78.

ON CROCCHI'S THEOREM.

[*Johns Hopkins University Circulars*, II. (1883), p. 2.]

In *Battaglini's Journal* for July, 1880, Signor Crocchi has given a theorem which may be stated in the following terms. If s_i, σ_i, h_i denote respectively the sum of the elementary combinations, of the powers, and of the homogeneous products each of the ith order of any number of elements, then h_i is the same function of $\sigma_1, \overline{\sigma}_2, \sigma_3, \overline{\sigma}_4, \ldots$ that s_i is of $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots$.

Signor Crocchi's proof is very elegant but a little circuitous. An instantaneous proof may be derived from the relation of reciprocity which connects s and h, namely, that if

$$h_i = f(s_1, s_2, \ldots s_i) \text{ then } s_i = f(h_1, h_2, \ldots h_i),$$

which is an immediate deduction from the well-known fact that

$$(1 + s_1 y + s_2 y^2 + s_3 y^3 + \ldots)(1 - h_1 y + h_2 y^2 - h_3 y^3 + \ldots) = 1.$$

For from this relation spring the equations

$$s_1 - h_1 = 0, \quad s_2 - h_1 s_1 + h_2 = 0, \quad s_3 - h_1 s_2 + h_2 s_1 - h_3 = 0 \ldots$$

which equations continue unaltered when the letters s and h are interchanged; for when such interchange takes place, the functions equated to zero of an even rank remain unaltered and those of an odd rank merely change their sign.

Returning to the immediate object in view, if $a, b, c, \ldots$ are the elements subject to the s, h, σ symbols, we may write

$$\Sigma \log(1 + ay) = \log(1 + s_1 y + s_2 y^2 + s_3 y^3 + s_4 y^4 + \ldots)$$

or,

$$\Sigma \log(1 - ay) = -\log(1 + h_1 y + h_2 y^2 + h_3 y^3 + h_4 y^4 + \ldots).$$

The first equation by differentiation performed in each side gives

$$\sigma_1 - \sigma_2 y + \sigma_3 y^2 - \sigma_4 y^3 + \ldots = \frac{s_1 + 2s_2 y + 3s_3 y^2 + 4s_4 y^3 + \ldots}{1 + s_1 y + s_2 y^2 + s_3 y^3 + \ldots},$$

and similarly the second equation gives

$$\sigma_1 + \sigma_2 y + \sigma_3 y^2 + \sigma_4 y^3 + \ldots = \frac{h_1 + 2h_2 y + 3h_3 y^2 + 4h_4 y^3 + \ldots}{1 + h_1 y + h_2 y^2 + h_3 y^3 + \ldots},$$

that is, $(\sigma_1 - \sigma_2 y + \sigma_3 y^2 - \ldots)(1 + s_1 y + s_2 y^2 + \ldots) = s_1 + 2s_2 y + 3s_3 y^2 + \ldots$

and $\quad (\sigma_1 + \sigma_2 y + \sigma_3 y^2 + \ldots)(1 + h_1 y + h_2 y^2 + \ldots) = h_1 + 2h_2 y + 3h_3 y^2 + \ldots.$

By comparison of coefficients of the powers of y, the first of these two equations affords the means of finding any σ in terms of the s quantities, and the second of these any σ in terms of the h quantities. But if we change s into h and $\sigma_2, \sigma_4, \ldots$ into $-\sigma_2, -\sigma_4, \ldots$ the first equation becomes the second. Hence if

$$s = f(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots)$$

$$h = f(\sigma_1, \bar{\sigma}_2, \sigma_3, \bar{\sigma}_4, \ldots). \quad \text{Q.E.D.}$$

It is not without interest to set out the reciprocity of the 6 relations which exist between s, σ, h. The synoptical scheme of such reciprocity may be exhibited symbolically as follows:

$$h/s = s/h, \quad h/\sigma = s/\pm\sigma, \quad \sigma/h = \pm\sigma/s.$$

As an illustration of the second of these symbolic equalities take

$$s_3 = \frac{\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3}{6}, \qquad s_4 = \frac{\sigma_1^4 - 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 - 6\sigma_4}{24};$$

the corresponding equations are

$$h_3 = \frac{\sigma_1^3 + 3\sigma_1\sigma_2 + 2\sigma_3}{6}, \qquad h_4 = \frac{\sigma_1^4 + 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 + 6\sigma_4}{24},$$

and it is worthy of observation that the sum of the numerical coefficients is always (as in the above examples) zero for the function of the σ quantities which gives an s of any order, and unity for the function of the same which expresses any h*.

* This statement is proved instantaneously by taking one of the elements equal to unity and all the rest zero; and the latter part of it gives a new proof of Cauchy's theorem which he obtains by a consideration of all the possible cyclic representations of the substitutions of n elements. The theorem is that if n elements be divided in every possible way into λ set of l, μ set of m, ν set of n ... elements, then

$$\Sigma \frac{1}{\pi\lambda\pi\mu\pi\nu \ldots l^\lambda m^\mu n^\nu \ldots} = 1.$$

For we know by a theorem of Waring that

$$s_n = \Sigma \pm \frac{1}{\pi\lambda\,\pi\mu \ldots l^\lambda m^\mu \ldots} \sigma_l^{\lambda} \sigma_m^{\mu} \ldots.$$

Hence by Crocchi's theorem the sum of the coefficients in h_n expressed in σ's is equal to

$$\Sigma \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m \ \ldots}$$

but it is also equal to unity. Cauchy's theorem is therefore proved.

Frequent occasion presents itself (especially in the theory of numbers) for expressing any s in terms of σ's, but probably up to the time when Signor Crocchi wrote on the subject there had never been any occasion to express h in terms of the σ's: for had such occasion ever arisen it seems almost impossible that the relation between the two corresponding sets of formulæ could have escaped observation.

In some recent researches, however, of the writer of this note on the irreducible semi-invariants of a quantic of an unlimited order, it becomes indispensable to convert homogeneous products into sums of powers, and Crocchi's theorem comes into play. (See sec. 4 of Article on Subinvariants, *Am. Math. Journ.*, Vol. V., part 2 [p. 597, above].)

The relation $\sigma/h = \pm\,\sigma/s$ is interesting under the point of view that virtually it contains an example of a sort of *invariance* of form which may possibly contain within itself the germ of an important theory. It informs us that if, in the function of h's which expresses any σ, in lieu of each h the function of s quantities to which it is equal be substituted, the form of the σ function will remain unchanged, except that when the order of the σ is an even number, its algebraical sign is reversed. Thus, for example,

$$\sigma_3 = h_1^3 - 3h_1h_2 + 3h_3, \quad h_1 = s_1, \quad h_2 = s_1^2 - s_2, \quad h_3 = s_1^3 - 2s_1s_2 + s_3.$$

Consequently if we write $\phi = x^3 - 3xy + 3z$, and for y and z substitute $x^2 - y$, $x^3 - 2xy + z$, respectively, the value of ϕ remains unaltered. So in like manner if we write

$$\phi = x^4 - 4x^2y + 4xz + 2y^2 - 4t,$$

and substitute for y, z, t;

$$x^2 - y, \quad x^3 - 2xy + z, \quad x^4 - 3x^2y + 2xz + y^2 - t,$$

respectively, no change ensues in ϕ except that it undergoes a change of sign.

So in general the σ functions with even and those with odd subindices may be regarded as the analogues of symmetrical and skew-invariants, respectively.

Again in the formula for s the sign *plus* or *minus* depends on the oddness or evenness of $\lambda + \mu + \ldots$. Hence if in

$$\Sigma \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m^\mu \ldots}$$

only those values of λ, μ, ... are admitted which make $\lambda + \mu + \ldots$ always odd or always even, either sum so formed will be equal to $\frac{1}{2}$, because the difference of the two sums is zero and their sum unity.

This theorem can, of course, be deduced like the former one from the method of cycles applied now, not to the entire number of the substitutions, but to that half of them which correspond to the *alternate group* of each, of which the number of representative cycles (monomial ones included) is always odd or else always even, according as the number of elements is one or the other.