67.

ON SUBINVARIANTS, THAT IS, SEMI-INVARIANTS TO BINARY QUANTICS OF AN UNLIMITED ORDER.

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Er macht kein System, sondern es wird, es concrescirt in ihm, wie das Kind im Mutterleibe. (Schopenhauer) *Deutsche Rundschau*, July, 1882, p. 69.

§ 1. Proem.

Any rational integer function ϕ of the letters $a, b, c, \ldots$ indefinitely continued, which satisfies the partial differential equation

$$(a\delta_b + 2b\delta_c + 3c\delta_d \ldots)\,\phi = 0$$

may be termed a subinvariant in respect to the elements $a, b, c, \ldots$ or simply a subinvariant to or *quâ* those elements. It follows from this definition that any rational integer function of one or more subinvariants is itself one.

The same function of the letters $a, b, c, \ldots$ which, when regarded as the coefficient of the highest power of the first variable x in a covariant to the quantic $(a, b, c, \ldots \between x, y)^i$ or the polynomial $(a, b, c, \ldots \between x, 1)^i$ is termed a differenciant of the quantic or polynomial, when regarded as an individual of the infinite scale to which ϕ belongs, assumes the name of a subinvariant in respect to the letters $a, b, c, \ldots$.

Of course a differenciant derives its name from reference to the fact that when multiplied by a suitable power of a it may be regarded as a function of the differences of the roots of any one of the infinite series of polynomials, of some covariant of each of which it is the principal coefficient.

It follows also from the definition that if any composite function is a subinvariant, each of its factors must be so too. For if the function be $P^\alpha . Q^\beta . R^\gamma \ldots$ writing $a\delta_b + 2b\delta_c + \ldots = E$, we must have

$$\alpha \frac{EP}{P} + \beta \frac{EQ}{Q} + \gamma \frac{ER}{R} + \ldots = 0,$$

which for denominators $P, Q, R, \ldots$ relatively prime to each other is obviously impossible unless $EP = 0$, $EQ = 0$, $ER = 0 \ldots$, that is, $P, Q, R \ldots$ are subinvariants.

Again, suppose U, V, Ω to be three subinvariants so related that the equation $XU + VU = \Omega$ is capable of being satisfied at all. I say that it must be capable of being satisfied by subinvariantive values of X, Y*.

For from the equation it follows that $EX \,.\, U + EY \,.\, V = 0$, of which the most general solution is

$$EX = K\left(\frac{V}{\Delta}\right), \quad EY = -K\left(\frac{U}{\Delta}\right).$$

Hence $$X = \left(\frac{V}{\Delta}\right) E^{-1} K + U_1, \quad Y = -\left(\frac{U}{\Delta}\right) E^{-1} K + V_1,$$

where U_1, V_1 are subinvariants. Substituting these values of X, Y in the original equation, there results $U_1 U + V_1 V = \Omega$, as was to be shown possible. The same or a similar manner of proof will serve to show that if for three functions U, V, W, $XU + YV + ZW = 0$, X, Y, Z, are, or may be replaced by subinvariants. I do not know for certain, but think that the proposition may be extended to any number of given functions $U, V, W, \ldots$.

It is scarcely necessary to add the fundamental theorem that if for the elements $a, b, c, \ldots$ be substituted the elements a, $a\lambda + b$, $a\lambda^2 + 2\lambda b + c, \ldots$ where λ is arbitrary, any subinvariant remains unchanged; the proof being that if such a change be made in the elements of any function F, ΔF (the change in F) is expressible by $(e^E - 1) F$, which, when F is a subinvariant, so that $EF = 0$, vanishes identically. Hence it is that subinvariants become differenciants†.

It may be worth while here to notice that if in place of the operator on ϕ in the above equation *any* numerical linear function of $a\delta_b$, $b\delta_c$, $c\delta_d \ldots$ be substituted‡, the value of ϕ which satisfies the transformed equation will be a subinvariant *quâ* the elements $a, b, c, \ldots$ divided respectively by appropriate

* For instance, in the above equation, U, V may be supposed to be two subinvariants of equal extent, exceeding by a unit that of Ω, their resultant in respect to their final letter. We know, by a principle demonstrated further on in the text, that Ω must be a subinvariant. The present theorem shows that X and Y also are (or may be replaced by) subinvariants.

† Or more simply for any number of letters $\alpha_1, \alpha_2, \ldots \alpha_i$, not fewer than the number of ratios between $a, b, c, \ldots$, if

$$a\Sigma\alpha_1 = ib, \quad a\Sigma\alpha_1\alpha_2 = \frac{i(i-1)}{2} c, \quad a\Sigma\alpha_1\alpha_2\alpha_3 = \frac{i(i-1)(i-2)}{2 \,.\, 3} d \ldots \text{ then } a\delta_b + 2b\delta_c + 3c\delta_d \ldots = \Sigma\frac{d}{d\alpha},$$

because $$\Sigma\frac{db}{d\alpha} = a, \quad \Sigma\frac{dc}{d\alpha} = 2b, \quad \Sigma\frac{dd}{d\alpha} = 3c \ldots.$$

Hence any subinvariant to the letters $a, b, c, \ldots$ is a function of the differences of $\alpha_1, \alpha_2, \ldots \alpha_i$.

‡ So, for example, $(a\delta_b + b\delta_c + c\delta_d \ldots)^{-1}\, 0$ is a subinvariant *quâ* the elements

$$a, \; b, \; \frac{c}{1 \,.\, 2}, \; \frac{d}{1 \,.\, 2 \,.\, 3} \ldots.$$

numbers; namely, if the linear function be $pa\delta_b + qb\delta_c + rc\delta_d$, these numbers will be 1, p, $\frac{p \cdot q}{1 \cdot 2}$, $\frac{p \cdot q \cdot r}{1 \cdot 2 \cdot 3}$, as will be evident by making

$$\alpha = a, \ \ p\beta = b, \ \ \frac{p \cdot q}{1 \cdot 2}\gamma = c, \ \ \frac{p \cdot q \cdot r}{1 \cdot 2 \cdot 3}\delta = d, \ldots$$

which being done the operator last above written may be changed into

$$\alpha\delta_\beta + 2\beta\delta_\gamma + 3\gamma\delta_\delta \ldots.$$

As a consequence of this it will readily be seen that if $\phi(a, b, c, d, \ldots)$ be a subinvariant to the elements $a, b, c, d \ldots$

$$\phi(0, b, c, d, \ldots), \ \ \phi(0, 0, c, d, \ldots), \ \ \phi(0, 0, 0, d, \ldots)$$

will respectively be subinvariants *quâ* the elements

$$b, \frac{c}{2}, \frac{d}{3}, \frac{e}{4}, \ldots,$$

$$c, \frac{d}{3}, \frac{e}{6}, \ldots,$$

$$d, \frac{e}{4}, \ldots,$$

and so on, the denominators following the law of figurate numbers.

This theorem, although foreign to the original and primary object of the present paper, as given in § 4, is of some considerable importance to the method of deduction. I mean the method (theoretically perfect but practically very difficult of application for quantics beyond the 4th order) according to which all the groundforms of a quantic, or which is the same thing, their ground-differenciants*, may be deduced by an exhaustive algebraical process in successive strata or categories from one another beginning with the known forms a, $ac - b^2$, $a^2d - 3abc + 2b^3, \ldots$ as the first category. See § 3.

It follows from the definition above given that a subinvariant may contain any given number of letters, and the number which it actually contains, less one (that is, the weight of the most advanced letter which appears in it), may be called its *extent*. Any subinvariant will then be a differenciant to a quantic whose order is not less than such extent.

Of course the definition of subinvariant may be extended to sets of letters $a, b, c \ldots; a', b', c' \ldots; a'', b'', c'' \ldots.$ Any function ϕ of these sets of letters may be called a subinvariant, or when necessary, by way of distinction, a pluri-subinvariant, which satisfies the equality

$$(a\delta_b + 2b\delta_c + \ldots + a'\delta_{b'} + 2b'\delta_{c'} + \ldots + a''\delta_{b''} + 2b''\delta_{c''} \ldots)\,\phi = 0.$$

* I shall frequently use the term groundform to signify the leading coefficient of what is ordinarily so termed.

But for greater simplicity, except when a necessity arises for enlarging the horizon, I shall, in what follows, confine myself to the case of a single set of letters, that is, of uni-subinvariants*.

By an irreducible subinvariant is of course to be understood one which cannot be expressed as a rational integer function of any others. A differenciant to an irreducible quantic is of necessity a subinvariant, but not necessarily or even generally an irreducible subinvariant in the absolute sense in which the word is employed above; it will, however, be inexpressible as a rational integer function of any other subinvariants whose extent does not exceed the order of the quantic concerned, and may thus be said to be *relatively* irreducible. Thus, for example, the subinvariant

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$$

is irreducible, relatively to the extent 3 or *quâ* the letters a, b, c, d, that is to say, cannot be expressed as a rational integer function of subinvariants whose elements are limited to a, b, c, d, but it is not an irreducible subinvariant in the absolute sense of the term, because it can be represented by a combination of the subinvariants

$$a,\ ac - b^2,\ ae - 4bd + 3c^2,\ (ac - b^2)\,e + 2bcd - ad^2 - c^3,$$

the letter e being eliminated by the process of taking the difference between the product of the 2nd and 3rd and that of the 1st and 4th of the preceding groundforms†.

Here I may take occasion to state a theorem of wide generality suggested by the above decomposition. It is well known that if ϕ be a subinvariant extending to the letter l as the highest letter which it contains, all the successive derivatives of ϕ in respect to l will also be subinvariants, as is evident from the fact that if $(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\,\phi$ is zero, the same must be true of $(\delta_l)(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\,\phi$, or what is the same thing, of

$$(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\,\delta_l\phi.$$

Suppose then that ϕ, ψ, ω, ... are any number of subinvariants limited to l as their highest letter, and regarded, each of them, as a homogeneous function of l and 1, then I say that any differenciant in respect to l of this system of quantics will be a subinvariant *quâ* the elements $a, b, c, \ldots k$. For we know that any differenciant of the system $\phi(x)$, $\psi(x)$, ... say

$$(\alpha,\ \beta,\ \gamma \ldots \lambda \between x,\ 1)^i;\ \ (\alpha',\ \beta',\ \gamma' \ldots \lambda' \between x,\ 1)^{i'}, \ldots$$

* Eventually I am inclined to substitute the word binariant for subinvariant, and to speak cf simple, double, treble or multiple binariants. The functions similarly related to ternary forms will then be styled simple or multiple ternariants, and so in general.

† So it may be shown that the subinvariants of deg-orders 5 . 7, 5 . 1, 5 . 5 to the Quintic (which are perfectly determinate), may be regarded as the resultants in respect to g of the sextic groundforms 2 . 0 and 4 . 6, 2 . 0 and 4 . 0, 2 . 0 and 4 . 4 respectively, all four of which are linear in g. See Sextic Germ Table, § 2. [p. 578, below.]

remains unaltered when α, $\alpha+\beta x$, $\alpha+2\beta x+\gamma x^2 \ldots \phi$, respectively, are substituted for $\alpha, \beta, \gamma \ldots \lambda$, and at the same time

$$\alpha', \alpha'+\beta' x, \ldots \psi, \text{ for } \alpha', \beta', \ldots \lambda',$$

respectively, and so on; that is to say, any subinvariant of the equation above written may be regarded as a function of

$$\phi x, \phi' x, \phi'' x, \ldots; \psi x, \psi' x, \psi'' x, \ldots; \ldots.$$

Hence in regard of the system of subinvariants any of its differenciants is a function of the members of the system, and the successive derivatives in respect to l of each member, all of which are subinvariants. Hence the differenciant in question may be regarded as a function exclusively of subinvariants, and is therefore a subinvariant of the letters $a, b, c, \ldots k$. As a particular application of the theorem we see that the resultant in regard to their last letter of two subinvariants of like extent and the discriminant of any subinvariant in regard to its last letter are subinvariants. Thus, for example, if the discriminant of a cubic be exhibited as a quadratic function of d, namely, under the form $a^2d^2+(4b^3-6abc)\,d+(4ac^3-3b^2c^2)$, its discriminant, namely,

$$(2b^3-3abc)^2-a^2(4ac^3-3b^2c^2), \text{ that is, } 4\,(b^3-3ab^3c+3a^2b^2c^2-a^3x^3)$$

is as it ought to be a subinvariant, namely, it is $4\,(b^2-ac)^3$. So more generally, if we regard any number of pluri-subinvariants (all of the same extent in each set of letters) as a system of multi-partite polynomials in the extreme letter of each set, any differenciant of such system will be a subinvariant (of course with diminished extent in each set) in regard to the original letters. The simple instance already given will serve as a diagram to make the reason self-evident. The invariant in respect to d of the discriminant of the cubic is the same as in respect to x of

$$a^2(x+d)^2+(4b^3-6abc)(x+d)+(4ac^3-3b^2c^2),$$

that is, of $a^2x^2+2\,(a^2d-3abc+2b^3)\,x+(a^2d^2+4b^3d-6abcd+4ac^3-3b^2c^2)$,

hence being a function of the three coefficients, which are all of them subinvariants, it is itself a subinvariant*.

It has been shown above that the same form which regarded as a differenciant is irreducible, that is, is incapable of being decomposed into products of other differenciants of no higher extent than its own, when regarded as a subinvariant may be, and as a matter of fact, far oftener than not will be decomposable into products of subinvariants of higher extent. Thus the irreducible differenciants to any quantic naturally resolve themselves into two classes, those which are absolutely irreducible and those which are only relatively so; and it would seem that in any natural method of proof of Gordan's theorem these would, it is likely, have to be considered separately.

* The method of proof here employed, it will be seen, is the same in kind as that employed in the ordinary proof of Taylor's theorem.

There is comparatively little difficulty in proving that the first class are finite in number; the proof of the second class being likewise finite, must depend upon the fact that they are the resultants of a finite number of functions.

I use the word resultant in the above paragraph in an enlarged sense. If U, V, W, ... are any given polynomials in x, y, ... z, t, ... u, I call any quantity not containing x, y, ... z capable of being exhibited under the form of the syzygetic function $U_1U + V_1V + W_1W \ldots$ a resultant of the given polynomials in respect to x, y, ... z. For resultants thus defined, the following important proposition admits of easy proof, namely: Every such resultant is capable of being represented as a sum of products U_1U, V_1V, ... of which the orders in x, y, ... z are *limited* in extent, and consequently the most general representation of such resultant can contain only a finite number of arbitrary parameters. When the number of the eliminables x, y, ... is one less than the number of the given functions which contain them, we fall back upon the ordinary kind of resultant, having only one arbitrary parameter. When there is but one eliminable x, and any number of polynomials U, V, W, ... of orders α, β, γ, ... in x, the order in x of each syzygetic product U_1U, V_1V, ... in a syzygetic function of U, V, W, ... which is competent to represent any resultant of the system, is (if I mistake not) at most one unit less than the sum of the two highest (or of the two as high as any) of the numbers α, β, γ ....

The orders of the syzygetic multipliers being once determined, the number of indeterminate constants is known, and these will be subject to satisfy a known number of *linear* equations, namely, a number greater by unity than the order of the $U_1U + V_1V$ ... polynomial, and thus the problem of finding the complete system of resultants of the original system of polynomials in one variable is brought to depend upon the problem of finding the complete system of resultants of a system of homogeneous *linear* functions of several variables, a problem of which the solution and the number of arbitrary parameters which at most can appear in it are perfectly well known and need not be here set forth.

The syzygetic products U_1U, V_1V, ... whose sum is competent to express every resultant of U, V ..., I have said, need none of them be taken of an order so high as the sum of the two greatest of the quantities α, β, γ.... Thus for instance in the case of U, V, W, ... being linear functions, the syzygetic multipliers, as is well known, need only to be taken as constants; or again when α, β, γ, ... form a descending series, the syzygetic products need only to be all of them made of the same order as the highest of the given functions. Take, to fix the ideas, three functions, U, V, W, all of them quadratics in x. The syzygetic multipliers may be taken all linear functions in x: there will thus arise six disposable constants subject to three conditions, inasmuch as the coefficients of x^3, x^2, x, must vanish in the sum of the products: if two of the multipliers, say of U, V, were made quadratic functions, there would be eight disposable constants subject to four conditions, since an additional coefficient, namely, of x^4, would have to vanish in the sum of the products: there would therefore be one additional arbitrary parameter, namely, $8-4$ instead of $6-3$, but the form of the *resultant* would be not more general than on the preceding supposition, because if to U_1, V_1 (the most general values of the linear multipliers of U, V), λV, $-\lambda U$ respectively be added, there will then be four arbitrary parameters, and consequently the solution must be the same as on the second supposition, but the value of the resultant remains unaltered by the change made in U_1, V_1.

Or again if U, V, W were the two first quadratics and the second a linear function in x, their syzygetic multipliers might be taken two constants and a linear function respectively: by raising the orders of any two of these multipliers by a unit, an additional arbitrary constant would be gained, but the sum of the products resulting therefrom would not thereby gain in generality, as may be shown by the same method as in the preceding example.

It might probably not be difficult to give a universal rule for determining the lowest orders of the syzygetic multipliers required for expressing the resultant in its most general form, of functions of one or even of several variables, but this is an inquiry which it is necessary to postpone, as it might lead to too long a deviation from the immediate purpose in view, and there are some difficulties attending the subject more than present themselves at first sight.

It is enough to know, and that only for the case of a single eliminable, the existence of a limit to the orders of the multipliers, which it is quite easy to demonstrate. That being premised, it will follow as an easy consequence, that any combination *inter se* of subinvariants of any given extent and each containing the highest letter corresponding thereto can only give rise to a limited number of subinvariants of lower extent, and from that it is easy by repeated applications of the same *principle of the limit* to infer that only a finite number of *relatively* irreducible subinvariants of any given extent (that is, irreducible into combinations of subinvariants of the same or lower extent) can arise from the combinations of a finite number of subinvariants of any given higher extent; but it will appear in the sequel that the degree and consequently that the number of irreducible subinvariants of any given extent is subject to a limit; consequently if the number of relatively irreducible subinvariants of any given extent (or which is the same thing, if the covariants of a quantic of any given order) were unlimited in number, this could only be in consequence of there being no extent so large but that subinvariants of that extent and containing the most advanced letter corresponding thereto, would be needed in order to exhibit the composition of the relatively irreducible, but in an absolute sense, reducible subinvariants referred to.

In § 4 I propose to show how to obtain the types (that is, deg-weights) of the absolutely irreducible subinvariants of the first few degrees. Besides the intrinsic interest of the inquiry, the result obtained without going beyond subinvariants of the 7th degree will serve to show conclusively that *it is not true* "that syzygants and groundforms of the same degree and order cannot appertain to the same binary quantic," but that when the order of the quantic is sufficiently elevated there *must* appertain to it, syzygants (*compound* ones) and groundforms of the same degree and order.

Let it be observed that the proposition here about to be disproved is not coextensive with the law of parsimony, but goes considerably beyond it—that is, implies much more than that law gives warrant for.

Let us for the moment call the number of linearly independent forms of the deg-order (j, ω) to a given quantic given by Cayley's rule, the denumerator to the type (j, ω), and the number of forms of such type that can be obtained by compounding together groundforms of lower types, the aggregator to the same type. Let us further suppose that the duad (j, ω) may be compounded of (j', ω'), (j'', ω'')*.

Suppose further that the aggregator to the type (j', ω') exceeds its denumerator, and also that there exists one or more, say Δ' linearly independent invariantive forms of the deg-order (ω'', j''), but that (*if possible*) the aggregator to the type (j, ω) is equal to or less than its denumerator, the difference being Δ. Obviously if such a case can occur, the law of parsimony (that is, the Newtonian rule of not assuming more causes to exist than are necessary to the explanation of a phenomenon or set of phenomena) will, on such a supposition, lead to the conclusion, not that there are Δ groundforms and *no* syzygies, but $\Delta + \Delta'$ groundforms and Δ' syzygies. Such a case does not present itself for quantics of the lower orders; it seems natural and logical therefore to seek for it in the case of a quantic of an infinite order, that is, in the case of subinvariants unlimited in *extent*. If it can be shown (as in § 4 it will be shown) that with an unlimited number of letters, an irreducible subinvariant and a compound syzygy of subinvariants coexist for a given degree and for the weight ω, it will follow from the nature of the process employed in what follows, that the same conclusion must hold when the *extent* of the subinvariants is limited, provided (at the very worst) that the limit is not less than ω, for it will be seen that no letter of higher weight than ω enters into the process which leads to the result under con-

* I mean that $j=j'+j''$, $\omega=\omega'+\omega''$.

sideration. It is in all human probability true that the proposition holds good in the form in which it was originally presented, namely, that *irreducible* syzygants and irreducible invariantive derivatives of the same type, to the same quantic cannot coexist; but whether the proposition so limited is sufficient to support the substitution of the process of tamisage performed upon the numerator of the representative generating fraction, in lieu of tamisage performed upon the development of that fraction in an infinite series, or how the method of substitutive tamisage, if at present inexact, may be modified *pari passu* with the needful modification in brute tamisage so as to recover its validity, is a matter which must be reserved for future consideration.

§ 2. Germs.

Before proceeding to the more immediate object of this paper I think it will be profitable to insert the following table of the multipliers of the highest letter or power of the highest letter f in the relatively irreducible subinvariants of the extent 5 (that is, the leading coefficients in the groundforms of the quintic), and a similar table for the groundforms of the sextic arranged according to the powers of g*. For many purposes these tables will be found as serviceable as the entire function of the letters or even as the entire covariant written out at length. Those relating to the quintic may be verified by comparison with the tables (as far as they extend) contained in the *Formes Binaires* of M. Faà de Bruno, but the order of arrangement of the terms in those tables is not what my method of representation points out as the most natural, and proceeds upon some principle not easy to divine. It is also necessary to state that there are very many errors and misprints in those tables. With regard to the particular choice of the groundforms of any deg-order I believe that in all cases but one the tables of M. de Bruno are in accordance with those employed by myself, and which are on the face of them the *simplest* that can be employed, with one exception, namely, in the expression for the covariant of deg-order 9 . 3 the multiplier of the power of f, or *germ* as it may well be styled, is $(ac-b^2)^3$, whereas in the extended tables of M. de Bruno the germ will be found to be some numerical linear function (its exact value I have forgotten) of

$$(ac-b^2)^3,\ a^2(ac-e^2)(ae-4bd+3c^2),\ \text{and}\ a^3\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

or which comes to the same thing, of the two former and

$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd;$$

the covariant thus given of deg-order 9 . 3 is accordingly more complicated than it need have been.

* Any such multiplier I call the *germ* of the form to which it appertains.

It may be well to notice that whenever two consecutive terms in either table occur with the same germ but different powers of the last letter, the complete subinvariant of the antecedent is (to a numerical factor *près*) the differential derivative of the consequent in respect to that letter; thus, for example, the leading coefficient in the covariant to the quintic of the deg-order 7 . 5 will be found by simply differentiating the invariant of the degree 8 and dividing the result by the number 3.

In the table immediately following (c), (d), (e), (e'), Δ stand for

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ ae-4bd+3c^2,\ ace-ad^2+2bcd-c^3-ad^2$$

and $$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd$$

respectively. The quantities which appear in the outside vertical column are the germs; the double figures which fill the occupied spaces are the deg-orders. Thus, for example, 7 . 5 being opposite to the germ $(c)(d)$ and in the column headed by f^2, indicates that the covariant to the quintic of degree 7 and order 5 has for its differenciant a quantity of the form

$$(ac-b^2)(a^2d-3abc+2b^3)^2+\text{a linear function of } f,$$

and so in general.

GERM TABLE TO THE QUINTIC.

	1	f	f^2	f^3	f^4	f^7
a	1 . 5					
(c)	2 . 6					
(d)	3 . 9	4 . 4				
(e)	2 . 2					
(e')	3 . 3					
a^2		3 . 5	4 . 0			
$a(c)$		4 . 6	5 . 1			
$a(d)$			6 . 4			
$(c)^2$		5 . 7	6 . 2			
$(c)(d)$			7 . 5	8 . 0		
$3a(e')-2(c)(e)$		5 . 3				
$a^2(c)$				7 . 1		
$a(c)^2$				8 . 2		
$(c)^3$				9 . 3		
$(c)^2(d)$					11 . 1	
$(c)^2$					12 . 0	
$(c)^3(e')$					13 . 1	
$a(c)^5$						18 . 0

In the annexed table $(c)\,(d)\,(e)\,(e')\,(f)\,(\Delta)$ retain their previous significations. The additional symbols (cf), (c^2f), (df), (cef) represent respectively the differenciants to the quintic of the deg-orders $4\,.\,6$, $5\,.\,7$, $4\,.\,4$, $5\,.\,3$, all of which are linear functions of f (see preceding table).

Germ Table to the Sextic.

	1	g	g^2	g^3	g^5	g^6
a	1.6	2.0				
(c)	2.8	3.2				
(d)	3.12	4.6				
(Δ)			6.0			
(e)	2.4					
(e')	3.6	4.0				
(f)	3.8					
$a\,(c)$			5.2			
$a\,(d)$			6.6			
$a\,(e)$		4.4				
$(c)\,(d)$				8.2		
$(d)\,(e)$			7.4			
$(c)\,(f)$	4.10	5.4				
$(d)^3$						15.0
$a\,(d)\,(e)$				9.4		
$(c)^2\,(f)$		6.6*				
$a\,(d)\,(f)$			7.2			
$(c)(c)^2\,(f)$				10.2		
$(c)\,(e)\,(f)$	5.8					
$a^2\,(c)\,(d)$					12.2	
a^5					10.0	
$(c)\,(c^2f)$				10.2		

§ 3. Groundforms.

Quantitative Deduction of their Categories.

I will now proceed to explain what I mean by the exhaustive or quantitative method of deducing the ground differenciants to a given quantic, referred to in the course of the preceding observations.

The well-known functions of alternately the second and third degrees $ac-b^2$, $a^2d-3abc+2b^3$, $ae-4bd+3c^2$, ... limited in extent to the order of the quantic under consideration, may be called the protomorphs or primaries.

Suppose then the groundforms to the cubic are to be deduced. The primaries or protomorphs, omitting a, are $ac-b^2$, $a^2d-3abc+2b^3$, and the residues (meaning thereby the remainders when these quantities are divided by a) are $-b^2$, $2b^3$. Hence $(a^2d-3abc+2b^3)^2+4(ac-b^2)^3$ will divide out by a (as it happens by a^2) and give the new groundform

$$a^2d^2+4ac^3+6abcd+4b^3d-3b^2c^2.$$

Between its residue $4b^3d-3b^2c^2$, and the two former, it is obvious that no new relation can arise. Hence the four forms

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ a^2d^2+4ac^3-6abcd+4b^3d+3b^2c^2$$

constitute the complete system of ground differenciants, and the corresponding co- and- invariants comprehend the complete system of such for the cubic.

Proceeding to the quartic, a new protomorph or base-form comes into view, namely, $ae-4bd+3c^2$, whose residue is $-4bd+3c^2$ in addition to the antecedent ones $4b^3d-3b^2c^2$, $2b^3$, $-b^2$, and since the second of these is the product of the first and last it follows that

$$-(a^2d^2+\ldots)+(ac-b^2)(ae-4bd+3c^2)$$

must contain the factor a, and on performing the division there emerges the new groundform

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

so that $(a^2d^2+\ldots)$ being equal to this multiplied by a less the product of two other groundforms, ceases itself to be one, and the groundforms now subsisting are the one last named in addition to the base-forms

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ ae-4bd+3c^2,$$

which, since the new one is the only one of the *five* containing the letter e, can enter into no combination with them of which the residue is zero, and consequently the *deduction* is at an end and the five named constitute the complete system of groundforms.

Beyond this point the method of deduction has not hitherto been pushed, nor could it have been, without the use of the theorem concerning the subinvariantive character of the residues, in consequence of their enormous complexity when regarded as simple functions of the letters. In what follows the deduction is extended to the case of the quintic*.

Algebraical Deduction of the Groundforms of the Quintic†.

The complete system of groundforms to be deduced may be denoted by the deg-order or the deg-weight: viewed as subinvariants, the latter is the more natural mode of designation: if j and ω are the degree and weight, the order ϵ will be $5j-2\omega$. For greater facility of reference to the known list of groundforms, it will be convenient to set out the order as well as the degree; the complete system of the designating j; $\epsilon.\omega$, of the twenty-three groundforms, that is, of the twenty-three relatively irreducible subinvariants of *extent* not exceeding five, will then be as follows: $1;5.0$, $2;2.4$, $2;6.2$,

* In Salmon's *Modern Algebra*, 3rd Ed., pp. 170—1, 195—6, the base-forms employed in the deduction of the quartic groundforms are not identical with those employed above, the third one being of the fourth instead of the second degree in the letters, and consequently not a groundform, whereby the deduction is rendered somewhat longer than that given in the text. The most eligible base-forms to employ in any case are alternately of the second and third degrees, whereas those given by Prof. Cayley, the author of this important method, are of degrees continually increasing by a unit.

† By algebraical, I mean in this connection, that which deals only with the ordinary algebraical processes of addition, multiplication and division, as contradistinguished from transcendental processes involving differential operation, or which is substantially the same thing, symbolical resolution.

The preceding deduction for the Cubic and the Quartic is by far the simplest mode of obtaining the complete systems of groundforms for these quantics, and proving their completeness, which, at an earlier period of the theory, was regarded as a problem of some little difficulty. See Faà de Bruno's *Formes Binaires*, Chapter 7, pp. 260—263, where the same results are obtained through the medium of "*Formes Associées*." I cannot but think that sooner or later this method, first discovered by the eagle-gaze of Cayley, will lead to the object which I presume he had in view when he originated it, namely, a proof of Gordan's theorem by ordinary algebra.

I think I see looming in the not far distance such a proof, depending ultimately upon the fact of a certain succession of increasing integer multiplets, subject to stated laws of limitation, not being capable of being indefinitely produced. To render sensible the sort of arithmetical theorem which I have in view, I subjoin a theorem *ejusdem generis* concerning singlets (simple integers), which, as far as I know, is new, and admits of easy proof.

A succession of integers of which no one is a multiple of one nor the sum of the multiples of two others cannot be continued ad infinitum.

To prove this we may begin with the case where one of the integers written down is a prime number, for which case the proof is immediate. Then it is easy from this to show that if the theorem is true for the case where one of the integers is a product of only i-primes, it must be true for the case where one of the integers is a product of only $(i+1)$ primes; for this case, by virtue of the supposition made, may easily be reduced to the case where one of the numbers is a relative prime to all the others, for which case the theorem is true, for the same reason as if the number in question were an absolute prime. Consequently the theorem is true universally.

By the quotient of a duad (in what follows) is to be understood the quotient of the second element by the first; by the sum of two duads, the duad whose elements are the sums of the

3;5.5, 3;9.3, 3;3.6, 4;0.10, 4;4.8, 4;6.7, 5;1.12, 5;3.11, 5;7.9, 6;2.14, 6;4.13, 7;1.17, 7;5.15, 8;0.20, 8;2.19, 9;3.21, 11;1.27, 12;0.30, 13;1.32, 18;0.45. The protomorphs or base-forms are the five first of these, namely,

1;5.0 is a, 2;6.2 is $ac-b^2$, 3;9.3 is $a^2d-3abc+2b^3$,

2;2.4 is $ae-4bd+3c^2$, 3;5.5 is $a^2f-5abe+2acd+8b^2d-6bc^2$.

Again, 3;3.6 is the determinant

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

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corresponding elements of the two, and by a multiple of a duad the duad whose elements are the elements of that duad multiplied each by the same *integer*. The foregoing theorem may then be extended as follows:

A succession of duads, the quotients of all which but two are intermediate to the quotients of those two, and such that no duad is a multiple of any one or the sum of the multiples of any two or three of the others, cannot be indefinitely continued.

Again, one couple of quantities may be said to be *intermediate* to three others when the point representing the first is situated within the triangle whose apices represent the other three; a point being said to represent the two quantities which are equal to its two coordinates in respect to any two given axes. So a triplet of quantities, by aid of an analogous representation in space, may be said to be intermediate to four others when its representative point lies inside the pyramid whose apices represent those four.

It will readily be understood that these definitions may be translated into conditions of inequality between determinants, and thus translated may be extended so as to yield a definition of one pollad of $n-1$ elements being *intermediate* to n, or indeed to any number of other such pollads. Also the quotient-system of an n-ad will be understood to mean the system of $(n-1)$ quotients got by dividing the first element of the n-ad into the $n-1$ others. The following general theorem may then be enunciated:

A succession of n*-ads such that the quotient-systems of all but* n *of them are intermediate to the quotient-systems of those* n cannot be indefinitely continued, if every n-ad which is either a multiple of some one or a sum of multiples of 2, 3, ..., n or $n+1$ of the others, is excluded from the succession.

More generally, and with a less stringent negative condition, *a succession of* n*-ads such that the quotient-systems of all but* ν *given ones* (ν *being any number*) *are intermediate to the quotient-systems of those* ν, *cannot be indefinitely continued, if every* n*-ad which is a multiple or a sum of multiples of any or all of the* n*-ads of a group of* $\nu+1$ *others* (*whereof* ν *are the given ones*) *is excluded from the succession.*

The hypothetical ground of connection between this theorem and Gordan's algebraical one is as follows: It may be shown to be implied in the method of deduction, that if the number of groundforms to the quintic were infinite, then there must exist a certain infinite succession of products, *[some of the form $b^x Q^y R^z S^t$, the others of the form $b^\xi Q^\eta R^\zeta S^\tau T$, such that neither any product $b^x Q^y R^z S^t$ nor any product $b^\xi Q^\eta R^\zeta S^\tau$ could be (a power of one or) a product of powers of any number of the products not involving T. If then it could be shown that there exists a set of quadruplets of the kind x, y, z, t such that every other one of that kind and also every one of the kind ξ, η, ζ, τ is *intermediate* to that] set, the existence of such a succession would be impossible by virtue of the arithmetical theorem, and the possibility of the existence of an infinite number of groundforms would consequently be disproved. A similar kind of proof could conceivably, but with more difficulty, be extended to quantics of any order.

[* For the words placed in square brackets, see the correction, p. 621, below.]

This, not involving the letter f, has been previously deduced, and it has been shown that its integrating factor (that is, the power of a by which it must be multiplied to give a rational integer function of the base-forms) is a^3; it has, in fact, been shown (dropping the second integer and dealing only with deg-weights) that

$$(1\,.\,0)^3(3\,.\,6)=(1\,.\,0)^2(2\,.\,2)(2\,.\,4)-4(2\,.\,2)^3+(3\,.\,3)^2.$$

I shall denote the residue of any form ϕ by the symbol $\Re\phi$; each such residue is a function of the *five* letters b, c, d, e, f, being in fact a subinvariant in regard to the letters $b, \frac{c}{2}, \frac{d}{3}, \frac{e}{4}, \frac{f}{5}$, and therefore of the four groundforms proper to the diminished extent 4, that is, of the five following functions

$$b,\ \frac{bd}{3}-\frac{c^2}{4},\ \frac{b^2e}{2}-3\,\frac{bcd}{6}+2\,\frac{c^3}{8},\ \frac{bf}{5}-4\,\frac{ce}{8}+3\,\frac{d^2}{9},$$

$$\begin{vmatrix} b, & \frac{c}{2}, & \frac{d}{3} \\ \frac{c}{2}, & \frac{d}{3}, & \frac{e}{4} \\ \frac{d}{3}, & \frac{e}{4}, & \frac{f}{5} \end{vmatrix}$$

or (getting rid of the denominators) of

$$b,\ 4bd-3c^2,\ 2b^2e-bcd+2c^3,\ 6bf-15ce+10d^2,$$

$$\begin{vmatrix} 3b, & 3c, & 2d \\ 3c, & 4d, & 3e \\ 10d, & 15e, & 12f \end{vmatrix}$$

of which the deg-weights are $1\,.\,1$, $2\,.\,4$, $3\,.\,6$, $2\,.\,6$, $3\,.\,9$ respectively; the first of these is b, the others I shall call Q, T, R, S respectively. In all that follows I shall denote a numerical linear function of two or more quantities by enclosing them in brackets with commas interposed*; thus, for example, (ϕ, ψ, θ) will mean $\lambda\phi+\mu\psi+\nu\theta$, where λ, μ, ν are certain determinate (but unexpressed) numbers.

We know from the theory of the groundforms of extent 4 (that is, differenciants of a quartic) that the above five quantities are not algebraically independent, but are connected by an equation of the form

$$T^2=(Q^3,\ b^3S,\ b^2QR).$$

We have also the following expressions for the residues of the groundforms denoted by their deg-orders, and their first deduct, namely,

$$\Re(2\,;6)=b^2,\ \Re(3\,;9)=b^3,\ \Re(2\,;2)=Q,\ \Re(3\,;5)=bQ,\ \Re(3\,;3)=T,$$

or, using deg-weights instead of deg-orders,

$$\Re(2\,.\,2)=b^2,\ \Re(3\,.\,3)=b^3,\ \Re(2\,.\,4)=Q,\ \Re(3\,.\,5)=bQ,\ \Re(3\,.\,6)=T.$$

* **The brackets will sometimes for convenience be omitted.**

Since $b^3 . Q = b^2 . bQ$, that is, $\mathfrak{R}(3,\ 9)\,\mathfrak{R}(2,\ 2) - \mathfrak{R}(2\,.\,6)\,\mathfrak{R}(3,\ 5) = 0$, it follows that $((3\,;9)(2\,;2),\ (2\,;6)(3\,;5))$ must contain a.

Also it is obvious that the effect of throwing out a from a differenciant to the quintic which contains it, is to diminish the degree by one unit, leaving the weight unaltered, and therefore diminishes the order by five units.

Hence $$\frac{1}{a}((3\,;9)(2\,;2),\ (2\,;6)(3\,;5)) = 4\,;6.$$

It will be more convenient here and hereafter to use exclusively deg-weights instead of deg-orders to denote the forms; the above equation thus expressed becomes

$$\frac{1}{a}((3\,.\,3)(2\,.\,4),\ (2\,.\,2)(3\,.\,5)) = 4\,.\,7.$$

Turning now to the deg-weights of the residues, it will be seen that $4\,.\,7$ can only be composed of $1\,.\,1$ and $3\,.\,6$.

Hence $\mathfrak{R}(4\,.\,7) = bT$, which is not a product of residues; so $4\,.\,7$ must be a new groundform. Again, (adhering to the use of deg-weights) we have

$$(\mathfrak{R}(3\,.\,5))^2 = b^2Q^2 = \mathfrak{R}(2\,.\,2)\,(\mathfrak{R}(2\,.\,4))^2.$$

Hence $$\frac{1}{a}((3\,.\,5)^2,\ (2\,.\,2)\,.\,(2\,.\,4)^2) = 5\,.\,10.$$

The only mode of resolving $5\,.\,10$ into sums of the duads $1\,.\,1$, $2\,.\,4$, $3\,.\,6$, $2\,.\,6$, $3\,.\,9$, is by the addition of $2\,.\,4$ and $3\,.\,6$.

Hence $\mathfrak{R}(5\,.\,10)$ is a numerical multiple of QT, that is, of $\mathfrak{R}(2\,.\,4)$ and $\mathfrak{R}(3\,.\,6)$. Hence $((5\,.\,10),\ (2\,.\,4)(3\,.\,6))$ contains a; consequently $5\,.\,10$ is not a groundform, but we shall have $\frac{1}{a}((5\,.\,10),\ (2\,.\,4)(3\,.\,6)) = 4\,.\,10$, and $4\,.\,10$ can be resolved into $1\,.\,1 + 3\,.\,9$ and $2\,.\,4 + 3\,.\,6$. Hence $\mathfrak{R}(4\,.\,10) = (bS,\ QR)$ and $4\,.\,10$* will be a new groundform.

So again $(3\,.\,3)(3\,.\,5) = b^3\,.\,bQ$, and $(2\,.\,2)^2(2\,.\,4) = (b^2)^2 Q$. Hence

$$\frac{1}{a}\{(3\,.\,3)(3\,.\,5),\ (2\,.\,2)^2(2\,.\,4)\} = 5\,.\,8,$$

which can be resolved in only one way into a sum of the duads $1\,.\,1$, $2\,.\,4$, $3\,.\,6$, $2\,.\,6$, $3\,.\,9$, namely, into $1\,.\,1 + 1\,.\,1 + 3\,.\,6$. Hence

$$\mathfrak{R}(5\,.\,8) = \mathfrak{R}(2\,.\,2)\,\mathfrak{R}(3\,.\,6),$$

and consequently $5\,.\,8$ is not a groundform, but

$$\frac{1}{a}\{5\,.\,8,\ (2\,.\,2)(3\,.\,6)\} = [4\,.\,8],$$

which, in respect to the duads above mentioned, is resoluble (and only resoluble) into $2\,.\,4 + 2\,.\,4$ and $1\,.\,1 + 1\,.\,1 + 2\,.\,6$.

* $4\,.\,10$ which is the same (using deg-orders) as $4\,.\,0$ obviously cannot undergo further depression, and is consequently a groundform.

Hence $\mathfrak{R}(4.8) = (Q^2, b^2R)$, and since $Q = \mathfrak{R}(2.4)$ we have
$$\mathfrak{R}\,((4.8), (2.4)^2) = b^2R;$$
and since $([4.8], (2.4)^2)$ is of the deg-weight 4.8, we see that there is a form 4.8 such that $\mathfrak{R}(4.8) = b^2R$, which is consequently a groundform, since b^2R is not a rational integer function of any of the previous residues. Thus, then, from the base-forms 2.2, 3.3, 2.4, 3.5, besides the groundform not containing f, namely, 3.6, we have derived the three additional groundforms 4.7, 4.10, 4.8. Of these 4.7 and 4.8 belong to the same category as 3.6, being like it derived immediately from the base-forms. Whereas, in obtaining 4.10 it has been necessary to employ 3.6, so that it belongs to a more distant category. If we call the base-forms primaries, 3.6, 4.7, 4.8 will be secondaries, and 4.10 a tertiary. So again we shall find
$$\mathfrak{R}(3.3)\,\mathfrak{R}(3.6) = b^3.T, \text{ and } \mathfrak{R}(2.2)\,\mathfrak{R}(4.7) = b^2.bT.$$
Hence $\frac{1}{a}\{(3.3)(3.6), (2.2)(4.7)\} = 5.9$, and $\mathfrak{R}(5.9) = b^3.R$,

which cannot be compounded out of the preceding residues, so that (5.9) is another tertiary.

Again $\mathfrak{R}(4.7)\,\mathfrak{R}(2.4) = Q.bT$, and $\mathfrak{R}(3.5)\,\mathfrak{R}(3.6) = bQ.T$.

Hence $\frac{1}{a}\,((4.7)(2.4), (3.5)(3.6)) = 5.11$, and $\mathfrak{R}(5.11) = (b^2S, bQR)$,

for 5.11, in regard to the oft-quoted duads, is resoluble only into
$$1.1 + 1.1 + 3.6 \text{ and } 1.1 + 2.4 + 2.6.$$
Hence 5.11 is also a tertiary groundform.

Again
$$\mathfrak{R}(2.2)\,\mathfrak{R}(2.4)\,\mathfrak{R}(3.6) = b^2.Q.T, \text{ and } \mathfrak{R}\,4.7\;\mathfrak{R}\,3.5 = bT.bQ.$$
Hence $\frac{1}{a}\,((2.2)(2.4)(3.6), (4.7), (3.5)) = [6.12]$,

and the duad 6.12 is resoluble into
$$3.9 + (1.1)^3{}^* (2.6) + (2.4) + (1.1)^2 (3.6)^2 \text{ and } (2.4)^3,$$
corresponding to b^3S, b^2QR, Q^3, T^2. Now Q, T, b^2R are all residues, as already shown, and since b^2 and (bS, QR) are residues $(b^3S, b^2R.Q)$, and therefore b^3S is a residue.

Hence a form denotable by 6.12 which shall be a linear function of $[6.12]$ and of the combinations of inferior groundforms, will have a residue zero, and consequently $[6.12]$ will not be a groundform, but the 6.12 last spoken of will be divisible by a, and the quotient will give a groundform 5.12, whose residue corresponding to the composition $3.6 + 2.6$ is RT. We shall thus have obtained for our tertiary or third batch of groundforms (descendants,

* It will be often found convenient to use $(p.q)^i$ to mean the sum of i duads $p.q$.

that is, in the second degree from the base-forms) the subinvariants denoted by 4 . 10, 5 . 9, 5 . 11, 5 . 12.

Again $\mathfrak{R}(3.3)\,\mathfrak{R}(4.10)=b^3(bS,\ QR)$;

$$\mathfrak{R}(2.2)\,\mathfrak{R}(5.11)=b^2(b^2S,\ bQR);\ \mathfrak{R}(2.4)\,\mathfrak{R}(5.9)=Q(b^3R).$$

Hence between these three equations the two arguments b^4S, b^3QR may be eliminated, and there results

$$\frac{1}{a}\{(3.3)(4.10),\ (2.2)(5.11),\ (2.4)(5.9)\}=6.13,$$

and 6 . 13 will be resoluble only into

$$3.6+2.6+1.1, \text{ so that } \mathfrak{R}\,6.13=bRT.$$

Again $\mathfrak{R}\,3.5\,\mathfrak{R}\,5.12=bQ.RT$; $\mathfrak{R}\,3.6\,\mathfrak{R}\,5.11=T(b^2S,\ bQR)$;

$$(\mathfrak{R}\,4.7).(\mathfrak{R}\,4.10)=bT(bS.\ QR),$$

on the right-hand side of which three equations $bQRT$, b^2ST are the only two arguments appearing, so that

$$(\mathfrak{R}\,3.5,\ \mathfrak{R}\,5.12,\ \mathfrak{R}\,3.6,\ \mathfrak{R}\,5.11,\ \mathfrak{R}\,4.7,\ \mathfrak{R}\,4.10)$$

may be made equal to zero. Hence we have a new deduct 7 . 17, and $\mathfrak{R}\,7.17$ will be found $=(Q^2S,\ b^2RS,\ bQR)$, and 7 . 17 will be a groundform, as is apparent at once from the fact that it is the same (using a deg-order instead of deg-weight) as 7 ; 1 which is obviously indecomposable into any inferior forms.

But it may be objected that conceivably there might exist a syzygy between (3 . 5) (5 . 12), (3 . 6) (5 . 11), (4 . 7) (4 . 10), so that the form 7 . 17 obtained by dividing a linear combination of the three products by a may really be a null quantity. But not to mention the unlikelihood that a syzygy should occur between so low a number as only three products of groundforms of elevated degrees, the existence of such a syzygy may be directly disproved as follows: (3 . 6) (6 . 11) will contain only the first power of f, and writing

$$5.12=Lf^2+2Mf+N,\quad 4.10=Pf^2+2Qf+R,$$

we shall have $4.7=Lf+M,\qquad 3.5=Pf+Q,$

so that if the supposed syzygy exists we must have $LQ-MP=0$, but

$$L=-a^2,\ M=5abc-2acd+8b^2d+6bc^2,\ P=(a^2c-ab^2),\ Q=\ldots.$$

Hence since M does not contain a as a factor, MP cannot equal LQ, so that the conceivable syzygy does not exist, and the groundform 7 . 17 is correctly deduced*.

* I shall eventually supersede this proof of the non-existence of the syzygy under discussion by a method involving no algebraical computation. It is a remarkable feature in this deduction that although it is in its nature quantitative, no algebraical computations whatever need to nor will be employed in working it out and establishing its validity at each stage, thanks to the use made of the factors of integration, as will presently appear.

Again $\mathfrak{R}\, 5 . 9\, \mathfrak{R}\, 3 . 5 = b^3R . bQ$, $\mathfrak{R}\, 3 . 3\, \mathfrak{R}\, 5 . 11 = b^3 (b^2S,\ bQR)$,

$$(\mathfrak{R}\, 2 . 2)^2 . (\mathfrak{R}\, 4 . 10) = b^4 (bS,\ QR),$$

between which equations b^5S, b^4QR can be eliminated; thus there will be a form [7 . 14] deduced from

$$\frac{1}{a}\left((5 . 9)(3 . 5),\ (3 . 3)(5 . 11),\ (2 . 2)^2 (4 . 10)\right).$$

Also the sole components of $\mathfrak{R}\, 7 . 14$ will be easily seen to be

$$\left(3 . 6 \times (2 . 4)^2,\ 3 . 6 \times 2 . 6 \times (1 . 1)^2\right).$$

Hence $\mathfrak{R}\, [(7 . 14)] = (Q^2T,\ b^2RT)$, in which each of the two arguments is a residue*. Hence we may find a 7 . 14 which will be divisible by a and thus obtain a form 6 . 14, which (since 5 . 14 is necessarily non-existent) cannot be further depressed.

That this is not a *null* form will presently be demonstrated. It results that 6 . 14 is a new groundform, and we have now completed a new (quatertiary) group, that is, the third in order of descent from the primaries, namely, the group 5 . 12, 6 . 13, 7 . 17, 6 . 14.

Here, having reached the middle of this long deduction, it will be expedient to pause for a while and take stock of the relations so far established between the base-forms and their deducts.

I enclose, in what follows, the deg-weight numbers within square brackets, in order to indicate that the forms which they represent are not necessarily identical with the simplified forms represented by the same numbers, but are the immediate quotients which present themselves after dividing out by a or a power of a in the course of the deduction. We have thus

$$a^3 [3 . 6] - a^2 [2 . 2] . [2 . 4] = (2 . 2)^3,\ (3 . 3)^2 \qquad (3)$$

$$a\ [4 . 7] = [2 . 2]\ [3 . 5],\ [2 . 4]\ [3 . 3] \qquad (1)$$

$$a^2 [4 . 8] + a\,(?) = [3 . 3]\ [3 . 5],\ [2 . 2]^2 [2 . 4] \qquad (2)$$

$$a^2 [4 . 10] + a\,(?) = [3 . 5]^2,\ [2 . 4]^2 [2 . 2] \qquad (2)$$

$$a\ [5 . 9] = [4 . 7]\ [2 . 2],\ [3 . 3]\ [3 . 6] \qquad (4)$$

$$a\ [5 . 11] = [4 . 7]\ [2 . 4],\ [3 . 5]\ [3 . 6] \qquad (4)$$

$$a^2 [5 . 12] + a\,(?) = [3 . 6]\ [2 . 4]\ [2 . 2],\ [4 . 7]\ [3 . 5] \qquad (5)$$

$$a\ [7 . 17] = [5 . 12]\ [3 . 5],\ [3 . 6]\ [5 . 11],\ [4 . 10]\ [4 . 7] \qquad (6)$$

$$a\ [6 . 13] = [4 . 10]\ [3 . 3],\ [2 . 2]\ [5 . 11],\ [2 . 4]\ [5 . 9] \qquad (5)$$

$$a^2 [6 . 14] = [5 . 9]\ [3 . 5],\ [5 . 11]\ [3 . 3],\ [2 . 2]^2 [4 . 10] \qquad (6)$$

In the above table the quantities connected by one or more commas represent a linear function of themselves, and the sign of interrogation means

* For Q^2, b^2R, T are each of them residues.

"some known rational integral function of the base-forms." The numerals to the right (beginning with (3) and ending with (6)) indicate the power of (a) by which each corresponding deduct has to be multiplied in order to become an integral function of the base-forms, and which may be called its integrating factor. Thus for example the integrating factor of [5 . 9] is a^4, because the integrating factors of the two arguments in the linear function expressing a [5 . 9] are a, a^3 respectively; so again a^5 is the integrating factor of [5 . 12], because the integrating factors of the arguments of the linear function which expresses $a^2[5 . 12] + a(?)$ are a^3, a respectively. So again the arguments corresponding to a [7 . 17] having the integrating factors a^5, a^4, a^3 respectively, the integrating factor of [7 . 17] will be 1 + 5 (the dominant of the numbers 3, 4, 5), that is, 6. This will be sufficient to show how the integrating factors are to be successively obtained, it being of course borne in mind that the integrating factor of a product of deducts is the product of the integrating factors of the deducts taken separately. With the aid of this table we may see *à priori* that the linear forms representing [7 . 17], [6 . 13], [6 . 14] cannot be identically *nulls*. In the preceding cases no proof is required because we know subinvariants can only be decomposed in one way into factors.

Thus, firstly, for [7 . 17], the integrating factors of the three arguments being a^5, a^4, a^3; for if a syzygy existed between them we should have $B_1 + aB_2 + a^2B_3 = 0$, where each B is a rational integer function of the base-forms not containing a as a factor.

Secondly, for [6 . 13], the separate integrating factors being a^2, a^4, a^4 respectively, did a syzygy exist, we should have $a^2B + B_1 + B_2 = 0$, and consequently [2 . 2] [5 . 11] would be in syzygy with [2 . 4] [5 . 9], which is impossible.

Thirdly, for [6 . 14], the separate integrating factors being a^4, a^4, a^2, the syzygy is impossible, for the same reason as in the preceding case.

I pass on now to the fifth group, that is, to the deducts four degrees of succession removed from the base-forms.

$\mathfrak{R}\, 2 . 2\, \mathfrak{R}\, 6 . 13 = b^2 . bRT$, $\mathfrak{R}\, 3 . 6\, \mathfrak{R}\, 5 . 9 = T . b^3R$. Hence there is a deduct [7 . 15]. Its integrating factor will be a into the dominant of the integrating factors of 6 . 13, 5 . 9, which are a^4, a^5, that is, it will be a^6. Also in regard to the duads 1 . 1, 2 . 4, 2 . 6, 3 . 9, 3 . 6, the compositions of 7 . 15 are

$$(1 . 1)^3 + (2 . 6)^2,\ (1 . 1)^2 + (2 . 4) + (3 . 9),\ (1 . 1) + (2 . 4)^2 + (2 . 6),$$

or b^3R^2, b^2QS, bQ^2R, and the two latter being residues we may write $\mathfrak{R}\, 7 . 15 = b^3R^2$. Its integrating factor is a into the dominant of the integrating factors of 6 . 13, 5 . 9 (which are a^5, a^4), and is therefore a^6; 7 . 15 is necessarily a groundform, for b^3R^2 is obviously indecomposable into simpler residues.

Again $\mathfrak{R}\, 3 . 6\, \mathfrak{R}\, 6 . 13 = T . bRT$, and $\mathfrak{R}\, 5 . 12\, \mathfrak{R}\, 4 . 7 = RT . bT$. Hence 8.19 is a deduct, and its decompositions in respect to the customary duads being

3rd and 6th products have a common factor 3.6; hence the three cannot be syzygetically connected, and consequently 12.30 is a *bona-fide* existing deduct, and being incapable of further depression, is necessarily a ground-form.

The index of integration will be a unit greater than the dominant of the indices last found, that is, it is 14.

Its residue will be found to be of the form

$$(b^3S^3,\ b^3R^3S,\ b^2QR^4,\ b^2QRS^2,\ bQ^2R^2S,\ Q^3R^3,\ Q^3S^2).$$

Again, fourthly,

$$\begin{aligned}
\mathfrak{R}\,6.13\ \mathfrak{R}\,8.19 &= bRT.(QST,\ bR^2T)\\
\mathfrak{R}^2 5.12\ \mathfrak{R}\,4.\ \ 8 &= R^2T^2.b^2R\\
\mathfrak{R}^2 5.12\ \mathfrak{R}^2 2.\ \ 4 &= R^2T^2.Q^2\\
\mathfrak{R}^2 3.\ \ 6\ \mathfrak{R}^2 4.10 &= T^2.(bS,\ QR)^2\\
\mathfrak{R}2.4\ \mathfrak{R}3.6\ \mathfrak{R}5.12\ \mathfrak{R}4.10 &= QT.RT.(bS,\ QR).
\end{aligned}$$

In these five equations the arguments on the left-hand side are four in number, namely, $b^2R^3T^2$, $b^2S^2T^2$, $bQRST^2$, $Q^2R^2T^2$. Accordingly, a linear combination of the five quantities on the right-hand side will be zero, and there is a deduct 13.32 which cannot be further depressed (since 12.32 is necessarily non-existent), and may be easily seen to be an actual quantity and not a null, inasmuch as the indices of integration of the products of which the quantities to the left are the residues (the anti-residues as they may be termed), are

$$5+9,\ 5+5+2,\ 5+5,\ 3+3+2,\ 3+5+2,\ \text{that is, } 14,\ 12,\ 10,\ 8,\ 10,$$

of which only *a pair are equal*. Its index of integration is one unit more than the dominant of these numbers, that is, is 15.

Finally $\mathfrak{R}13.32 = (b^2RT,\ b^2RS^2T,\ bQR^2ST,\ Q^2R^3T,\ Q^2S^2T)$. The four last deducts 11.27, 9.21, 12.30, 13.32 form the batch fifth in descent from the primaries, and their indices of integration have been shown to be 12, 10, 14, 15.

We are now within sight of the goal of our wearisome pilgrimage. We may form eight equations leading to 18.45, the skew-invariant, as follows:

$$\mathfrak{R}4.10\ \mathfrak{R}7.17\ \mathfrak{R}3.6\ \mathfrak{R}5.12 = (bS,\ QR)(Q^2S,\ b^2RS,\ bQR^2).T.R.T \quad (1)$$

$$\mathfrak{R}^2 4.10\ \mathfrak{R}3.6\ \mathfrak{R}8.19 = (bS,\ QR)^2.T.(QST,\ bR^2T) \quad (2)$$

$$\mathfrak{R}^2 4.10\ \mathfrak{R}6.13\ \mathfrak{R}5.12 = (bS,\ QR)^2.bRT.RT \quad (3)$$

$$\mathfrak{R}8.20\ \mathfrak{R}3.6\ \mathfrak{R}8.19 = (b^2S^2,\ b^2R^3,\ bQRS,\ Q^2R^2)\,T(QST.bR^2T) \quad (4)$$

$$\mathfrak{R}8.20\ \mathfrak{R}6.13\ \mathfrak{R}5.12 = (b^2S^2,\ b^2R^3,\ bQRS,\ Q^2R^2)bRT.RT \quad (5)$$

$$\mathfrak{R}11.27\ \mathfrak{R}3.6\ \mathfrak{R}5.12 = (b^3R^4,\ bQ^2R^3,\ bQ^2S^2,\ b^2QSR^2)\,T.RT \quad (6)$$

$$\mathfrak{R}6.13\ \mathfrak{R}13.32 = bRT(b^2R^4T,\ b^2RS^2T,\ bQR^2ST,\ Q^2R^3T,\ Q^2S^2T) \quad (7)$$

$$\mathfrak{R}9.21\ \mathfrak{R}^2 5.12 = (b^3S^2,\ b^3R^3,\ b^2QRS,\ Q^3S)R^2T^2. \quad (8)$$

Hence $\Re[12.27]=(b^3S^2T,\ b^2QRST,\ Q^3ST,\ b^3R^3T)$. But b^3R^2, b^2QS, RT have all been seen to be residues, hence b^3R^3T, b^2QRST are residues.

Also $(\Re 4.10)^2=(b^2S^2,\ bQRS,\ Q^2R^2)$ is a residue, as is also bT. Hence $(b^3S^2T, b^2QS.RT, bQ^2R.RT)$ is a residue, and $bQ(bS, QR)$, $Q(b^2S, bQR)$ being each of them residues, b^2QS, bQ^2R are each of them separately residues. Hence b^3S^2T is a residue. Also $Q^2\ \Re 8.2=(Q^3ST,\ bQ^2R^2T)$ is a residue, and bQ^2R^2T is a residue, because bQ^2R, RT are residues. Hence Q^3ST is a residue. Hence all the arguments in expression for $\Re[12.27]$, namely, b^3R^3T, b^2QRST, $b^3S^2T^2$, Q^3ST are residues; consequently a deduct 12.27 may be found such that $\Re 12.27=0$, and there will be a deduct 11.27 which cannot be still further reducible, because 10.27 is necessarily non-existent. Its index of integration will be two greater than the dominant of those of $(5.12)(6.13)$ and 7.15, which are $5+5$ and 6, that is, it is 12. Its residue $\Re 11.27$ will easily be seen to be

$$(b^3R^4,\ b^3RS^2,\ bQ^2R^3,\ bQ^2S^2,\ b^2QSR^2,\ Q^3RS).$$

Again, secondly, $\Re 5.9\ \Re 5.12=RT.b^3R$,

$$\Re 3.6\ \Re 7.15=T.b^3R^2.$$

Hence there is a deduct 9.21 which cannot be further depressed, because 8.21 is necessarily non-existent, and it will readily be found that

$$\Re 9.21=(b^3S^2,\ b^3R^3,\ b^2QRS.Q^3S),$$

and that the index of integration is $1+4+5$, that is, is 10.

Again, thirdly,

$$\begin{aligned}
&\Re 6.13\ \Re 7.17=bRT(Q^2S,\ b^2RS,\ bQR^2)\\
&\Re 5.11\ \Re 8.19=(b^2S,\ bQR)(QST,\ bR^2T)\\
&\Re 3.6\ \Re^2 5.12=T.(RT)^2=R^2T(Q^3,\ b^2QR,\ b^3S)\\
&\Re 5.12\ \Re 4.8\ \Re 4.10=RT.b^2R(bS,\ QR)\\
&\Re 2.4\ \Re 3.6\ \Re^2 4.10=Q.T.(bS,\ QR)^2\\
&\Re 2.4\ \Re 3.6\ \Re 8.20=Q.T(b^2S^2,\ bQRS,\ Q^2RQ^2,\ b^2R^3).
\end{aligned}$$

Hence it will be seen that the arguments on the right-hand side of the equation are the five following, namely, bQ^2RST, b^3R^2ST, b^2QR^3T, b^2QS^2T, Q^3R^2T, and no others. Hence the six products on the left may be linearly combined so as to give a result zero, and there will consequently be a deduct 12.30.

To prove that this is not a null, take the integrating factors of

$(6.13)(7.17)$, $(5.11)(8.19)$, $(3.6)(5.12)^2$, $(5.12)(4.8)(4.10)$,

$(2.4)(3.6)(4.10)^2$, $(2.4)(3.6)(8.20)$.

These will be found to be

$5+6$, $4+9$, $3+5+5$, $5+2+2$, $3+2+2$, $3+10$, or $11, 13, 13, 9, 7, 13$.

Hence if there were any syzygy between these products it must be between the 2nd, 3rd and 6th, which have a common integrating factor a^{13}, but the

(5) $(a)(6;4)=(4;0)(3;9),\ (2;6)(5;3),\ (2;2)(5;7)$

(8) $a(7;1)=(3;5)(5;1),\ (3;3)(5;3),\ (4;6)(4;0)$

(6) $a^2(6;2)+a(?)=(5;7)(3;5),\ (3;9)(5;3),\ (2.6)^2(4.0)$

(8) $a(7;5)=(2;6)(6;4),\ (3;3)(5;7)$

(9) $a(8;2)=(3;3)(6;4),\ (5;1)(4;6)$

(10) $a^2(8;0)+a(?)=(4;6)(6;4),\ (5;1)(3;3)(2;6)$

(12) $a^2(11;1)+a(?)=(2;6)(5;1)(6;4),\ (3;3)^2(7;5)$

(10) $a(9;3)=(3;3)(7;5),\ (5;7)(5;1)$

(14) $a(12;0)=(6;4)(7;1),\ \ (5;3)(8;2),\ \ (3;3)(5;1)^2(5;1)(4.4)(4.0),$
$(2;2)(3;3)(8;0)$

(15) $a(13;1)=(6;4)(8;2),\ (5;1)^2(6;4),\ (5;1)^2(2;2)^2,\ (3;3)^2(4;0)^2,$
$(2;2)(3;3)(5;1)(4;0)$

(23) $18;0=(4;0)(7;1)(3;3)(5;1),\ (4;0)(3;3)(8;2),$
$(4;0)^2(6;4)(5;1),\ (8;0)(6;4)(5;1),\ (8;0)(3;3)(8;2),$
$(6;4)(13;1),\ (9;3),\ (5;1)^2$

In addition to the deducts which appear in the above table, the groundform 1 . 5 and the four protomorphs 2 ; 2 2 ; 6 3 ; 5 3 ; 9 have to be taken into account. Thus the twenty-three groundforms to the quintic will be seen to be distributed among seven batches or categories containing respectively 1, 4, 3, 4, 3, 3, 4, 1 individuals.

It was my intention to have simplified some of the steps of the deduction, and to have supplied the omissions, to show in one or two cases that the deducts as obtained are actual and not null forms*, but unfortunately the proof-sheets have been kept back, owing to the necessities of the printing-office, for some weeks, and in the meanwhile my attention has been drawn off to other parts of the subject, and I am unable to give sufficient time to call back to mind the intended ameliorations or rectifications of the text.

§ 4. Perpetuants.

On Absolutely Irreducible Binary Subinvariants.

Any rational integral value of $(\lambda a\delta_b+\mu b\delta_c+\nu c\delta_d\ldots)^{-1}0$ is a binary subinvariant. If none of the numerical coefficients $\lambda, \mu, \nu\ldots$ are zero, the subinvariant is simple. If in the series of coefficients $\lambda, \mu, \nu, \pi, \rho\ldots$, any number i of breaks occur in consequence of i non-contiguous terms $\nu, \rho\ldots$ vanishing, it becomes a multiple subinvariant corresponding to a semi-invariant

* When the deduct is a zero instead of a possible new groundform, it indicates a syzygy between anterior groundforms.

The arguments on the right-hand side of these equations will be seen to be the seven following: $T^2b^3R^5$, $T^2b^3R^2S^2$, $T^2b^2QR^3S$, $T^2b^2QS^3$, $T^2bQ^2R^4$, $T^2bQ^2RS^2$, $T^2Q^3R^2S$. Hence a linear function of the anti-residues to the eight products to the left can be made zero, and the sums of each set of duads being $19 . 45$, there emerges the deduct $18 . 45$ corresponding to the skew-invariant $18 ; 0$.

That this is not a null may be shown in the usual manner as follows: The indices of integration of the several anti-residues are

$$2+6+3+5,\ 2+2+3+9,\ 2+2+5+5,\ 10+3+9,\ 10+5+5,\ 12+3+5,$$
$$5+15,\ 10+5, \text{ that is, } 16,\ 16,\ 14,\ 22,\ 20,\ 20,\ 20,\ 15.$$

The 5th, 6th and 7th indices constitute the only triad of equal indices, but the 5th, 6th and 7th anti-residues cannot be in syzygy, inasmuch as the two first of them have the factor $5 . 12$ in common. Hence the value of $18 . 45$ found as above will not be null.

Its index of integration will be one unit more than the dominant of the above numbers, that is, it is 23, and its residue will be of the form

$$(b^3R^6T,\ b^3R^3S^2T,\ b^3S^4T,\ b^2QR^4ST,\ b^2QRS^3T,\ bQ^2R^5T,\ bQ^2R^2S^2T,$$
$$Q^3R^3ST,\ Q^3S^3T).$$

We ought now to be able to show that there exists no other deduct of which the residue is not a rational integral function of the 22 residues which have been determined in order to prove that the system of groundforms obtained is complete. But this inquiry is one of considerable difficulty, and must be reserved for future consideration.

I will now bring together the several steps of the deduction (several of which, especially in the earlier stages, would admit of abridgement), separating the successive strata from one another and substituting the more familiar designation of deg-orders for the equivalent deg-weights. The single numbers on the left-hand side are the indices of integration to the corresponding deducts.

TABLE OF DEDUCTION FOR THE QUINTIC.

(3)	$a^3(3;3)+a^2(?)=(2;6)^3,\ (3;9)^2$
(1)	$a(4;6)=(2;6)(3;5),\ (2;2)(3;9)$
(2)	$a^2(4;4)+a(?)=(3;9)(3;5);\ (2.6)^2(2.2)$
(2)	$a^2(4;0)+a(?)=(3;5)^2,\ (2;2)^2(2;6)$
(4)	$a(5;3)=(4;6)(2.2),\ (3;5)(3;3)$
(5)	$a^2(5;1)+a(?)=(3;3)(2;2)(2;6),\ (4.6)(3.5)$
(4)	$a(5;7)=(4;6)(2;6),\ (3;9)(3;3)$

(5) $(a)(6;4)=(4;0)(3;9),\ (2;6)(5;3),\ (2;2)(5;7)$

(8) $a(7;1)=(3;5)(5;1),\ (3;3)(5;3),\ (4;6)(4;0)$

(6) $a^2(6;2)+a(?)=(5;7)(3;5),\ (3;9)(5;3),\ (2.6)^2(4.0)$

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(10) $a^2(8;0)+a(?)=(4;6)(6;4),\ (5;1)(3;3)(2;6)$

(12) $a^2(11;1)+a(?)=(2;6)(5;1)(6;4),\ (3;3)^2(7;5)$

(10) $a(9;3)=(3;3)(7;5),\ (5;7)(5;1)$

(14) $a(12;0)=(6;4)(7;1),\quad (5;3)(8;2),\quad (3;3)(5;1)^2(5;1)(4.4)(4.0),$
$(2;2)(3;3)(8;0)$

(15) $a(13;1)=(6;4)(8;2),\ (5;1)^2(6;4),\ (5;1)^2(2;2)^2,\ (3;3)^2(4;0)^2,$
$(2;2)(3;3)(5;1)(4;0)$

(23) $18;0=(4;0)(7;1)(3;3)(5;1),\ (4;0)(3;3)(8;2),$
$(4;0)^2(6;4)(5;1),\ (8;0)(6;4)(5;1),\ (8;0)(3;3)(8;2),$
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* When the deduct is a zero instead of a possible new groundform, it indicates a syzygy between anterior groundforms.

of i distinct binary quantics. If, however, the subinvariant is to appertain to a system of quantics, all of unlimited order, it would be necessary for the breaks in the series to be each of them at an infinite distance from the initial term and from one another.

In what follows I shall confine my attention to simple binary subinvariants, and investigate the types, that is, the deg-weights (order ceases to be predicable) of those of them which are absolutely indecomposable, that is, incapable of being expressed as rational integral functions of others of lower types of any extent whatever.

It may be convenient to give a name to absolutely indecomposable subinvariants, and I propose, until an apter word presents itself, to call them perpetuants*. The present section then will be occupied with the successive determination of the types of all possible simple binary perpetuants up to a certain limit of degree.

We know, by Cayley's rule, that the number of linearly independent binariants of degree j and weight w is the difference between the number of partitions of w into j parts, and the number of partitions of $w-1$ into such parts, and therefore by Euler's law of reciprocity is the difference between the number of partitions of w into parts none exceeding j, and the number of partitions of $w-1$ into such parts; it is therefore the coefficient of x^w in

$$\left\{\frac{1}{(1-x)(1-x^2)\dots(1-x^j)}-\frac{x}{(1-x)(1-x^2)\dots(1-x^j)}\right\}$$

or the coefficient of x^w in $\dfrac{1}{(1-x^2)(1-x^3)\dots(1-x^j)}$, which I shall call the generating function for the degree j of the linearly independent subinvariants.

Thus for the degree 1 the generating function is simply 1, and there will be one subinvariant (a) of the degree 1 and weight zero.

For the degree 2 the generating function is $\dfrac{1}{1-x^2}$, which expanded gives the series $1+x^2+x^4+\dots$; there is consequently one semi-invariant of the degree 2 for every even weight 0, 2, 4, 6 ...; but the first of these will be merely the square of the one of degree 0 and weight 1; hence the generating function for the perpetuants of degree 2 is $\dfrac{1}{1-x^2}-1$ or $\dfrac{x^2}{1-x^2}$ giving rise to the deg-weights $2.2\ \ 2.4\ \ 2.6\dots$ corresponding to the well-known series of quadrinvariants or quadri-semi-invariants $ac-b^2$, $ac-4bd+3c^2$, ... Again,

* Perhaps *Revenants* would be more expressive to signify the forms (or ghosts of forms, if one pleases to say so) which never die out, but *continually return* as the leading coefficients of irreducible covariants. Such I need not say is not the case with conditionally irreducible integrals of the above partial differential equation (as for instance the discriminants to the cubic), which sooner or later die out and are seen no more as sources of irreducible covariants to quantics of a superior order.

for $j = 3$ the generating function to the linearly independent binariants, or for brevity sake say the *total* generating function is $\dfrac{1}{(1-x^2)(1-x^3)}$.

To find the irreducible forms, or say the *limited* generating function, we must take away the cube of the one of degree 1 and weight zero, and the product of this one and each indecomposable one of the degree 2, and consequently the limited generating function will be

$$\frac{1}{(1-x^2)(1-x^3)} - \left(\frac{x^2}{1-x^2} + 1\right) \text{ that is } \frac{x^3}{(1-x^2)(1-x^3)};$$

thus we obtain perpetuants of the deg-weights $3 \, . \, i$, where the least value of i is 3 and the number of such for $i = 3, 4, 5, 6, 7, 8$; $9, 10, 11, 12, 13, 14$; $15, 16, 17, \ldots$ will be 1, 0, 1, 1, 1, 1; 2 1 2 2 2 2; 3, 2, 3, ...

Again, for $j = 4$, the total generating function is $\dfrac{1}{(1-x^2)(1-x^3)(1-x^4)}$.

To determine the subtrahend consider the total partitions of 4 (the number itself not counting as a partition). These are 1^4, $1^2 . 2$, $1 . 3$, 2^2. The three former will give rise to the partial subtrahends $1, \dfrac{x^2}{1-x^2}, \dfrac{x^3}{(1-x^2)(1-x^3)}$, but for 2^2, that is, $2 . 2$ the case is different.

Taking the development of $\dfrac{x^2}{1-x^2}$, that is, $x^2 + x^4 + x^6 + x^8 + \ldots$ the function corresponding to $2 . 2$ to be subtracted is not $\left(\dfrac{x^2}{1-x^2}\right)^2$, but the sum of the *homogeneous products* of the second order of the infinite succession $x^2, x^4, x^6, x^8, \ldots$, or calling s_1 the sum of the terms and s_2 the sum of their squares, is $\dfrac{s_1{}^2 + s_2}{2}$, that is, is

$$\frac{1}{2}\left\{\left(\frac{x^2}{1-x^2}\right)^2 + \frac{x^4}{1-x^4}\right\} \text{ or } \frac{x^4(1+x^2) + x^4(1-x^2)}{(1-x^2)(1-x^4)},$$

that is,

$$\frac{x^4}{(1-x^2)(1-x^4)}.$$

Hence the *limited* generating function for the degree 4 is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)} - \left(\frac{x^3}{(1-x^2)(1-x^3)} + \frac{x^2}{1-x^2} + 1\right) - \frac{x^4}{(1-x^2)(1-x^4)},$$

which is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}\{1 - (1-x^4) - x^4(1-x^3)\},$$

that is

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Let us pause a moment in the deduction to draw an inference from this result. The lowest power of x in the development of the limited generating function for the degree 4 being x^7, we see that an absolutely indecomposable binariant of the 4th degree cannot be of lower weight than 7. Consider any semi-invariant of degree 4 to a quantic of order i. Its weight must be less than $2i$. Hence if it is indecomposable, 7 must be less than $2i$, or i must be at least 4. Thus we see that there can be no absolutely indecomposable binariant of the 4th degree appertaining to a cubic. This shows *à priori* that the discriminant to the cubic, regarded as a subinvariant, is decomposable, as we know is the case*.

So in general if we know that no perpetuant of the degree j is of lower weight than k, we may be assured that no invariant or semi-invariant to a quantic of the degree j can be absolutely indecomposable if the order of the quantic is less than $\frac{2k}{j}$.

Agreeing to call the weight of any subinvariant divided by its degree its relative weight, we may put this result into words, by saying no quantic can possess an absolutely indecomposable invariant or semi-invariant of a given degree unless its order is at least twice as great as the minimum relative weight of a perpetuant of that degree. We may see further that the quartic can have no indecomposable invariant or semi-invariant of the degree 4, for its weight would be 8, but x^8 does not appear in the development of

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Pass we on now to the case of the 5th degree.

The indefinite partitions of 5 (leaving 5 itself out of the number) are 4.1, 3.2, 3.1.1, 2.2.1, 2.1³, 1⁵ which obviously give rise to the subtrahends

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)},\quad \frac{x^3}{(1-x^2)(1-x^3)}.\frac{x^2}{1-x^2},\quad \frac{x^3}{(1-x^2)(1-x^3)},$$
$$\frac{x^4}{(1-x^2)(1-x^4)},\quad \frac{x^2}{1-x^2},\ 1.$$

But from the mode in which the deduction has been carried on, it will be obvious on reflection that the sum of all these except the second which corresponds to a partition not ending with a unit will be equal to the total generating function for the case of the degree 4. So that the total subtrahend is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}+\frac{x^5}{(1-x^2)(1-x^2)(1-x^3)}.$$

Hence the limited generating function for the degree 5 is

$$\frac{x^5}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}-\frac{x^5}{(1-x^2)(1-x^2)(1-x^3)},$$

that is, is $\dfrac{x^5\{1-(1+x^2)(1-x^5)\}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$, which is $\dfrac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$, where for brevity I use in general (q) to denote $1-x^q$.

* It may easily be collected from the course of the ensuing investigation that *every* binary discriminant is decomposable into subinvariants of lower degrees than its own.

Here, for the first time, a new feature presents itself, namely, the presence of a negative coefficient in the numerator, and consequently of a series of such in the development in an infinite series of the generating function.

Each negative term $-kx^t$ in the development will obviously indicate the existence of k general syzygies of the degree 5 and weight t, or as we might call them, *privative* groundforms. The number of such terms will be finite, and they will be most readily obtained by writing the *l. g. f.* (limited generating function) under the form

$$\frac{-x^7(1-x^3)(1-x^5)+x^{15}}{(2)(3)(4)(5)}, \text{ that is } \frac{-x^7}{(2)(4)}+\frac{x^{15}}{(2)(3)(4)(5)}.$$

To find them it will be observed that the number of ways of composing 0, 2, 4, 6, 8, 10, 12, 14, 16 with the elements 2 and 4 are respectively 1, 1, 2, 2, 3, 3, 4, 4, 5, and that 1, 1, 2, 3, 5 are the number of ways of composing 0, 2, 4, 6, 8, with the elements 2, 3, 4, 5. Hence there will exist the negative terms

$$-x^7,\ -x^9,\ -2x^{11},\ -2x^{13},\ -2x^{15},\ -2x^{17},\ -2x^{19},\ -x^{21}\,*,$$

the sum of which is

$$-\frac{x^7+x^{11}-x^{21}-x^{23}}{1-x^2}.$$

Adding this with its sign changed to $\dfrac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$ there results

$$\frac{x^{18}+x^{20}-x^{21}-x^{23}+x^{24}+x^{25}+2x^{26}-x^{29}-2x^{30}-x^{31}-x^{32}+x^{33}+x^{35}}{(2)(3)(4)(5)},$$

which may be thrown under the form

$$x^{18}\left\{\frac{(3)+x^2(2)(3)+x^4+x^6(8)+x^8(3)(4)+x^8(4)(5)}{(2)(3)(4)(5)}\right\}.$$

It is therefore omni-positive in its development, which shows that no negative terms have been omitted, but that the 13 syzygies of odd weights ranging from 7 to 21 typically represented by $-\dfrac{x^7+x^{11}-x^{21}-x^{23}}{1-x^2}$ (say $-R_5$) constitute their entire aggregate. We see also that the minimum weight of a perpetuant of the 5th degree is 18, so that the double of the minimum relative weight is $\dfrac{36}{5}$, and accordingly there can exist no absolutely indecomposable binary subinvariants of the 5th degree, until we come to Quantics of the 8th order or upwards.

Proceeding to the degree 6, the total subtrahend from the *t. g. f.* (total generating function) for that degree would be *ut suprâ* the *t. g. f.* for the

* The numbers 1 1 2 2 2 . 2 1 are got by subtracting from the figures 1 1 2 2 3 3 4 4 5 the figures 1 1 2 3 5

degree one below (here 5), less expressions depending on the partitions of 6 not concluding with a unit, were it not for the presence of the negative terms represented by $-R_5$; the quantity to be subtracted corresponding to the partition 5.1, being now not the *l. g. f.* for degree 5, $\frac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$, but this quantity rendered omni-positive in its development by the addition of R_5.

Hence the total subtrahend will be $\frac{1}{(2)(3)(4)(5)}+R_5+$ the quantities depending on the partitions 2.4 2.2.2 3.3.

To 2.4 will correspond the subtrahend $\frac{x^2}{(2)}\cdot\frac{x^7}{(2)(3)(4)}$.

To 3.3 will correspond $\frac{\phi x^2+(\phi x)^2}{2}$ where $\phi x=\frac{x^3}{(1-x^2)(1-x^3)}$, and to 2.2.2 by Crocchi's theorem*, will correspond the representative of the homogeneous products of the 3rd order of the terms in

$$\psi x=\frac{x^2}{1-x^2}, \text{ that is, } \frac{(\psi x)^3+3\psi x\psi x^2+2\phi x^3}{2.3}.$$

There might for a moment be felt a hesitation in applying the formula for homogeneous products to ϕx, in consequence of the coefficients in its development being no longer exclusively unities; but the force of this objection vanishes as soon as it is borne in mind that we may replace any term kx^t in the development of ϕx by k separate terms x^t, each of which corresponds to a distinct subinvariant.

Thus then to 3.3 will correspond the partial subtrahend

$$\frac{x^6}{2}\left\{\frac{1}{(1-x^2)^2(1-x^3)^2}+\frac{1}{(1-x^4)(1-x^6)}\right\} \text{ or } x^6\frac{(1+x^2)(1+x^3)+(1-x^2)(1-x^3)}{2(2)(3)(4)(6)},$$

that is,

$$\frac{x^6+x^{11}}{(2)(3)(4)(6)},$$

and to 2.2.2 will correspond

$$x^6\frac{(1+x^2)(1+x^2+x^4)+3(1-x^6)+2(1-x^2)(1-x^4)}{6(2)(4)(6)}, \text{ or } \frac{x^6}{(2)(4)(6)}.$$

It may be remarked, in passing, that for any degree $2i$ the subtrahend corresponding to the partition consisting of i parts (each of the value 2), is $\frac{x^{2i}}{(2)(4)\dots(2i)}$, as may be shown, *à priori*, thus: using y in place of x^2 we have to find the sum of all the quantities ky^t where k is the number of ways of generating y^t as a product of i of the powers $1, y, y^2, y^3 \dots$, that is, k is the number of ways of composing t with i or less than i of the indefinite

* See for an instantaneous proof of this theorem, the *Johns Hopkins University Circular* for November 1882 [below, p. 653].

series of natural numbers, which by Euler's theorem, already cited, is the same as that of compounding t out of any number of parts none exceeding i. Hence the denominator of the subtrahend required will be

$$\frac{1}{(1-y)(1-y^2)\dots(1-y^i)}, \text{ that is, } \frac{1}{(2)(4)\dots(2i)}.$$

The numerator is obviously x^{2i}, and the complete value $\frac{x^{2i}}{(2)(4)\dots(2i)}$ as was to be found.

I may add, that this theorem (which is one concerning homogeneous product-sums expressed as functions of power-sums of the same elements), by an easy deduction from Crocchi's theorem, serves to show if the ith power-sum of a set of elements is $\frac{1}{1-c^i}$ (I substitute c for y) then the ith elementary symmetric function of the elements is

$$\frac{c^{\frac{i^2-i}{2}}}{(1-c)(1-c^2)\dots(1-c^i)}$$

and reversing the terms of this proposition we may say, that if

$$z^q - \frac{1}{1-c}z^{q-1} + \frac{c}{(1-c)(1-c^2)}z^{q-2}\dots \pm \frac{c^{\frac{n^2-n}{2}}}{(1-c)(1-c^2)\dots(1-c^n)}z^{q-12} + \dots = 0,$$

then the sum of the ith powers of z (q being not less than i) is $\frac{1}{1-c^i}$, to which may be added that the sum of the ith homogeneous products of z is

$$\frac{1}{(1-c)(1-c^2)\dots(1-c^i)},$$

as, for example, if $i=2$ the first of these sums, namely,

$$\frac{1}{(1-c)^2} - 2\frac{c}{(1-c)(1-c^2)} = \frac{1}{1-c^2}$$

and the other, namely,

$$\frac{1}{(1-c)^2} - \frac{c}{(1-c)(1-c^2)} = \frac{1}{(1-c)(1-c^2)}.$$

But this is a mere digression, a wild flower gathered on the wayside. Returning to the determination of the *l. g. f.** for the degree 6, we see that it will be

$$\frac{1}{(2)(3)(4)(5)(6)} - \frac{1}{(2)(3)(4)(5)} - \frac{x^9}{(2)(2)(3)(4)} - \frac{x^6+x^{11}}{(2)(3)(4)(6)} - \frac{x^6}{(2)(4)(6)} - R_5,$$

or $\frac{N}{(2)(3)(4)(5)(6)} - R_5$, where

* I repeat that *t. g. f.* stands for total generating function, and *l. g. f.* for limited generating function.

$$\begin{aligned} N &= x^6 - (1+x^2+x^4)(1-x^5)x^9 - (x^6-x^{16}) - x^6(1-x^3)(1-x^5) \\ &= x^6 + x^{14} + x^{16} + x^{18} + x^{16} + x^9 + x^{11} \\ &\quad - x^9 - x^{11} - x^{13} - x^6 - x^6 - x^{14} \\ &= -x^6 - x^{13} + 2x^{16} + x^{18}. \end{aligned}$$

Thus the *l. g. f.* for the degree 6 is

$$-R_5 + \frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{(2)(3)(4)(5)(6)}.$$

$-R_5$ represents the fourteen compound syzygants of the degree 6; the fraction to which $-R_5$ is annexed, when developed, will give rise to only a *limited* number of terms with negative coefficients corresponding to the ground-syzygies; the remainder of the terms, infinite in number, will represent the infinite succession of groundforms. It may be well here to notice, as a universal fact, that in the development of the fraction $\frac{R(x)}{(2)(3)\ldots(n)}$ (where $R(x)$ is rational integral function of x) the number of negative terms or the number of positive terms will be finite according as $R(1)$ is positive or negative, and, as in the above fraction, $R(1)=1$, it follows that there are only a finite number of negative terms, and consequently only a limited number of ground-syzygies, an important conclusion which will easily be seen to apply not only to the use of the degree 5 (in which syzygies first make their appearance) and 6, as here shown, but for all higher degrees, it being a universal law that the irreducible syzygies for subinvariants of any given degree, and therefore of any degree not exceeding a given limit, are finite in number.

The law that the development of $\frac{R(x)}{(1-x^2)(1-x^3)\ldots(1-x^n)}$, commencing from a certain point is omni-positive or omni-negative, according as $\phi 1$ is positive or negative when n exceeds 2, admits of easy proof. Of course the law could not be true when $n=2$, as, for example, for $\frac{1-2x}{1-x^2}$ which remains *neutral*, that is, neither omni-positive nor omni-negative (which latter, if the law did apply, it ought eventually to become) throughout its entire extent.

Beginning with $\frac{Rx}{(1-x^2)(1-x^3)}$ the coefficient of x^i [where $i = 6t + \tau$ $(\tau < 6)$] will be not less than t, and not greater than $t+1$ in the development of

$$\frac{1}{(1-x^2)(1-x^3)}.$$

Hence in the development $\frac{-K + (K+\epsilon)x^\delta}{(1-x^2)(1-x^3)}$ the coefficient of x^i will be not less than $-K(t+1) + (K+\epsilon)\left(t - \frac{\delta}{6} - 1\right)$, and consequently for a sufficiently large value of i must be positive. *A fortiori* the same will be true for

$\dfrac{R(x)}{(1-x^2)(1-x^3)}$ when $K+\epsilon$ is the sum of the positive coefficients in Rx of powers of x none of whose indices are higher than δ, and K the sum of the negative coefficients of any powers of x; this proves the law for $\dfrac{R(x)}{(1-x^2)(1-x^3)}$ when $R(1)$ is supposed to be positive, and moreover the series will be omni-positive after a certain point in the strict sense of the following coefficients being neither negative nor zero.

Hence the law will be true for $\dfrac{Rx}{(1-x^2)(1-x^3)(1-x^4)}$, for we may divide $\dfrac{Rx}{(1-x^2)(1-x^3)}$, when expanded, into four series, whose indices $\equiv 0, 1, 2, 3$ respectively to modulus 4, and the negative terms in each of these being finite in number, it is clear that the effect of dividing any one of them by $1-x^4$ will be to give rise to a series omni-positive after a certain point, because each coefficient in the quotient of any one of the series divided by $\dfrac{1}{1-x^4}$ will at worst contain only the sum of a finite number of given negative coefficients, and a number of terms all greater than zero, whose sum, when that number is taken great enough, must exceed the arithmetical value of the former sum. Hence $\dfrac{R(x)}{(1-x^2)(1-x^3)(1-x^4)}$ will be the sum of four series, each omni-positive from a certain point, and will therefore be omni-positive from the most advanced of those points. In like manner

$$\frac{Rx}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$

may be shown to be the sum of five series, each with an infinite omni-positive branch, and consequently will be itself of the same character, and so in general. Of course the same reasoning would show the truth of the law when $R(x^i)$ is negative, and that it may be extended to any denominator of the form $(1-x^i)(1-x^j)(1-x^k)\ldots$ provided any two of the indices $i, j, k\ldots$ are prime to one another. And of course a similar conclusion obtains (*mutatis mutandis*) when $R(x)$ is negative. The law might be proved more scientifically and more briefly as a consequence of the general algebraical representation of the denumerant of any equation in integers

$$[l_1x_1 + l_2x_2 + \ldots + l_ix_i = n]$$

as a sum of a non-periodical and of periodical parts, whereof the former is always of a higher dimension in n than any of the latter, except when all the l quantities have a common factor. See the annexed Excursus [p. 605, below].

I now proceed to find the lowest power of x in the fraction

$$\frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{(2)(3)(4)(5)(6)},$$

say F, in which the coefficient is positive, in order to ascertain the minimum weight of an absolutely irreducible subinvariant of the 6th degree.

I think the easiest practical mode of proceeding to effect this is to use the table in my possession (having been previously calculated for me by Mr Franklin for another purpose) which gives the coefficients of the powers of x in $\frac{1}{(2)(3)(4)(5)(6)}$; those coefficients used as they stand, then advanced seven steps, then five steps further, then taken back two steps, and at the same time doubled, will give four series of numbers, the sum of the 1st and 2nd of which subtracted from the sum of the 3rd and 4th will give the successive coefficients from x^6 upwards in the development of F.

The four series are as underwritten:

(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9,	14,	13,	19,	20,	26,	27,	36,	36;	47,
							1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9,	14,	13,
												1,	0,	1,	1,	2,	2,	4,	3,	6,
										2,	0,	2,	2,	4,	4,	8,	6,	12,	12,	18,

(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)
49,	60,	63,	78,	80,	97,	102,	120,	126;	149,	154,	180,	189,	216,	227,	260,
19,	20,	26,	27,	36,	36;	47,	49,	60,	63,	78,	80,	97,	102,	120,	126;
6;	9,	9,	14,	13,	19,	20,	26,	27,	36,	36;	47,	49,	60,	63,	78,
18,	28,	26,	38,	40,	52,	54,	72,	72;	94,	98,	120,	126,	156,	160,	194,

(37)	(38)	(39)	(40)	(41)	(42)	(43)	(44)	(45)	(46)	(47)	(48)	(49)	(50)	(51)
270,	307,	322;	361,	378,	424,	441,	492,	515,	568,	594,	656,	682;	750,	783,
149,	154,	180,	189,	216,	227,	260,	270,	307,	322;	361,	378,	414,	441,	492,
80,	97,	102,	120,	126;	149,	154,	180,	189,	216,	227,	260,	270,	307,	322;
204,	240,	252;	298,	308,	360,	378,	432,	454,	520,	540,	614,	644;	722,	756,

(52)	(53)	(54)	(55)	(56)	(57)	(58)	(59)	(60)	(61)	(62)	(63)
854,	891,	972,	1010,	1098,	1144,	1236,	1287;	1391,	1443,	1555,	1617,
515,	568,	594,	656,	682;	750,	783,	854,	891,	972,	1010,	1098,
361,	378,	424,	441,	492,	515,	568,	594,	656,	682;	750,	783,
848,	882,	984,	1030,	1136,	1188,	1312,	1364;	1500,	1566,	1708,	1782,

(64)	(65)	(66)	(67)	(68)	(69)	(70)	(71)	(72)	(73)	(74)
1734,	1802,	1932,	2002,	2142,	2223;	2369,	2457,	2618,	2709,	2881,
1144,	1236,	1287;	1391,	1443,	1555,	1617,	1734,	1802,	1932,	2002,
854,	891,	972,	1010,	1098,	1144,	1236,	1287;	1391,	1443,	1555,
1944,	2020,	2196,	2288,	2472,	2574;	2782,	2886,	3100,	3234,	3468,

(75)	(76)	(77)	(78)	(79)
2985,	3164,	3276,	3472,	3588;
2142,	2223;	2369,	2457,	2618,
1617,	1734,	1802,	1932,	2002,
3604,	3864,	4004,	4284,	4446;

The first group of four numbers in which the 3rd and 4th terms combined exceed the 1st and 2nd will easily be seen to be, $\left.\begin{matrix}2369\\1617\\1236\\2782\end{matrix}\right\}$ which is 70 places from the first term, and for which the difference is 4018 less 3986 or 32. Starting from this point the series for F will be seen to be

$$32x^{76}-18x^{77}+81x^{78}+36x^{79}+188x^{80}+94x^{81}+211x^{82}+161x^{83}$$
$$+287x^{84}+242x^{85}+\ldots;$$

so that there can be no practical doubt of the series being omni-positive from and after the 78th power of x*.

The relative weight of any one of the irreducible subinvariants corresponding to $32x^{76}$ is $\frac{76}{6}$, the double of which is $25\frac{1}{3}$. Hence there can be no irreducible semi-invariant of the 6th degree to a quantic below the 26th order, and, on account of the coefficient of x^{77} being negative, we see that a quantic of the 26th order can have no groundforms of the 6th degree in the coefficients except such as are invariants or quart-invariants.

As regards the syzygies irrespective of the compound ones represented by $-R_5$, we see that there will be primitive ones of all weights from 6 to 77 inclusive, with the exception of the weights 7 and 76, but that there will be no syzygies, whether reducible or irreducible, of the same weights as the irreducible subinvariants. Let us now pass on to the case of the 7th degree†.

The partitions of seven itself and those ending in unity excluded are 5.2 4.3 2.2.3.

Hence calling R_6 the sum of the negative terms in $\frac{-x^6-x^{13}+2x^{16}+x^{18}}{(2)(3)(4)(5)(6)}$, the *l. g. f.* for 7 will be

$$\frac{x^7}{(2)(3)(4)(5)(6)(7)}-\frac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}\,\frac{x^2}{(2)}-\frac{x^7}{(2)(3)(4)}\,\frac{x^3}{(2)(3)}$$
$$-\frac{x^4}{(2)(4)}\,\frac{x^3}{(2)(3)}-R_5\frac{x^2}{1-x^2}-R_6.$$

If we call this $\qquad \dfrac{x^7+N}{(2)(3)(4)(5)(6)(7)}-R_5\dfrac{x^2}{1-x^2}-R_6,$

$$N=x^7(1-x^7)\{(1+x^2+x^4)(-x^9+x^{12}+x^{14})+x^{10}(1+x+x^2+x^3+x^4)$$
$$(1-x+x^2)+x^7(1-x^5)(1+x^2+x^4)\}=-(1-x^7)\,P,$$

where $P=\Sigma x^t-\Sigma x^\tau$, t having the values 12 14 16, 14 16 18; 10 11 12 13 14, 12 13 14 15 16; 7 9 11, and τ having the values 9 11 13; 11 12 13 14 15; 12 14 16.

* This conclusion will be strictly proved in the sequel with the aid of my general partition formulæ, in Section V.

[† For the 7th degree, cf. J. Hammond, *American Journal*, Vol. v. (1882), p. 225, under the heading: Disproof of Prof. Sylvester's Fundamental Postulate.]

Hence $$P = x^7 + x^{10} + x^{12} + 2x^{14} + 2x^{16} + x^{18},$$
and $$x^7 + N = -x^{10} - x^{12} - x^{14} - 2x^{16} - x^{18} + x^{17} + x^{19} + 2x^{21} + 2x^{23} + x^{25}.$$

The first term in the development of $\dfrac{x^7+N}{(2)(3)\dots(7)}$ is $-x^{12}$, indicating that the first irreducible syzygy is of the weight 12; it is not until a very high power of x is reached that a positive coefficient corresponding to a perpetuant makes its appearance.

The tables set out in a subsequent section exhibit *inter alia* the coefficients in the developments of $\dfrac{1}{(2)(3)\dots(7)}$ and $\dfrac{1}{(2)(3)\dots(6)}$, say F_7 and F_6 as far as the 174th power of x. Using instead of $\dfrac{x^7+N}{(2)\dots(7)}$ the equivalent value $x^7F_7 - PF_6$, if the coefficient of x^{q+7} in this is positive, the coefficient of x^q in F_7 must be greater than that of x^q in $(1 + x^3 + x^5 + 2x^7 + 2x^9 + x^{11})F_6$, and *à fortiori* greater than that of x^q in $8x^{11}F_6$, that is, greater than 8 times that of x^{q-11} in F_6. But a glance at the tables* for the developments of F_7, F_6 will show that this is never the case within the limits of q, furnished by the tables, that is, for any value of q not exceeding 174. It is certain, therefore, that the value of the lowest index of x^r, for which in $\dfrac{N}{(2)\dots(7)}$ the coefficient is positive, must considerably exceed 181, as indeed one might have anticipated from the series of similar exponents 2, 3, 7, 18, 76 corresponding to the cases previously considered, the ratio of increase in these numbers going on continually increasing†. To ascertain the value of the exponent in question there is left no resource but to endeavour to elicit it (as I shall presently proceed to do) from the general algebraical value of the coefficient. But before doing so it will be well to notice a very important inference that may be drawn from the form of the generating function, namely,

$$\frac{N}{(2)(3)(4)(5)(6)(7)} - \frac{R_5}{(2)} - R_6.$$

$\dfrac{R_5}{(2)}$ or $(1 + x^2 + x^4 + \dots)(x^7 + x^9 + 2x^{11} + 2x^{13} + 2x^{15} + 2x^{17} + 2x^{19} + x^{21})$ will represent the deg-weights of the compound syzygies corresponding to the multiplication of the syzygies of the deg-weights 5.7 5.9 5.11 5.13 5.15 5.17 5.19 5.21 5.23 by the groundforms of every even weight.

There will thus be seen to exist compound syzygies of every odd weight (no less than 13 in fact of weight 21 or any higher odd number). If then ω' be the lowest power of x in $\dfrac{N}{(2)(3)(4)(5)(6)(7)}$ with a positive coefficient and

* *Vide* the numerical tables at end of Section V of this Memoir.

† Subsequent calculations, however, have revealed to me that this ratio does not go on continually increasing.

with an odd exponent, there will coexist groundforms and syzygies of the same degree and weight appertaining to the quantic of an infinite order for every weight denoted by an odd number not less than ω'. From this it is easy to infer that there must exist syzygies and groundforms of the same deg-weight (and therefore of the same deg-order) for one or more quantics of an order not exceeding ω'; [and it may be added that ω' being a high number (not a number less than 23) there will be 13 syzygies of every odd weight equal to or greater than ω'].

For suppose that Q is a quantic of order i. In determining its ground-semi-invariants of the successive degrees the same process may be applied as in calculating the perpetuants, that is, the ground semi-invariants to a quantic of an unlimited order, except that in lieu of the complete development of the generating function $\dfrac{1}{(1-x^2)(1-x^3)\ldots(1-x^j)}$ only such powers of x must be retained as are not higher than x^i. For the number of linearly independent subinvariants of the weight w and degree j will now be the difference between the number of ways of making up w with j parts none greater than i, less the number of ways of so making up $(w-1)$ which will be the difference between the number of ways of making up w and of making up $(w-1)$ with i parts none greater than j, which, if w does not exceed i, will be the same as if i were infinite. So far then as weights not superior in value to i are concerned, the total generating function for a quantic of the order i will be the same as for a quantic of an unlimited order, and consequently up to the weight i (inclusive) the generating functions for the ground subinvariants (to be obtained, be it remembered, by combining the total generating functions in the same manner, whatever the value of i may be) will be the same for a quantic of the ith as for the quantic of an unlimited order. Hence there must of necessity appertain irreducible covariants and compound syzygants of the same degree and order (namely, of the deg-order $7 \,.\, 5\omega'$) to a quantic of the order ω', and of course there is nothing to prevent such coexistence holding good for a quantic of an order very much lower than ω', the least value of which number say i, as far as I am able at present to see, can only be determined by putting each quantic of an order inferior to i successively upon its trial, a work of exceedingly great labour to undertake.

I use ω' to signify the lowest *odd* power of x in the development of the *g.f.* to perpetuants of the 7th degree affected with a positive coefficient, reserving ω to signify the lowest power (whether odd or even) so affected. Until further investigation we cannot say whether ω is equal to or less than ω', but we know that no absolutely irreducible subinvariant of the 7th degree can appertain to a quantic of an order lower than $\dfrac{2\omega}{7}$, a number whose exact value we shall eventually succeed in ascertaining with the aid of a partition formula obtained by the method which will form the subject of the annexed "excursus."

Inasmuch as the theory is precisely the same for fractions in general as for those which correspond to denumerants (the name I give to the number of solutions in integers of one or more linear equations), I shall show how to find the general term in the development of any rational fraction, limiting myself however, for the present, to the theory of rational functions of a single variable, which covers the case with which alone we are here concerned, of denumerants of a single linear equation, or which is the same thing, the problem of exhibiting the number of modes of composing a general number n with given smaller numbers as an algebraico-exponential function of n.

When analysis is sufficiently advanced to admit of a perfectly methodical distribution of its subject-matter, the theorem for the expansion of rational functions, about to be given, will, it seems to me, take its place immediately after Newton's binomial theorem, as the second leading theorem of Algebra; my method of partitions (as stated and applied in Tortolini's *Ann.* Vol. VIII. 1856, and in the *Quarterly Mathematical Journal*, 1855, Vol. I. p. 141, to neither of which I have at present means of access*, but the latter of which is referred to by Prof. Cayley in the *Phil. Trans.* for 1880, footnote p. 47) virtually amounted to an enunciation of the theorem for the case of the reciprocal of a rational integral function all of whose roots are roots of unity, under such a form as almost of necessity to lead to the supposition of its remaining true (*mutatis mutandis*) in the general case; the actual averment of the generalization was, I believe, first made by Prof. Cayley†.

Excursus.

On Rational Fractions and Partitions.

The method of finding the general term in the development of a rational fraction of a single variable in a series of ascending powers of the same may be regarded as a corollary to the following lemma, the proof of which is an instantaneous consequence of the fact that the coefficient of $\frac{1}{x}$, or to use Cauchy's word, the residue of $\frac{1}{(1-e^x)^i}$ developed in ascending powers of x

[* See Vol. II. of this Reprint, p. 90.]

† On second thoughts, and after more deliberate reflection, it occurs to me that I may have overstated in the text above the importance of the general theorem viewed as a theorem *an sich*; and that it is only from its special application to rational fractions whose infinity-roots are all of them roots of unity, that it derives its claim to be regarded as a cardinal theorem in Algebra.

when i is any positive integer is always -1: that this is so will be seen at once from the fact that the effect of changing i into $i+1$ in the above fraction is to increase it by $\frac{e^x}{(1-e^x)^{i+1}}$, that is, by the differential derivative of $\frac{1}{i(1-e^x)^i}$, whose residue is obviously zero, so that the residue of $\frac{1}{(1-e^x)^i}$ will be unaffected by continually decreasing i by a unit until it becomes unity; and obviously therefore the residue in question is always -1.

The lemma may be stated as follows:

The constant term in any proper algebraical fraction developed in ascending powers of its variable is the same as the residue with its sign changed of the sum of the fractions obtained by substituting in the given fraction in lieu of the variable its exponential multiplied in succession by each of its values (zero excepted, if there be such) which makes the given fraction infinite.

Any value of a variable which makes a function infinite may conveniently be called an infinity root, and if it is not zero, a finite-infinity root. So too, a factor whose vanishing makes a function vanish may be termed an infinity factor.

Suppose Fx is a *proper* Algebraical fraction, then we may write

$$Fx = \Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - x)^\lambda} + \Sigma \frac{\gamma_\lambda}{x^\lambda},$$

where $\lambda = 1, 2, \ldots$; $\mu = 1, 2, \ldots j$ and of course any of the coefficients in either sum may be made zero, and then (using in general here and hereafter co_n to signify the coefficient of x^n in an ascending expansion of the function with which it is in regimen) we have

$$\text{co}_{-1}\,\Sigma F(a_\nu e^x) \text{ [where } \nu = 1, 2, \ldots j]$$

$$= \text{co}_{-1}\,\Sigma\Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - a_\nu e^x)^\lambda} + \text{co}_{-1}\Sigma\Sigma \frac{\gamma_\lambda}{a_\nu^\lambda e^{\lambda x}}$$

$$= \text{co}_{-1}\,\Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - a_\mu e^x)^\lambda} = -\frac{c_{\lambda,\mu}}{a_\mu^\lambda} = -\text{co}_0\, Fx$$

which proves the lemma.

Hence the coefficient of x^n in a rational function fx, which is the same as $\text{co}_0 \frac{fx}{x^n}$ will be $-\text{co}_{-1}\Sigma\,(r^{-n} e^{-nx} fre^x)$ or $\text{co}_{-1}\Sigma\,\{r^{-n} e^{nx} f(re^{-x})\}$, [$r$ meaning each finite-infinity root of fx taken in turn], provided only that $\frac{fx}{x^n}$ is a proper algebraical function, that is, provided that n is greater than the degree of $f(x)$.

As for instance, if the degree of the fraction is zero, the theorem will not give the constant, but will give every coefficient of positive powers in the ascending expansion of fx, and if it is negative, the theorem will give all but the coefficients of negative powers.

This theorem, as observed by Prof. Cayley, *Phil. Trans.*, 1856, p. 139, may be obtained "from the known theorem," that if fx be resolved into simple partial fractions, the sum of those which have any power of $a-x$ in their denominator will be the residue of

$$\frac{f(a+\zeta)}{x-a-\zeta}\,*.$$

Prof. Cayley quotes as "a theorem of Cauchy's and Jacobi's, that the coefficient of $\frac{1}{z}$ in $Fz =$ coefficient of $\frac{1}{t}$ in $\psi' tF\psi t$."

This is obviously not true in general, for we might take $Fz = \frac{1}{z}$ and $\psi t = a + t$ or t^2 and the alleged equality would not exist. It is, however, true whenever ψt is of the form $at + bt^2 + \text{etc.}$, as may be proved instantaneously by supposing Fz resolved into partial fractions, and making $z = \psi t$, so that $\int dz\, Fz = \int dt\, \psi' tF\psi t$, and observing that if the expansion of $\psi' tF\psi t$ contains $\frac{k}{t}$, that of $\int dz Fz$ must contain $\frac{k}{z}$, since otherwise when this integral is expressed as a function of t, it would not contain (as it is bound to do) the term $k \log t$. The theorem so limited is sufficient for the purpose in view, since on writing, in place of ζ, $-a(1-e^{-t})$ we see that the residue of $\frac{f(a+\zeta)}{x-a-\zeta}$ is the same as the residue of $\frac{f(ae^{-t})}{(1-ae^{-t}x)}$, and consequently the coefficient of x^n in so far as it depends on the infinity root a, will be the residue of $(a^{-n}e^{nt})f(ae^{-t})$ as has been shown above to be the case. It may, possibly, be thought somewhat surprising that those familiar with the known theorem referred to and the general principle of transformation of residues should not have recognized, previous to the divulgation of my theorem, that the two things put together were competent to give a complete solution of the much ventilated problem of simple denumeration. But, perhaps, even supposing the mental conjunction of the two facts to have taken place, there would still have been needed an act of imagination (such as Kant justly remarks is at the bottom of every advance in geometry, where in reality the proof lies in the construction †) to have led to the choice of the particular transformation

* In his *Cours d'Algèbre*, Edition 1877, Vol. I. pp. 497—499, M. Serret obtains the same result under the form of the value (for $\zeta =$ zero) of

$$\frac{1}{\pi(m-1)}\left[\left(\frac{d}{d\zeta}\right)^{m-1}\frac{\zeta^m f(a+\zeta)}{x-a-\zeta}\right],$$

where m is the degree to which $(x-a)$ rises in the denominator of fx.

† Take as an example the theorem that the sum of the three angles of a triangle is equal to two right-angles: as soon as by a stroke of the imagination a line is conceived as drawn from one angle parallel to the opposite side, the truth of the proposition becomes virtually self-evident.

employed in this case, and to have entailed the consequences that are implied in it*.

In applying this theorem to finding the value of the denumerant to the equation $ax + by + \dots + lt = n$, which I denote by $\dfrac{n}{a, b, \dots l,}$ and is the same thing as the coefficient of x^n in the expansion of the rational fraction

$$\frac{1}{(1-x^a)(1-x^b)\dots(1-x^l)}$$

or more generally to finding the value of the denumerant

$$\frac{n}{a_1, a_2, \dots a_\alpha, b_1, b_2, \dots b_\beta, \dots l_1, l_2, \dots l_\lambda,}$$

(where each letter has a fixed value independent of its subindex), that is, the coefficient of x^n in the development of $\dfrac{1}{(1-x^a)^\alpha (1-x^b)^\beta \dots (1-x^l)^\lambda}$, say Fx, the first thing to be done is to determine and arrange in convenient groups the infinity roots of these functions. To effect this we have only to write down all the divisors of the set of numbers $a, b, \dots l$, that is, all the integers which divide one or more of those numbers, say $\delta_1, \delta_2, \dots \delta_\mu$. These divisors necessarily include the indices $a, b, \dots l$ and *unity*, which latter we may suppose to be δ_1.

Giving then i every value from 1 to μ, the primitive δ_ith roots of unity will obviously be the infinity roots required, and we may separate the required function of n into μ distinct portions or waves, as I term them, where supposing $\nu_1, \nu_2, \dots \nu_{\phi(\delta_i)}$ [$\phi(\delta_i)$ being the *totient* of δ_i, that is, the number of integers less than δ_i and prime to it] to be the primitive δ_ith roots of unity, the ith period or wave, say W_i, will be equal to the residue of

$$\Sigma r_q^{-n} e^{nt} F(r_q e^{-t}) \;\; [q = 1, 2, \dots, (\phi\delta_i)].$$

Since every primitive root r_q is either equal to or is mated with its reciprocal, the above expression may be replaced by the somewhat more convenient one $\Sigma (r_q{}^n e^{nt}) F(r_q{}^{-1} e^t)$.

This again admits of a very important transformation, namely, we may write $\nu = n + \frac{1}{2}(\alpha a + \beta b + \dots + \lambda l)$ and then

$$W_i = \text{co}_{-1} \Sigma \frac{r_q{}^\nu e^{\nu t}}{P\left(r_q{}^{\frac{a}{2}} e^{\frac{at}{2}} - r_q{}^{-\frac{a}{2}} e^{-\frac{at}{2}}\right)^\alpha}$$

* Thus, for example, the supposed investigator might have chosen to write $\sin t$ or $\log(1+t)$ in lieu of $1-e^t$ and the theorem thereby obtained would have been perfectly valid, but of little if any use, and the great bulk of transformations would certainly be of no use whatever; indeed, it is safe to say that the substitution practised, namely, that of $1-e^{\lambda t}$ [λ being taken at will] is the only one that would lead to a practical solution of the question.

(where P is used to signify that the product is to be taken of terms of like form to the one which is in regimen with it).

From this it follows that every wave W_i expressed as a function of ν, when ν is changed into $-\nu$, becomes $(-)^{\alpha+\beta+\dots+\lambda-1}W_i$, that is, retains its value absolutely or else merely changes its algebraic sign. To prove this it may be observed that whatever the index of the wave the above sum may be replaced by

$$\frac{1}{2}\,\mathrm{co}_{-1}\,\Sigma\left\{\frac{r_q{}^{\nu}e^{\nu t}}{P\left(r_q{}^{\frac{a}{2}}e^{\frac{at}{2}}-r_q{}^{-\frac{a}{2}}e^{-\frac{at}{2}}\right)^{\alpha}}+\frac{r_q{}^{-\nu}e^{\nu t}}{P\left(r_q{}^{-\frac{a}{2}}e^{\frac{at}{2}}-r_q{}^{\frac{a}{2}}e^{-\frac{at}{2}}\right)^{\alpha}}\right\}.$$

This is a consequence of r being either identical with $\frac{1}{r}$ as is the case for W_1 and W_2, or else being mated with it as belonging to the same group of primitive roots of unity.

Hence r_q may be changed into $r_q{}^{-1}$, and the expression to be residuated will undergo no change.

Again, if t is changed into $-t$, the residue changes its sign, and finally if r_q, t, and ν are simultaneously changed into $r_q{}^{-1}$, $-t$, $-\nu$ the expression to be residuated remains unaltered, except that it takes up a factor $(-)^{\Sigma\alpha}$. Consequently the effect of changing ν into $-\nu$, leaving everything else unaltered, will be to introduce the factor $(-)^{\Sigma\alpha-1}$; and this being true of every portion of the value of $\dfrac{n}{a\dots,\,b\dots,\,l\dots,}$ it follows that when that denumerant is expressed under the form $F\nu$, where $\nu=n+\frac{1}{2}\Sigma\alpha a$, $F(-\nu)=(-)^{-1+\Sigma\alpha}F(\nu)$.

There is consequently an enormous advantage gained, as well in the abbreviation of the calculations as in the conciseness of the result, by putting such a denumerant under the form of a function of the *augmented* argument ν instead of the original argument n; when so expressed I speak of the denumerant being in its canonical form.

In future, for greater simplicity, I shall disuse the indices α, β ... it being understood (unless the contrary is stated) that any of the indices a, b, c ... in the denominator of the denumerant $\dfrac{n}{a,\,b,\,c,\dots l,}$ or in its generating function $\dfrac{1}{(1-x^a)(1-x^b)\dots(1-x^l)}$ may be made equal to one another.

It is perhaps not unworthy of notice that the denumerant $\dfrac{n}{a,\,b,\dots,\,l,}$ may be expressed as the residue of a double sum without knowing the divisors of the indices. For it is obvious that we may express it as the sum of an infinite number of waves whose indices take in all values from unity up to infinity (since all those whose indices are non-divisors will be equal to zero)*,

* By a process, so to say, of *natural selection.*

and consequently as the residue of a sum of quantities obtained by substituting for r in the expression

$$\frac{r^\nu e^{\nu x}}{P\left(r^{\frac{a}{2}} e^{\frac{x}{2}} - r^{-\frac{a}{2}} e^{-\frac{x}{2}}\right)},$$

every primitive root of unity of every order up to the ωth inclusive, where ω is any number not less than the greatest of the quantities a, and therefore, if we please, equal to Σa, which saves the necessity of distinguishing the relative magnitudes of the several quantities a (ω it should be noticed must not be taken infinity, because that would render the sum to be residuated infinite). Thus then we see that the denumerant $\dfrac{n}{a, b, \ldots, l,}$ is the residue of

$$\Sigma \frac{e^{(t+2\pi ik)\nu}}{P\left\{e^{a\left(\frac{t}{2}+\pi ik\right)} - e^{-a\left(\frac{t}{2}+\pi ik\right)}\right\}},$$

where k represents every distinct quantity expressible by a proper fraction whose denominator is equal to or less than Σa*.

The result previously found concerning the relation of $F\nu$ to $F{-\nu}$ is in accordance with the observation due, I believe, to Jacobi, that if ϕn, ψn be the coefficients of x^n [n positive or negative] in the ascending and descending expansions of a proper rational fraction, then $\psi n = -\phi n$. For, in the particular fraction we are considering, it is obvious that calling the number of the factors (our former $\alpha + \beta + \ldots + \lambda$) i and $a + b + \ldots + l = s$, we shall have

$$\psi(-n-s) = (-)^i \phi n.$$

Therefore $\phi n = (-)^{i-1} \phi(-n-s)$ by Jacobi's observation.

If then $\nu = n + \dfrac{s}{2}$ and $\phi n = F\nu$ so that $\phi(-n-s) = F\left(-n-\dfrac{s}{2}\right) = F(-\nu)$ we shall have $F\nu = (-)^{i-1} F(-\nu)$, as already shown.

It is also a part of the same observation and shown in the same way that ϕn, used in the same sense as above, is zero for all values of negative n between *zero* and the *degree* of the fraction (*exclusive*); hence $F(\pm\nu)$ is zero for all values of ν from 0 to $\dfrac{s}{2} - 1$ inclusive if s be even, and from $\dfrac{1}{2}$ to $\dfrac{s}{2} - 1$ inclusive if s be odd†.

This fact alone is sufficient to give exactly the number of homogeneous equations required to determine (to a numerical factor près) the algebraico-

* The number of terms in this sum will be the sum of the totients of all the numbers up to the limit, an empirical expression for which (if my memory is not in fault) has been recently investigated by Mr Merrifield.

† In order not to break up the text, the footnote (which ought to come here) regarding the two statements above, as to the coefficient-functions of any proper fraction, is transferred to the last page of this Excursus [p. 621 below].

exponential form $F(\nu)$, that is, the effective* *trivial* zero values of $F(\nu)$ are exactly equal in number to the number of terms which that form contains, as I will proceed to show.

The number of the indices $a, b, c, \ldots$ in which any divisor is contained may be termed its frequency in respect to those numbers, and it is a very simple arithmetical fact that if the totient of every divisor of a set of given numbers be multiplied by its frequency in respect to the set, the sum of the products so obtained will be equal to the sum of the given numbers. When the set reduces to a single term this theorem becomes the familiar one, that any number is equal to the sum of the totients of all its several divisors, and from this to the general case there is but a step, for we may suppose the set of numbers written out in a line, and under every one of them which contains a divisor j the totient of j to be written, and every value from 1 upwards as far as the highest number of the set to be given to j. The rectangle (partly filled with totients and partly vacant) so formed, read off in columns, will, by the preceding case, give the sum of the set of numbers, and read off in lines, the sum of the products of each divisor by its frequency.

Let us now inquire into the number of the terms contained in the several waves. W_1, which always exists, will be the coefficient of $\frac{1}{t}$ in $\frac{e^{\nu t}}{P\left(e^{\frac{at}{2}}-e^{-\frac{at}{2}}\right)}$, and therefore (always supposing the number of indices a to be i) will be the coefficient of t^{i-1} in the product of $\left(1+\nu t+\nu^2\frac{t^2}{1.2}+\ldots\right)$ into the ascending development of $\frac{1}{P\left(\frac{e^{\frac{at}{2}}-e^{-\frac{at}{2}}}{t}\right)}$, and will therefore be a function of ν consisting of multiples of $\nu^{i-1}, \nu^{i-3}, \ldots$ until a multiple of ν or a constant is reached, and therefore containing $E\frac{i+1}{2}$ terms, the first of which it may be well to notice (using $a_1, a_2 \ldots a_i$ in lieu of $a, b, \ldots l$ as the indices) will obviously always be $\frac{1}{\Pi(i-1)a_1.a_2.\ldots a_i}$†.

In like manner it will be obvious that for W_2 the degree of ν will be the frequency of 2 diminished by a unit, and the form of W_2 will be $(-)^n$ into a polynomial function of ν of that degree.

* I say *effective* because it will presently be seen that in a certain case one of the trivial zero values will be ineffective, that is, will only lead to an identity and not to an equation between the coefficients in question.

† The highest power of ν in any other wave (which is its frequency diminished by unity) will *in general be* less than $i-1$, and consequently the sign of the terms in the development of any rational fraction beyond a certain point must be unvarying, and the development from that

Again, any other wave W_i of frequency f_i will consist of a set of products of polynomial functions of ν of the degree $f_i - 1$ each multiplied by a sum of exponential quantities consisting of pairs of the form $c\Sigma(\rho^{\nu+\delta} + \rho^{\nu-\delta})$ or $c\Sigma(\rho^{\nu+\delta} - \rho^{\nu-\delta})$ according as $i - f_i$ is even or odd, where δ will be half the number of primitive ith roots of unity, say $\frac{\tau(i)}{2}$, where the numerator is the totient of i.

Hence the total number of constants to be determined in the algebraico-exponential function representing $\frac{n}{a_1, a_2, \ldots a_i}$ will be

$$E\frac{f_1+1}{2} + E\frac{f_2+1}{2} + \Sigma\frac{\phi\lambda f_\lambda}{2} \quad [\lambda = 3, 4, \ldots \infty].$$

(1) Suppose that f_1 and f_2 are not both even.

Then remembering that $\frac{f_1}{2} + \frac{f_2}{2} + \frac{f_3 \cdot \tau_3}{2} + \frac{f_4 \cdot \tau_4}{2} + \ldots = \frac{s}{2}$, the antecedent expression $= E\left(\frac{s}{2} + 1\right)$, for when f_1, f_2 are both odd, the two first terms on the left-hand side of this equation exceed the corresponding ones in the equation above it by $\frac{1}{2}, \frac{1}{2}$ respectively, and $E\left(\frac{s}{2} + 1\right)$ will exceed $\frac{s}{2}$ by unity (because $f_1 - f_2$ the number of the odd elements in the sum of all of them being even, s is even). And if f_1, f_2 are one odd and the other even, the right as well as the left-hand side of each equation will be increased $\frac{1}{2}$, for s will be now odd.

(2) Suppose that f_1, f_2 are both even, then

$$E\frac{f_1+1}{2} + E\frac{f_2+1}{2} + \frac{f_3\tau(3)}{2} + \ldots = \frac{f_1}{2} + \frac{f_2}{2} + \frac{f_3\tau(3)}{2} + \ldots = \frac{s}{2}.$$

Hence the number of constants to be determined is $1 + E\frac{s}{2}$, except when f_1, f_2 are both even, in which case it is $\frac{s}{2}$.

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point omni-positive or omni-negative, according as the numerator, on substituting unity for the variable, is positive or negative. The case of exception is when all the indices have a common numerant, say δ, for then the frequency of δ will be the same as of unity, and W_δ be of the same degree as W_1 in ν, so that the reason for uniformity of sign (at a sufficient distance from the origin) no longer subsists. This is the proof referred to at p. [600], in what precedes.

It is worth while imprinting on the memory the rule that the asymptotic value of

$$\frac{n}{a_1, a_2, \ldots a_i}, \div n^{i-1} \text{ is } \frac{1}{\{1 \cdot 2 \cdot 3 \ldots (i-1)\}\, a_1 \cdot a_2 \ldots a_i},$$

which ought, I imagine, to be susceptible of some simple proof or illustration by the method of nodes or cross-gratings, such as employed by Eisenstein to prove the law of reciprocity for quadratic residues, and by myself (*Johns Hopkins Circulars*, Nos. 13 and 14, pp. 179, 180, 209)* to demonstrate the impossibility of the existence of trebly periodic functions.

[* Below, pp. 635, 644.]

On the first supposition the trivial values of ν which make $F(\nu)$ zero are $0, 1, 2, \ldots \frac{s}{2}-1$ when s is even, and $\frac{1}{2}, \frac{3}{2}, \ldots \left(\frac{s}{2}-1\right)$ when s is odd, the number of such being $E\left(\frac{s}{2}\right)$ in either case, and there will be $E\left(\frac{s}{2}\right)$ homogeneous equations for finding the ratios of $E\left(\frac{s}{2}\right)+1$ coefficients, which is exactly the right number.

On the second supposition, that is, when f_1, f_2 are both even, the number of the trivial values in question will be $\frac{s}{2}$, the same as the number of the coefficients, so that at first sight there would appear to be one superfluous equation—such, however, is not really the case—because the value 0 attributed to ν will lead not to a homogeneous equation between the coefficients but to the identity $0=0$. For evidently W_1, W_2 becoming odd functions of ν, will vanish when $\nu=0$, and every other wave will also vanish; for when $\nu=0$ it will consist exclusively of pairs of terms of the form $c(\rho^{\delta}-\rho^{-\delta})$ (because by hypothesis f_1 the number of the elements is even), and since ρ and $\frac{1}{\rho}$ may be interchanged, it follows that the sum of such pairs must be zero. Hence whatever the relation of the number of odd and the number of even elements to the modulus 2, there will be just as many homogeneous equations as are required for determining the ratios of the coefficients in the form which expresses the denumerant. The absolute values of the coefficients may be found by writing $F\left(\frac{s}{2}\right)=$ coefficient of x^0 in the generating function $=1$, or by virtue of the observation made above, that the leading coefficient in W_1 for the elements $a_1, a_2, \ldots a_i$ is $\frac{1}{\pi(i-1)\, a_1, a_2, \ldots a_i}$.

When the denumerant is regarded as a function of n and not of ν, it is obvious *à priori* that being a particular integral of an equation in finite differences of the order s, its coefficients must be determinable in relative magnitude by the knowledge of $(s-1)$ values of the variable for which it vanishes, and this is almost but not quite sufficient in itself to establish the preceding result regarding the canonical form.

I will illustrate this method presently by one or two easy examples, but previously it will, I think, be desirable to give greater precision and uniformity to the nomenclature of simple denumerants.

If any such be denoted by $\frac{n}{a, b, \ldots l}$, (I have sometimes here or elsewhere referred to n as the numerator or denumerator or partible number, and to $a, b, \ldots l$, variously as the denominators or as the indices or as the elements of the denumerant), in future I shall call n the componend, and $a, b, \ldots l$ the components of the denumerant.

A denumerant with a single component as $\dfrac{n}{a,}$ which I call an elementary denumerant, deserves special attention, for it will presently be seen that every given simple denumerant is expressible as a sum of powers of its componend multiplied respectively by linear functions of elementary denumerants whose several components are the divisors of the components of the given one.

The elementary denumerant $\dfrac{n}{a,}$ being the number of solutions in positive integers of the equation $ax=n$, is obviously 1 or 0 according as n does or does not contain a. But we may also regard $\dfrac{n}{a,}$ as an analytical function and define it as the mean of the a values of ρ^n where ρ is any root of the equation $\rho^a-1=0$, and so construed it will preserve a meaning even when n is taken a negative integer, and will mean 1 or 0, provided that n be an integer of either kind, according as it does or does not contain a without a remainder. It is in this extended sense that $\dfrac{n}{a,}$ or $\dfrac{\nu}{a,}$ will be employed in what follows.

Supposing r to be a primitive ith root of unity, W_i will consist of a sum of powers of ν each multiplied by the sum of quantities of the form $cr^{n+\delta}$ (where for the moment for greater clearness of elucidation I purposely retain n instead of using its augmentative ν). On giving n all values from $-\delta$ to $-\delta+i-1$ inclusive, this sum will take i successive values to be determined from the equation containing the primitive roots, say $\epsilon_0, \epsilon_1, \dots \epsilon_{i-1}$, so that its general value will be expressible under the form

$$\epsilon_0 \frac{n+\delta}{i,} + \epsilon_1 \frac{n+\delta-i}{i,} + \dots + \epsilon_{i-1} \frac{n+\delta-i+1}{i,}.$$

We may then replace n by $\nu-\dfrac{s}{2}$, and on so doing and further replacing (where requisite) any numerator by its residue in respect to i, shall obtain a sum of the form

$$\eta_0 \frac{\nu}{i,} + \eta_1 \frac{\nu-1}{i,} + \dots + \eta_{i-1} \frac{\nu-i+1}{i,} \text{ when } s \text{ is even,}$$

and of the form

$$\eta_0 \frac{\nu-\frac{1}{2}}{i,} + \eta_1 \frac{\nu-\frac{3}{2}}{i,} + \dots + \eta_{i-1} \frac{\nu-i+\frac{1}{2}}{i,} \text{ if } s \text{ is odd.}$$

On this being done, remembering the extension given to the sense of an elementary denumerant and the theorem that the analytical value $F\nu$ of a denumerant is equal to $\pm F(-\nu)$, we see that in either case the above sums will be reducible to a sum of pairs of terms of the form $\eta\left(\dfrac{\nu+k}{i,} \pm \dfrac{\nu'-k}{i,}\right)$

[the same + or − sign subsisting throughout the whole series for any specified power of ν] but subject to the exception that when i is even, two of the pairs will be replaced by single terms, multiples of $\frac{\nu \pm \frac{i}{2}}{i,}$ and of $\frac{\nu}{i,}$ respectively, which become zero when the negative sign is the one to be employed*.

Thus taking $i=2$, W_2 takes the form $(-)^n R\nu$, that is, $\frac{n}{2,}-\frac{n-1}{2,}$. W_1 it is scarcely necessary to repeat will contain no elementary denumerants, being purely an algebraical function of the resolvent. W_2 is such a function multiplied by $(-1)^n$. This multiplier is expressible under the form $\left(\frac{n}{2,}-\frac{n \pm 1}{2,}\right)$ which is always a function of n that remains unchanged when n is changed into $-n$. But when the two denumerants are expressed as functions of ν the case is different; if s (the sum of the components) is an even number, the above pair of terms becomes $(-)^{\frac{s}{2}}\left(\frac{\nu}{2,}-\frac{\nu \pm 1}{2,}\right)$ which is unaltered by the change of ν into $-\nu$, but when s is odd it becomes $(-)^{\frac{s-1}{2}}\left(\frac{\nu-\frac{1}{2}}{2,}-\frac{\nu+\frac{1}{2}}{2,}\right)$ which changes its sign when ν is changed into $-\nu$.

Before quitting the subject of nomenclature I may just observe that it will be convenient to call denumerants, when their resolvents are the natural numbers commencing with unity, *natural denumerants*, and when the natural numbers commencing with 2, curtate natural, or for greater brevity simply *curtate* denumerants, the highest number reached in either case being termed the order; D_i and Δ_i may then be used to denote natural and curtate denumerants of the order i†.

I now return to the application of the method of indeterminate coefficients to finding the value of denumerants whose components are given. This method is not practically applicable when the sum of the components is considerable, because that sum measures the number of linear equations to be solved. In the following section I shall work out in full, by the regular process, the case where the components are 2, 3, 4, 5, 6, 7, of which the result

* The sign is positive or negative according as the number of the components less the power of ν in question is odd or even, and it is easy also to see that the sum of all the coefficients of the elementary denumerants in the multiplier of each power of ν will be always zero.

† It is curtate denumerants which are almost exclusively required in the applications to the theory of invariants. If necessary to bring into evidence the componend we may use the more explicit notation $\overset{n}{D}_i$, $\overset{n}{\Delta}_i$ to signify natural and curtate denumerants of the order i with the componend n. Thus we may write $\overset{n}{D}_i-\overset{n-1}{D}_i=\overset{n}{\Delta}_i$ and $\overset{n}{D}_i-\overset{n}{D}_{i-1}=\overset{n-1}{\Delta}_i$.

It may be as well to notice that for curtate, as well as for natural denumerants, the divisors of the components are the natural numbers from unity to the order of the denumerant inclusive, so that the number of the waves for either of these sort of denumerants is equal to the order.

is more especially required for the purposes of the preceding section, and which has not previously been calculated. The other algebraical formulae for denumerants in their canonical form I shall give without exhibiting the work; the accuracy of most of them can be ascertained by comparison with Prof. Cayley's values of the same, exhibited as functions of the unaugmented componend in the *Phil. Trans.* for 1856 and 1858.

Let us suppose 1, 2, 3 to be the components,

we may write $\dfrac{n}{1,\,2,\,3,} = A\nu^2 + B + (-)^\nu C + \Sigma(\rho^{\nu+1} + \rho^{\nu-1})D,$

where $\rho^2 + \rho + 1 = 0$, or more simply, $A\nu^2 + B + (-)^\nu C - D\Sigma\rho^\nu = 0$.

Hence making $\nu = 0, 1, 2$ we have

$$B + C - 2D = 0$$
$$A + B - C + D = 0$$
$$4A + B + C + D = 0,$$

so that $2C + 3A = 0 \quad 3D + 4A = 0 \quad B + \left(\dfrac{8}{3} - \dfrac{3}{2}\right)A = 0,$

or $A = 6\sigma \quad B = -7\sigma \quad C = -9\sigma \quad D = -8\sigma;$

and to find σ, making $\nu = 3$, we obtain

$$(54 - 7 + 9 + 16)\sigma = 1 \quad \text{or } \sigma = \frac{1}{72}.$$

Hence $\dfrac{n}{1,\,2,\,3,} = \dfrac{\nu^2}{12} - \dfrac{7}{72} - \dfrac{1}{8}\left(\dfrac{\nu}{2,} - \dfrac{\nu-1}{2,}\right) + \dfrac{1}{9}\left(2\dfrac{\nu}{3,} - \dfrac{\nu+1}{3,} - \dfrac{\nu-1}{3,}\right)$,

monomial denumerants being used to replace the exponential quantities $(-1)^\nu$; $\Sigma\rho^\nu$.

The leading coefficient $\dfrac{1}{12}$ it will be observed $= \dfrac{1}{(1\,.\,2)\,(1\,.\,2\,.\,3)}$, as it ought to be by the general rule.

The maximum negative value of $\dfrac{n}{1,\,2,\,3,} - \dfrac{\nu^2}{12}$ is $\dfrac{7}{72} + \dfrac{1}{8} - \dfrac{1}{9}$ or $\dfrac{1}{9}$, and its maximum positive value $\dfrac{2}{9} + \dfrac{1}{8} - \dfrac{7}{72}$ or $\dfrac{1}{4}$. Hence the value of $\dfrac{n}{1,\,2,\,3,}$ is always the nearest integer to $\dfrac{(n+3)^2}{12}$.

But by Euler's theorem of reciprocity $\dfrac{n}{1,\,2,\,3,}$ is the number of ways of resolving n into three or less than three parts, and consequently $\dfrac{n-3}{1,\,2,\,3,}$ is the number of ways of resolving n into exactly three parts, this therefore is always the nearest integer to $\dfrac{n^2}{12}$, as first observed I believe by the late lamented Prof. De Morgan.

Take as another case the components 1, 2, 3, 4 which give $\nu = n + 5$. We may write

$$\frac{n}{1, 2, 3, 4,} = A\nu^3 + B\nu + (-)^\nu C\nu + D\Sigma(\rho^{\nu+1} - \rho^{\nu-1}) + E\Sigma(i^{\nu+1} - i^{\nu-1})$$

where $\rho^2 + \rho + 1 = 0$, $i^2 + 1 = 0$. Hence giving ν the successive values 1, 2, 3, 4, (omitting $\nu = 0$, which would lead to $0 = 0$) we obtain

$$A + B - C - 3D - 4E = 0$$
$$8A + 2B + 2C + 3D = 0$$
$$27A + 3B - 3C + 4E = 0$$
$$64A + 4B + 4C - 3D = 0.$$

Hence $\quad 72A + 6B + 6C = 0$, and $36A + 6B - 2C = 0$,

consequently $\quad 2C + 9A = 0 \quad 2B + 15A = 0 \quad -3D + 16A = 0$

or $\quad A = 6\sigma \quad B = -45\sigma \quad C = -27\sigma \quad D = 32\sigma \quad E = -27\sigma$.

Finally making $\nu = 5 \quad \sigma(750 - 225 + 135 + 96 + 108) = 1$, or $\sigma = \frac{1}{864}$,

and
$$\frac{n}{1, 2, 3, 4,} = \frac{1}{144}\nu^3 - \frac{5}{96}\nu - \frac{1}{32}\left(\frac{\nu}{2,} - \frac{\nu - 1}{2,}\right) + \frac{1}{9}\left(\frac{\nu - 1}{3,} - \frac{\nu + 1}{3,}\right) + \frac{1}{8}\left(\frac{\nu - 1}{4,} - \frac{\nu - 3}{4,}\right).$$

The principal coefficient is $\frac{1}{144}$ or $\frac{1}{\Pi 3 . 1 . 2 . 3 . 4}$, as it ought to be, according to the general rule, and this serves as a verification of the correctness of the whole work.

It will be found convenient to append here, instead of reserving for the following section, the analytical expression for the first wave of a general denumerant, which stands out markedly from the rest, inasmuch as it can be expressed once for all as an algebraical function of the componend and components without any regard being had to the arithmetical form of the latter.

Let $C(\tau_1\tau_2 \ldots \tau_j)$, $H(\tau_1\tau_2 \ldots \tau_j)$ or more briefly $C_j\tau$ $H_j\tau$ be understood to denote the perfectly well-known functions of $\tau_1, \tau_2, \ldots \tau_j$ which represent the elementary symmetric function and the sum of the homogeneous products of the jth order of those quantities of which τ_q represents the sum of the qth powers, so that, for example, $C_2\tau$, $H_2\tau$ will serve to denote $\frac{\tau_1{}^2 - \tau_2}{2}$, $\frac{\tau_1{}^2 + \tau_2}{2}$ respectively, upon which supposition we may write

$$e^{\tau_1 t + \tau_2\frac{t^2}{2} + \tau_3\frac{t^3}{3} + \ldots} = 1 + \tau_1 t + \frac{\tau_1{}^2 + \tau_2}{2} t^2 + \ldots + H_q\tau t^q + \ldots$$

Also let it be observed preliminarily that as a direct inference from Maclaurin's theorem, if ϕ represent any function of x but does not contain ν,

$$\text{co}_j\, e^{\nu x + \phi} = \text{co}_j\, e^\phi + \text{co}_{j-1}\, e^\phi \nu + \text{co}_{j-2}\, e^\phi \frac{\nu^2}{1 . 2} + \ldots$$

Furthermore for greater brevity let us agree to express the W_1 for j components $a_1, a_2, \ldots, a_j$ under the form $W_{1,j}$, and write it equal to $\frac{V_j}{\pi_j}$ where π_j indicates the product of the j components.

We may then write

$$V_j = \pi_j \,\text{co}_{-1} \frac{e^{\nu x}}{P(e^{a\frac{x}{2}} - e^{-a\frac{x}{2}})}.$$

Now from the known expression for $\log \sin \theta$, we may write

$$\log (e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}}) = \log \theta + \beta_1 \theta^2 - \beta_2 \theta^4 + \ldots \pm \beta_q \theta^{2q} + \ldots$$

where $$\beta_q = \frac{1}{\Pi 2q} \cdot \frac{B_{2q-1}}{2q}.$$

Hence $$V_j = \text{co}_{j-1}\, e^{\nu x - 2\tau_1 \frac{x^2}{2} + 2\tau_2 \frac{x^4}{4} - 2\tau_3 \frac{x^6}{6} \cdots}$$

where $2\tau_q = \frac{B_{2q-1}}{\Pi 2q} \sigma_{2q}$ and the latter factor indicates the sum of the $2q$th powers of the components.

Hence writing $x^2 = t$ we have $V_j = \text{co}_{j-1}\, e^{\nu x - \tau_1 t + \tau_2 \frac{t^2}{2} - \tau_3 \frac{t^3}{3} \cdots}$

and consequently making $T = -\tau_1 t + \tau_2 \frac{t^2}{2} - \tau_3 \frac{t^3}{3} \cdots$

$$V_j = \text{co}_{j-1} T + \text{co}_{j-2} T \,.\, \nu + \text{co}_{j-3} T \,.\, \frac{\nu^2}{1\,.\,2} + \text{co}_{j-4} T \,.\, \frac{\nu^3}{1\,.\,2\,.\,3} \cdots$$

$$= \frac{\nu^{j-1}}{\Pi\,(j-1)} - H_1 \tau \frac{\nu^{j-3}}{\Pi\,(j-3)} + H_2 \tau \frac{\nu^{j-3}}{\Pi\,(j-3)} \cdots$$

the series ending with ν or with a constant according as j is even or odd.

Thus $$V_2 = \nu,$$

$$V_3 = \frac{\nu^2}{2} - H_1(\tau),$$

$$V_4 = \frac{\nu^3}{6} - H_1(\tau)\nu,$$

$$V_5 = \frac{\nu^4}{24} - H_1(\tau)\frac{\nu^2}{2} + H_2(\tau),$$

$$V_6 = \frac{\nu^5}{120} - H_1(\tau)\frac{\nu^3}{6} + H_2(\tau)\,\nu, \text{ and so on,}$$

each V being an integral with respect to ν of the one which precedes it.

Substituting for each τ its value in terms of the Bernouillian numbers B and the σ's, and giving the former their arithmetical values we shall obtain

$V_2 = \nu,$

$V_3 = \frac{\nu^2}{2} - \frac{\sigma_2}{24},$

$V_4 = \frac{\nu^3}{6} - \frac{\sigma_2}{24}\nu,$

$V_5 = \frac{\nu^4}{24} - \frac{\sigma_2}{48}\nu^2 + \left(\frac{\sigma_2{}^2}{1152} + \frac{\sigma_4}{2880}\right),$

$V_6 = \frac{\nu^5}{120} - \frac{\sigma_2}{144}\nu^3 + \left(\frac{\sigma_2{}^2}{1152} + \frac{\sigma_4}{2880}\right)\nu,$

$V_7 = \frac{\nu^6}{720} - \frac{\sigma_2}{576}\nu^4 + \left(\frac{\sigma_2{}^2}{2304} + \frac{\sigma_4}{5160}\right)\nu^2 - \left(\frac{\sigma_2{}^3}{82944} + \frac{\sigma_2\sigma_4}{103680} + \frac{\sigma_6}{181440}\right),$

$V_8 = \int_\gamma^0 d\nu\, V,$ and so on.

Such are the expressions for V best adapted for actual use, since it is desirable to express $W_{1,j}$, that is, $\frac{V_j}{a_1 . a_2 \ldots a_j}$ explicitly in terms of powers of ν; but there is another somewhat noteworthy form which can be given to the V with an even subindex as follows:

It is obvious that

$$V_{2k} = \text{co}_{-1}\frac{\frac{1}{2}(e^{\nu x} - e^{-\nu x}) + \frac{1}{2}(e^{\nu x} + e^{-\nu x})}{P\left(e^{\frac{a}{2}x} - e^{-\frac{a}{2}x}\right)} = \text{co}_{-1}\frac{\frac{1}{2}(e^{\nu x} - e^{-\nu x})}{P\left(e^{\frac{a}{2}x} - e^{-\frac{a}{2}x}\right)}$$

for the neglected part of the numerator will contribute nothing to the residue*.

We may now calculate the logarithm of the entire quantity to be residuated instead of merely the denominator, and take the residue of its exponential ν; on so doing it will be obvious on reflection that we shall obtain the product of ν into a quantity of the very same form as the constant term in V_{2k-1}, when instead of σ_{2q} in the value of τ_q we substitute $-(2n)^{2q} + \sigma_{2q}$. If then we write $2U_q = \frac{B_{2q-1}}{\Pi 2q}\{(2n)^{2q} - \sigma_{2q}\}$ it is easy to see that we shall have $V_{2k} = \nu C_{k-1}(U)$.

* For V_{2k} the *effective* numerator of the residuand is a *sine* form, and may be subjected to the same treatment as its fellows in the denominator. The case is different with V_{2k-1}, for which the effective numerator of the residuand is a *cosine* form. But we may write

$$V_{2k-1} = \frac{d}{d\nu}V_{2k} = C_{k-1}U + \nu\delta C_{k-1}U,$$

and if we turn to account the fact that in $C_{k-1}U$ along with $(2\nu)^2, (2\nu)^4, \ldots (2\nu)^q \ldots$ are associated $-s_2, -s_4, \ldots -s_{2q} \ldots$ and choose to write $-\nu^{2q}\frac{d}{ds_{2q}} = \Delta^q$, it will be found that the above expression may be transformed so as to give the symbolical equation (more curious perhaps than useful) $V_{2k-1} = \left(\frac{1+\Delta}{1-\Delta}\right)^2 C_{k-1}U$, whereas as previously found $V_{2k} = \nu C_{k-1}U$.

Thus, for example, suppose $2k=6$, we may write V_6 under the form

$$\nu\left\{\frac{(4\nu^2-s_2)^2}{1152}-\frac{(16\nu^4-s_4)}{2880}\right\}$$

to verify which it will be observed that

$$\frac{16}{1152}-\frac{16}{2880}=\frac{1}{72}-\frac{1}{180}=\frac{1}{120} \text{ and } \frac{8}{1152}=\frac{1}{144},$$

so that

$$V_6=\frac{\nu^5}{120}-\frac{s_2}{144}\nu^3+\left(\frac{s_2{}^2}{1152}+\frac{s_4}{2880}\right)\nu, \text{ as previously found.}$$

Before having done with this outline it may be well to call attention to the circumstance that the distribution of the infinity-roots into groups determined by the divisors of the components is not in all cases the best mode of grouping to adopt.

Thus suppose that the components $(a_1, a_2, \ldots a_i)$ are all prime relatively to each other, it will in such case be most expeditious, after taking out the algebraical part W_1, to separate what remains into i portions, referring respectively to *all* the non-unity a_1th, a_2th, ... a_ith roots of unity*.

This view enables us to give a concise answer to a question of some interest, namely, as to what is the number of solutions of the inequality

$$a_1x_1+a_2x_2+\ldots+a_ix_i<\mu\,(a_1a_2\ldots a_i),$$

say $\mu\pi_i$, where μ is any positive integer and the coefficients are relatively prime each to each.

Certainly this number is no other than the denumerant $\dfrac{\mu\pi_i}{1, a_1, a_2, \ldots a_i}$ which might be calculated by the general formula, but would give a result neither concise nor elegant; we may on the other hand regard it as a sum of denumerants, say $\Sigma\dfrac{\mu\pi_i-\delta}{a_1, a_2, \ldots a_i}$, where δ takes all values from 0 to $\mu\pi_i-1$. Now each such denumerant will consist of a purely algebraical and a purely periodical part, and it is very easy to see according to the view just indicated that the sum of all the latter will be zero. Hence the number required will be

$$\Sigma^0{}_{\mu\pi_i-1}\frac{V_1}{\pi_i}.$$

I may illustrate this by the very simplest imaginable case, where there are but two components p, q and the number required is that of the solutions in integers of the inequality $px+qy<pq$ where p and q are relative primes.

Calling $pq=n$, the rule laid down will give for the number sought

$$\Sigma_0{}^{n-1}\frac{\nu}{n}, \text{ that is, } \Sigma_0{}^{n-1}\frac{n+\dfrac{p+q}{2}}{n}=\frac{pq-p-q-1}{2}.$$

* This is tantamount to blending into one all the waves corresponding to the non-unity divisors of each component.

This result admits of a somewhat *piquant* verification. The number of integers less than pq and containing neither p nor q is $(p-1)(q-1)$, and if every two of these which are supplementary to one another (I mean whose sum is pq) be made into a pair, it is an easily demonstrable, but by no means an unimportant fact, that one of the pair will be a compound and the other a non-compound of p and q. Hence the total number of non-compounds is $\frac{1}{2}(p-1)(q-1)$, and therefore the total number of solutions of $px+qy<pq$ will be the remainder when the above is subtracted from pq, that is, $\frac{1}{2}(pq+p+q-1)$ as previously determined.

I will embrace this opportunity of noticing a correction that should be made to the long footnote in Section 3 given in the preceding number of the *Journal*. In lieu of the words* in the last paragraph of that note following the word *products*, line 3 and preceding the word *set*, line 8, read as follows:

Of the form $b^xQ^yR^zS^tT^u$ *such that no one of them could be* (*a power of one or*) *a product of powers of any of the others. If then it could be shown that there exists in the succession a set of quintuplets* x, y, z, t, u, *such that the quotient system of every other quintuplet in the succession is intermediate to the quotient system of that*

It may also be as well to notice here that the method of expressing in terms of ordinary space the intermediateness of a quadruplet, a triplet or a couplet, to four, three or two other such respective multiplets, may be profitably simplified by the use of quadriplanar, trilinear and bi-punctual coordinates, in flat spaces of three, two and one dimension respectively; for we may then without having recourse to quotient-systems regard each element of the multiplet as a coordinate of its representative point, inasmuch as the affection concerned being one relative exclusively to the inwardness or outwardness of a point in regard to a closed environment, obviously remains unchanged by projection.

What follows is the footnote referred to at foot of page [610] *where it was meant to be inserted.*

Each of the two statements regarding the coefficient-functions becomes next to self-evident when the coefficient of x^n in the reciprocal of $(1-\alpha x)(1-\beta x)\dots(1-\lambda x)$ is put under the form of a sum of terms similar to $\alpha^n \div \left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right)\dots\left(1-\frac{\lambda}{\alpha}\right)$ interpreted (when necessary) as meaning the function of (n; α, $\beta \dots \lambda$) indefinitely near to the value of what such sum becomes when any equal elements are made to undergo arbitrary infinitesimal variations. Jacobi's proof of the theorem, I rather think, is got by proving it directly for each of the simple partial fractions into which any given proper fraction may be supposed to have been resolved.

A third method is to form the equation between $u_n, u_{n-1}, \dots u_{n+j-1}$, and between

$$v_{-n}, v_{-n-1}, \dots v_{-n-j-1}$$

[* p. 581 above.]

[u_n being the general coefficient in the ascending and v_{-n} in the descending development of $1 \div R(x)$]; the two equations become identical on changing u and n into v and $-n$, and $j-1$ homogeneous equations which help to determine the constants will be the same in both, namely, those got by making $n=-1, -2, \ldots -(j-1)$, consequently the two particular integrals u_n, v_n can differ only by a factor independent of n; if we write then $u_n : v_n :: P : Q$ and call the first and last coefficients in the denominator A and L, and pay attention to the fact that u_n, v_n can only become infinite when A, L vanish, and also to the indifference of the relation of R regarded as a quantic in x and 1 to the two sorts of development, it is plain to see that $P : Q :: A^\mu : \pm L^\mu$, but the x-weight of u_n is n and of v_{-n} is $-n$; hence $\mu=0$ and $u_n : v_n$ is independent, not only of n but of the coefficients in R, and to determine its value we may make $R=x^{j-1}-x^j$, which gives at once $u_n=-v_n$. This being true for all values of n, it is obvious that the relation will continue to subsist, when instead of unity any polynomial function of x of lower degree than that of the denominator (see below) is taken for the numerator.

Moreover, if the degree of the numerator be $j-\delta$, u_q and v_q will be seen (from what goes before) to vanish for every value of q common to the series

$$-1, -2, \ldots -(j-1) : 0, -1, \ldots -(j-2) : \ldots : (j-\delta-1), (j-\delta-2), \ldots -(\delta-1),$$

namely, for the values $-1, -2, \ldots -(\delta-1)$ or in other words either coefficient-function of the index of any power of the variable which appears neither in the ascending nor the descending development of a rational fraction is equal to zero.

Unless the fraction is a proper one u_n and v_n (the coefficient-functions) will not be continuous functions of n throughout; hence arises the necessity of this limitation in dealing with the generalized equation $u_n=-v_n$. Thus, for example, for the improper fraction $\frac{1+2x^2}{(1-x)^2}$, u_0 and v_0 are 1 and 2, but for any positive or negative value of n other than 0, u_n and v_n will be $3n-1$ and $-(3n-1)$ respectively. It may be added that the theorem will continue to subsist even for an improper fraction, provided that on freeing its numerator from a power of the variable, it becomes a proper one, for then the coefficient-functions remain continuous throughout.

This last proof, although more laboured than the preceding ones, seems to me the best because it goes straight to the heart of the question and does not depend on any apparently accidental results of calculation, but (so to say) compares the two twin functions in their nascent state, in the very act of birth.

The relation of the two coefficient-functions to one another and to the two general terms in the actual expansions becomes more clear if we use ϕn, ψn to denote the two former, reserving u_n, v_n for the two latter. Then besides the equation $\phi n+\psi n=0$ which is absolute, we have the equations $u_n=\phi n$, $v_n=\psi n$, limited as follows. Call Δ the deficiency of the numerator of the generating proper fraction, that is, the number of units that it stops short of its maximum possible value: then the first of these two equations holds good for all values of n not less than $-\Delta$, the latter for all values of n not greater than -1; if Δ is not zero, that is, if the degree of the numerator is not the integer next below that of the denominator, these two ranges will overlap for the values $-1, -2, \ldots, -\Delta$ of n, and for those values $\phi n=u_n=0$, $\psi n=v_n=0$. In the use made of these theorems in the text, the numerator is a mere constant, so that Δ has its maximum value, namely it is one unit less than the sum of the components (that sum being the degree of the generating function to a denumerant).

The general theorem may be brought into more distinct relief as follows: A finite fraction may be conceived as containing any number of powers of x positive or negative in numerator and denominator, and its two developments may be supposed to touch or be separate or to intersect one another. In the *last* case two coefficient-functions ϕn, $-\phi n$ exist applicable to all terms outside but inapplicable to any term inside the overlap. In the second case such functions exist which (besides being applicable, as in the case of contact, to all terms belonging to either of the two developments) vanish for all values of n in the chasm which separates them.