "the Quadratic elements," that is, we may make $\gamma_1 = 0$ and $\gamma_{1+r} = \frac{C-1+r}{C+r}$, and continuing this process, $1 + k$ (any number) of the initial G's may be converted into positive absolutes; that is to say, we may make $\gamma_1 = 0$, $\gamma_2 = 0$, ... $\gamma_k = 0$, $\gamma_{k+r} = \frac{C-1+r}{C+r}$.

[‡ Vol. II. of this Reprint, p. 501, *footnote*.]

It will have been noticed in what precedes, that I have made no allusion to special forms of an equation, whether absolute or having reference to the assumed arbitrary parameter in G, but have confined myself to the general case where only one term in the double series can vanish for any given value of x. Nor is it necessary to do more than this in treating the theory; for (1) if f contains no equal roots, we may, by infinitesimal or infinitely small variations attributed to the coefficients, cause those relations between them to subsist which are necessary in order that two or more of the terms may vanish simultaneously, and cannot thereby alter the character of the roots, which can only make the passage from real to imaginary, or *vice versâ*, after one or more pairs of them have passed through the state of equality; (2) if f contains equal roots, we may vary the coefficients in such a manner as not to disturb the equalities which subsist between them, and shall have independent relations enough to spare to abolish as before the relations implied in the fact of the simultaneous evanescence above referred to.

Thus it seems to me that we need trouble ourselves with the discussion of the consequences of such simultaneous evanescence only if we wish to know what inferences to draw if we are unfortunate enough to find that event occurring at one or the other of the actual limits λ, μ we may be dealing with, and for no other purpose.

Postscript.

As I was on the point of despatching what precedes by post to England, it occurred to me, in consequence of the previously unnoticed depression of the degrees of the terms in the G series, to examine more closely their constitution for the critical case, that namely where $\gamma_i = \frac{n-i+1}{n-i}$, and I have had the satisfaction of finding that every such G is proportional to the

If we make $k=n$, all the G's become positive absolutes, and *the theorem passes into Fourier's.* In connexion with this fact, it should be noticed that my theorem in its form as hitherto given does not logically contain Fourier's as a consequence; for it is possible that for certain values of λ and μ, $pP(\mu) - pP(\lambda)$ may be greater than $p(\mu) - p(\lambda)$, so that Fourier's theorem may indicate the passage of a smaller number of roots than the seemingly more stringent one; hence in applying my theorem, Fourier's should always be employed simultaneously with it, a practical direction which has hitherto been overlooked. Of course when the question concerns the total number of roots, Descartes' rule *is* logically contained in Newton's, or my generalisation of it as previously given.

It may be well to mention here, that a more general form of my theorem introducing a second arbitrary parameter will be found in some far back number of the *Educational Times* as the solution of a question proposed in a previous number. It is founded, if I recollect right, on the principle that if for the equation of the nth degree in x, say $fx=0$, we substitute $\epsilon x^{n+\nu} + fx = 0$, where ν is any positive integer (ϵ being an infinitesimal), no new real root is introduced if ν is even, provided ϵ be taken with the right sign, and only one (of infinite value) if ν is odd. [See below: Solutions contributed by the Author to the *Educational Times*.]

Hessian of the f antecedent to it, regarded as a homogeneous function of x and 1, being that Hessian multiplied by a negative number.

To prove this I have to show that if $F(x, y)$ is of the order λ, then

$$\lambda F \frac{d^2F}{dx^2} - (\lambda - 1)\left(\frac{dF}{dx}\right)^2$$

is a positive multiple of y^2 multiplied by the Hessian of F in regard to x, y.

Now $$\lambda F = x\frac{dF}{dx} + y\frac{dF}{dy},$$

and $$(\lambda - 1)\frac{dF}{dy} = x\frac{d}{dx}\frac{dF}{dy} + y\frac{d^2F}{dy^2}.$$

Hence $$y\frac{dF}{dy} = \lambda F - x\frac{dF}{dx},$$

$$y\frac{d^2F}{dx\,dy} = (\lambda - 1)\frac{dF}{dx} - x\frac{d^2F}{dx^2},$$

and $$y^2\frac{d^2F}{dy^2} = (\lambda - 1)\left(\lambda F - x\frac{dF}{dx}\right) - x\frac{d}{dx}\left(\lambda F - x\frac{dF}{dx}\right)$$

$$= (\lambda^2 - \lambda)F - (2\lambda - 2)x\frac{dF}{dx} + x^2\frac{d^2F}{dx^2}.$$

Hence $$y^2\left\{\frac{d^2F}{dx^2}\frac{d^2F}{dy^2} - \left(\frac{d^2F}{dx\,dy}\right)^2\right\}, \text{ that is, } -y^2H(F),$$

$$= (\lambda^2 - \lambda)\frac{d^2F}{dx^2}F - (2\lambda - 2)x\frac{d^2F}{dx^2}F + x^2\left(\frac{d^2F}{dx^2}\right)^2 - \left\{(\lambda - 1)\frac{dF}{dx} - x\frac{d^2F}{dx^2}\right\}^2$$

$$= (\lambda - 1)\left\{\lambda\frac{d^2F}{dx^2}F - (\lambda - 1)\left(\frac{dF}{dx}\right)^2\right\},$$

where the least value of λ is 2 so that $\lambda - 1$ is always positive.

Thus the f and G series may be put under the following form, where f_i of course means $\frac{d^if}{dx^i}$ and $H\phi x$ signifies the Hessian of ϕ regarded as a quantic in x and 1,

$$f : f_1 : f_2 : f_3 : \ldots : f_{n-1} : f_n,$$
$$-1 : Hf : Hf_1 : Hf_2 : \ldots : Hf_{n-2} : -1.$$

I anticipate that it will be found possible to extend the theorem by the addition of a third series for the case of $n = 4$ or 5, a third and fourth for that of $n = 6$ or 7, and, in general, by the use of $\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+1)$ series according as n is even or odd. And possibly it may turn out that the maximum number of series available for any given value of n will by the reckoning of the gain of complete permanences of sign (that is, treble, quadruple... permanences for 3, 4... series) as x increases from λ to μ, afford not merely a superior limit to, but the actual number of, real roots passed over in the interval.

As I find that Mr Todhunter uses a single symbol ϖ for the pP employed in my memoir in the second number of the *Proceedings of the London Mathematical Society**, it may be well to advise my readers that I use p, P to signify permanences of sign, and v, V variations of sign in the f and G series respectively; so that double permanences, permanence variations, variation permanences and variation variations would be denoted by the compound symbols pP, pV, vP, vV respectively.

The theorem above given is, I find, only a particular case of the one subjoined.

Let f_i denote $(a_0, a_1, a_2 \dots a_i \between x, y)^i$ and $H_\epsilon (f_{i+\epsilon})$ that covariant of $f_{i+\epsilon}$ whose highest power of x bears the coefficient

$$\begin{vmatrix} a_0, & a_1, & a_2, & \dots & a_\epsilon \\ a_1, & a_2, & a_3, & \dots & a_{\epsilon+1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_\epsilon, & a_{\epsilon+1}, & a_{\epsilon+2}, & \dots & a_{2\epsilon} \end{vmatrix};$$

then is

$$\begin{vmatrix} f_{i-\epsilon}, & f_{i-\epsilon+1}, & f_{i-\epsilon+2}, & \dots & f_{i+\epsilon} \\ f_{i-\epsilon+1}, & f_{i-\epsilon+2}, & f_{i-\epsilon+3}, & \dots & f_{i+\epsilon+1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ f_{i+\epsilon}, & f_{i+\epsilon+1}, & f_{i+\epsilon+2}, & \dots & f_{i+2\epsilon} \end{vmatrix}$$

equal to $y^{\epsilon^2+\epsilon} H_\epsilon (f_{i+\epsilon})$.

The order in (x, y) of $H_\epsilon f_{i+\epsilon}$, since the weight of its leading coefficient is $\epsilon^2 + \epsilon$ and its degree in the coefficients $\epsilon + 1$, will be $(\epsilon + 1)(i + \epsilon) - 2(\epsilon^2 + \epsilon)$, that is, $(\epsilon + 1)i - \epsilon^2 - \epsilon$, so that multiplied by $y^{\epsilon^2+\epsilon}$ the order becomes $(\epsilon + 1)i$, as it ought to be.

The theorem may be proved as follows:

Let ϕ be any homogeneous function of λ dimensions in x, y, and denote $\frac{d}{dx}, \frac{d}{dy}$ by X, Y.

(1) I shall show that in respect of ϕ,

$$y^i \,.\, Y^i = {}^i\lambda - i\,{}^{i-1}(\lambda - 1)\,xX + \frac{i \,.\, i-1}{2}\,{}^{i-2}(\lambda - 2)\,x^2X^2 \dots + (-)^i x^i X^i,$$

where ${}^im$ for any positive integer values of m and i denotes the factorial quantity $m(m-1)\dots(m-i+1)$.

Suppose the equation to be true for any assigned value of i, it will be true for $i + 1$. For $Y^i\phi$, it will be observed, is of $\lambda - i$ dimensions in x, y; hence

$$y^{i+1}\,Y^{i+1} = (\lambda - i - xX) * y^i Y^i$$

[* Vol. II. of this Reprint, p. 498.]

for $(\lambda - i) Y^i\phi = (xX + yY) Y^i\phi$ by Euler's well-known theorem on homogeneous functions.

The $(j+1)$th and $(j+2)$th terms in y^iY^i are respectively

$$\mp \frac{i(i-1)\ldots(i-j+1)}{1.2\ldots j}(\lambda-j)(\lambda-j-1)\ldots(\lambda-i+1)\,x^jX^j,$$

say

$$-Ax^jX^j$$

and $$\pm \frac{i(i-1)\ldots(i-j)}{1.2\ldots(j+1)}(\lambda-j-1)(\lambda-j-2)\ldots(\lambda-i+1)\,x^{j+1}X^{j+1},$$

say

$$Bx^{j+1}X^{j+1}.$$

Now $$xX * x^{j+1}X^{j+1} = x^{j+2}X^{j+2} + (j+1)\,x^{j+1}X^{j+1}.$$

Hence the $(j+2)$th term in $y^{i+1}Y^{i+1}$ will be

$$A + (\lambda - i - j - 1)\,B,$$

that is, putting $$\frac{i(i-1)\ldots(i-j+1)}{1.2\ldots j(j+1)} = B', \text{ is } \mu B',$$

where

$$\begin{aligned}\mu &= (j+1)(\lambda-j) + (i-j)(\lambda-i-1-j)\\ &= \{-j^2 + (\lambda-1)j + \lambda\} + \{j^2 - (\lambda-1)j + \lambda i - i^2 - i\}\\ &= (\lambda - i)(i+1).\end{aligned}$$

Thus the $(j+2)$th term in $y^{i+1}Y^{i+1}$ will be

$$\pm \frac{(i+1)i\ldots(i+1-j)}{1.2\ldots(i+1)}(\lambda-j-1)(\lambda-j-2)\ldots\{\lambda-(i+1)+1\};$$

and consequently the equation is true for $i+1$.

Hence, being true for $i = 1$, it is true universally.

(2) Consider a persymmetrical determinant of the order $\epsilon + 1$ formed with the distinct constituents $\phi_0, \phi_1, \phi_2, \ldots \phi_{2\epsilon}$, where ϕ_0 is a constant and in general $\frac{d}{dx}\phi_k = \pm k\phi_{k-1}$; as, for example, suppose $\epsilon = 2$, and let the determinant be

$$\begin{vmatrix} a, & ax+b, & P \\ ax+b, & P, & Q \\ P, & Q, & R \end{vmatrix},$$

where P, Q, R stand for

$$ax^2 + 2bx + c,\ ax^3 + 3bx^2 + 3cx + d,\ ax^4 + 4bx^3 + 6cx^2 + 4dx + e,$$

and $\frac{d\phi_k}{dx} = k\phi_{k-1}$. If we made $\frac{d\phi_k}{dx} = -k\phi_{k-1}$ the effect would be to change the signs of all the odd-degreed functions, but the value of the determinant would not be altered by this change. Calling the columns P_0, P_1, P_2,

$$P_0,\ P_1 - xP_0,\ P_2 - 2xP_1 + x^2P_0$$

will represent a determinant equal to the given one, but of the form

$$\begin{vmatrix} a, & b, & c \\ ax+b, & bx+c, & cx+d \\ ax^2+2bx+c, & bx^2+2cx+d, & cx^2+2dx+e \end{vmatrix},$$

and now, calling the lines L_0, L_1, L_2, the equivalent determinant

$$L_0,\ L_1 - xL_0,\ L_2 - 2xL_1 + x^2L_0$$

becomes

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

which is the same as if in the original form we made $x=0$.

So in general for the order $\epsilon+1$, calling the ϵ columns $P_0, P_1, P_2 \dots P_\epsilon$, we may pass to a new determinant by means of the combinations represented by

$$P_0,\ P_1 - xP_0,\ P_2 - 2xP_1 + x^2P_0,\ \dots\ P_\epsilon - \epsilon xP_{\epsilon-1} + \frac{\epsilon(\epsilon-1)}{2}x^2P_{\epsilon-2} \dots + (-)^\epsilon x^\epsilon P_0,$$

and calling the lines of these new determinants

$$L_0,\ L_1,\ L_2 \dots L_\epsilon,$$

$$L_0,\ L_1 - xL_0,\ L_2 - 2xL_1 + x^2L_0,\ \dots\ L_\epsilon - \epsilon xL_{\epsilon-1} + \frac{\epsilon(\epsilon-1)}{2}x^2L_{\epsilon-2} \dots + (-)^\epsilon x^\epsilon L_0,$$

will produce a determinant containing no power of x, and which is what the original one becomes on making $\epsilon=0$.

(3) If we take for our $2\epsilon+1$ distinct elements of the persymmetrical matrix, the quantities

$$X^{2\epsilon}\phi,\ yYX^{2\epsilon-1}\phi,\ y^2Y^2X^{2\epsilon-2}\phi, \dots\ Y^{2\epsilon}\phi,$$

where ϕ is of λ dimensions in x, y, we shall find by virtue of (1) that they will be represented by

$$A_0,\ A_1 - A_0x,\ A_2 - 2A_1x + A_0x^2, \dots$$
$$A_{2\epsilon} - 2\epsilon A_{2\epsilon-1}x + \frac{2\epsilon(2\epsilon-1)}{2}A_{2\epsilon-2}x^2 \dots + A_0x^{2\epsilon},$$

(where $\phi_k = -k\phi_{k-1}$) on making

$$A_0 = X^{2\epsilon}\phi,\ A_1 = (\lambda - 2\epsilon + 1)X^{2\epsilon-1}\phi,$$
$$A_2 = (\lambda - 2\epsilon + 2)(\lambda - 2\epsilon + 1)X^{2\epsilon-2}\phi,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$A_{2\epsilon} = \lambda(\lambda-1)\dots(\lambda - 2\epsilon + 1).$$

Now obviously the persymmetrical determinant in question on striking out each power of y from its several constituents will be diminished in the proportion of 1 to $y^{2+4\dots+2\epsilon}$, that is, $y^{\epsilon^2+\epsilon}$.

Hence

$$\begin{vmatrix} A_0, & A_1, & A_2, & \dots A_\epsilon \\ A_1, & A_2, & A_3, & \dots A_{\epsilon+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_\epsilon, & A_{\epsilon+1}, & A_{\epsilon+2}, & \dots A_{2\epsilon} \end{vmatrix}$$

$$= y^{\epsilon^2+\epsilon} \begin{vmatrix} X^{2\epsilon}\phi, & X^{2\epsilon-1}Y\phi, & \dots X^\epsilon Y^\epsilon \phi \\ X^{2\epsilon-1}Y\phi, & X^{2\epsilon-2}Y^2\phi, & \dots X^{\epsilon-1}Y^{\epsilon+1}\phi \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ X^\epsilon Y^\epsilon \phi, & X^{\epsilon-1}Y^{\epsilon+1}\phi, & \dots Y^{2\epsilon}\phi \end{vmatrix}.$$

This is true for any function ϕ homogeneous in x and y*.

If ϕ is a rational integral function of x, y, say

$$(a_0, a_1, a_2, \dots \between x, y)^\lambda,$$

the last written determinant becomes a covariant whose leading coefficient is the persymmetrical determinant formed with the elements $a_0, a_1, a_2, \dots a_{2\epsilon}$ multiplied by

$$\{\lambda(\lambda-1)\dots(\lambda-2\epsilon+1)\}^{\epsilon+1},$$

and if we write

$$\lambda(\lambda-1)\dots(\lambda-2\epsilon+1)B_0 = X^{2\epsilon}\phi,$$

$$\lambda(\lambda-1)\dots(\lambda-2\epsilon)B_1 = X^{2\epsilon-1}\phi,$$

$$\lambda(\lambda-1)\dots(\lambda-2\epsilon-1)B_2 = X^{2\epsilon-2}\phi,$$

$$\dots\dots\dots$$

$$B_{2\epsilon} = \phi,$$

we shall have the persymmetrical determinant formed with the elements $B_0, B_1, \dots B_{2\epsilon}$, equal to $y^{\epsilon^2+\epsilon}$ multiplied into the covariant of which the leading coefficient is the persymmetrical determinant formed with the elements $a_0, a_1, \dots a_{2\epsilon}$, as was to be proved.

* It seems to me very likely or almost certain that every covariant of $f(x, y)$, or what becomes such when f is a quantic, may in like manner be converted into a function of f and of its derivatives in respect to one of the variables alone divided by an appropriate power of the other; and, if true, as it can hardly help being, the proof ought not to be far to seek.

It is indeed virtually contained in a formula obvious from inspection of the expression for y^iY^i in (1), namely,

$$\left\{\frac{d}{dx}(y^iY^iX^j)\right\} * = -i\{y^{i-1}Y^{i-1}X^{j+1} *\},$$

whatever homogeneous function is supposed to follow the asterisk. In connexion with this it should be observed that the determinant in (2) is bound to vanish, from the mere fact that on putting $x=0$, it becomes an invariant of $(a_0, a_1, a_2, \dots a_{2\epsilon}\between \xi, \eta)^{2\epsilon}$, and that its several terms are what the a elements become when X becomes $\xi + x\eta$. We are thus led to view the whole subject of invariance under a somewhat broader aspect, as a theory not directly concerned with quantics, but with homogeneous functions in general.

Scholium. The theory of hyperdeterminants teaches us that every in- and co-variant has its source of being in a higher existence, namely, in a pure form typified by

$$F(X^\mu\phi,\ YX^{\mu-1}\phi,\ Y^2X^{\mu-2}\phi, \ldots Y^\mu\phi),$$

ϕ being a perfectly general operand, or as we may phrase it, an operand absolute. This enables me to express the idea which was struggling into light when I wrote the antecedent footnote. It is this: Let ϕ now be made to do duty for any given homogeneous function of given order λ in x, y.

The value of F will remain unaltered when we write

$$\frac{\lambda-\mu+1}{y} \text{ in place of } Y,$$

$$\frac{(\lambda-\mu+2)(\lambda-\mu+1)}{y^2} \quad \text{,,} \quad Y^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-\mu+1)}{y^\mu} \quad \text{,,} \quad Y^\mu.$$

This is an immediate consequence of the invariantive property of F combined with the fact that

$$\frac{d}{dx}y^iY^i = -iy^{i-1}Y^{i-1}X,$$

previously shown. The numerators in the above expressions are the first terms in the expression for y^iY^i as a function of xX modified by writing successively $\lambda-\mu+1$, $\lambda-\mu+2$, ... λ in place of λ on account of the powers of X which precede $Y, Y^2, \ldots Y^\mu$ in F and lowering the degrees of the operands in respect to these powers by $\mu-1$, $\mu-2$, ... 0 units respectively.

Thus, for example, the pure invariant

$$(X^4:)(Y^4:) - 4(X^3Y:)(XY^3:) + 3(X^2Y^2:)^2$$

where the colon (:) does duty for an operand absolute is equivalent to

$$\begin{aligned}\frac{1}{y^4}.\quad &(\lambda-3)(\lambda-2)(\lambda-1)\lambda\,(X^4:)(X^0:)\\ &-4(\lambda-3)^2(\lambda-2)(\lambda-1)\quad (X^3:)(X:)\\ &+3(\lambda-3)^2(\lambda-2)^2\quad (X^2:)(X^2:),\end{aligned}$$

the colon now representing a homogeneous function of order λ in x, y.

So in general we may say that a *pure invariant*, or it might be more correct to say the *Schema* of an invariant, *is a function of symbolic inverses* $(X, Y, \ldots)$ *to any number of letters and of any number of unconditional absolutes, possessing the property that when those absolutes become conditioned to stand for homogeneous functions of the letters, of* SPECIFIED *orders, it becomes a function of any one of the letters, of the symbolic inverses to the rest and of the absolutes so conditional.*

This property, which is certainly *necessary*, is in all probability sufficient to define a pure invariant, for I presume (nay I think it is obvious) that when it is satisfied, the only part the arbitrarily selected letter can play is that of contributing a power of itself as a factor to the function in which it figures. This definition of invariance, although it may appear abstruse, is in reality the most complete and simplest, in the sense of exemption from foreign ingredients and unnecessary specifications, that can be given, and may of course be extended without difficulty to systems of sets of letters $(x, y, \ldots)$. Nor should it be overlooked that in our great art, the *ars magna excogitandi*, a gain in expression is a gain in power*.

Returning from this rather wide excursus to our original theme of Newton's theorem, it may be useful to give the values of the G† series as far as required for equations of the 5th order inclusive corresponding to the critical value of the arbitrary parameter, that is, for the case of $C = -n$.

The given form being supposed to be $(a, b, c, \ldots \between x, y)^n$,

when $n = 2$, $-G_1 = ac - b^2$,

when $n = 3$, $-G_1 = (ac - b^2)\,x^2 + (ad - bc)\,x + (bd - c^2)$,

$-G_2 = ac - b^2$,

when $n = 4$, $-G_1 = (ac - b^2)\,x^4 + 2\,(ad - bc)\,x^3$
$+ (ae + 2bd - 3c^2)\,x^2 + 2\,(be - cd)\,x + (ce - d^2)$,

$-G_2 = (ac - b^2)\,x^2 + (ad - bc)\,x + (bd - c^2)$,

$-G_3 = ac - b^2$,

when $n = 5$, $-G_1 = (ac - b^2)\,x^6 + 3\,(ad - bc)\,x^5 + 3\,(ae + bd - 2c^2)\,x^4$
$+ (af + 7be - 8cd)\,x^3 + 3\,(bf + ce - 2d^2)\,x^2 + 3\,(cf - de)\,x + (df - c^2)$,

G_2, G_3, G_4 being the G_1, G_2, G_3 of the preceding case‡.

In applying the series of these G's combined with the f series to ascertain the maximum possible number of real roots passed over in going *up* from λ

* The object of pure Physic is the unfolding of the laws of the intelligible world. ["The unseen world" belongs to another province altogether.] The object of pure Mathematic (which is only another name for Algebra) that of unfolding the laws of the human intelligence. With Geometry it fares as it was thought to be probably about to fare with a certain distant land—it is "wiped out" between the two neighbouring powers. Algebra takes for its share Geometry in the abstract. Sensible or empirical Geometry (as, thanks to the Copernican genius of Lobatcheffsky and the sublimated practical sense of Helmholtz, is now beginning to be well understood) falls into the domain of Physic.

So already Logic is divided between Psychology and Algebra; and so eventually with Grammar, whilst Linguistic is handed over to History, Psychology and Physiology; its theoretical part, the laws of syntax, declension or conjugation, regimen and collocation, must be eventually absorbed into Algebra.

† [*In line* 12 *of* p. 415 *above, the first sign should be* $-$, *not* $+$.]

‡ It is thus seen that the G series is formed of the second alliances or "überschiebungen" of the given form (made homogeneous in x, y), and of its successive derivatives each with itself; and I have great reason to believe (as already hinted) that we may append a 3rd, 4th, ... series

to μ it is proper to use simultaneously the three *independent* superior limits (1) the gain of pP's, (2) the loss of vP's, (3) the gain of p's or loss of v's, which two latter numbers are of course identical.

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by substituting the 4th, 6th, ... of such alliances in lieu of the second, filling up the vacant spaces with positive absolutes, and always reckoning the gain of the permanence-permanence-permanence...s in going up from λ to μ as one superior limit, and, as a consequence thereof, the loss of the variation-permanence-permanence...s as another. Thus, for example, for the case of $n=4$, the series would be three in number, namely,

$$\begin{array}{ccccc} f, & f_1, & f_2, & f_3, & f_4, \\ 1, & -Hf, & -Hf_1, & -Hf_2, & 1, \\ 1, & 1, & s, & 1, & 1, \end{array}$$

where $s=ae-4bd+3c^2$ (and it may be noticed that we know from the expression for s in terms of the roots that when they are real, s must be positive).

For $n=5$ the series would be

$$\begin{array}{cccccc} f, & f_1, & f_2, & f_3, & f_4, & f_5, \\ 1, & -Hf, & -Hf_1, & -Hf_2, & -Hf_3, & 1, \\ 1, & 1, & s, & s', & 1, & 1, \end{array}$$

where $s'=ae-4bd+3c^2$,

and $s=(ae-4bd+3c^2)\,x^2+(af-3be+2cd)\,x+(bf-4ce+3d^2)$.

When $n=6$ or $n=7$ a new series would dawn into existence, and so on continually. Thus we set a number of sieves, as it were, successively under each other; it is certain, however, that by this method we can never be assured that no more than the actual number of real roots have fallen through; but there is another method which might be studied, and is, I think, not unworthy of investigation, that is, to take for our third series the covariants of f which have for their common leading coefficient the discriminant of the form $(a, b, c, d \between x, y)^3$, for the fourth series the covariants which have for their common leading coefficient the discriminant of $(a, b, c, d, e \between x, y)^4$, and so on indefinitely, always filling up the vacant spaces with positive absolutes.

In this way I think it not improbable that the gain of compound permanences may be found to give not merely a superior limit to, but the actual number of real roots passed over in any ascent from one value of x to another.

Such a theorem, however, would have no practical value as a method for separating the roots, as its application would entail much greater labour than the ordinary Sturmian process.