85.

PROOF OF A WELL-KNOWN DEVELOPMENT OF A CONTINUED PRODUCT IN A SERIES.

[*Johns Hopkins University Circulars*, II. (1883), p. 46.]

To prove that the general term in the development in a series of powers of a of the reciprocal of

$$(1-a)(1-ax)\dots(1-ax^i)$$

(say of Fx) is

$$(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i})\div(1-x)(1-x^2)\dots(1-x^i).a^j$$

say $X_j a^j$, I proceed as follows.

I call the coefficient of a^j in the development, J_x, and show that every linear factor of X_j is contained in J_x.

Any such factor is a primitive factor of x^r-1, where r is any integer such that

$$E\frac{i+j}{r}-E\frac{i}{r}-E\frac{j}{r}=1,$$

and it is unrepeated.

Let $x=\rho$, and let the negative minimum residue of $i+1$ in respect to r be $-\delta$.

Then $F\rho$ is equal to the product of δ linear functions of a divided by a power of $(1-a^r)$, and consequently the only powers of a (say a^θ) which appear in its development will be those for which the residue of θ in respect to r, is 0, 1, 2, ... δ, and consequently

$$E\frac{i+\theta}{r}-E\frac{i}{r}-E\frac{\theta}{r}=0.$$

Hence a^j will not appear therein: so that J_x vanishes when any factor of X_j is zero, and consequently since every such factor is unrepeated, J_x contains X_j.

But J_x is obviously of the degree ij in x, and X_j which is the sum of the j-ary homogeneous products of 1, x, x^2, ..., x^i is of the same degree. Hence the two functions of x can only differ by a constant factor. On making $x=1$, Fx becomes $(1-a)^{-(i+1)}$; so that X_j becomes

$$\frac{(j+1)(j+2)\dots(j+i)}{2\dots i}$$

and J_x becomes the product of vanishing fractions

$$\frac{1-x^{j+1}}{1-x}, \frac{1-x^{j+2}}{1-x^2}, \dots \frac{1-x^{j+i}}{1-x^i}, \text{ that is, } (j+1), \frac{j+2}{2}, \dots \frac{j+i}{i}.$$

Hence $X_j = J_x$. Q.E.D.

The expansion of

$$(1-ax)(1-ax^2)\dots(1-ax^i)$$

in terms of powers of a may be verified in like manner.

It is not without interest to observe (if the remark has not been made before) how this development is connected by the principle of correspondence with the preceding one.

Throwing out by multiplication the factor $(1-a)$ in the denominator of Fx we obtain the reciprocal of

$$(1-ax)(1-ax^2)\dots(1-ax^i),$$

say $\frac{1}{Gx}$, under the form

$$1+\dots+\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i-1})}{(1-x)(1-x^2)\dots(1-x^{i-1})}x^j a^j+\dots.$$

Consequently the number of ways in which n can be divided into exactly j parts 1, 2, ... i (repetitions admissible) is the coefficient of x^n in the expansion according to ascending powers in x of the above multiplier of a^j.

But if any such partition be arranged in ascending order, and 0, 1, 2, ... $(j-1)$ be added (each to each) to its components, it is converted into a partition without repetitions, and by a converse process of subtraction each such partition is convertible into one of the former, but in which either repetition or non-repetition is allowable. Hence the free partitions of $n-\frac{j^2-j}{2}$ into j parts limited not to exceed $i-j+1$, have a one-to-one correspondence with the unrepetitional partitions of n into j parts limited not to exceed i, and must be equal to them in number. Hence the coefficient of a^j in $G(-x)$ may be deduced from that of a^j in $(Gx)^{-1}$ by multiplying the latter by $x^{\frac{1}{2}(j^2-j)}$ and changing i into $i-j+1$. Hence the general term in $G(-x)$ will be

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^i)}{(1-x)(1-x^2)\dots(1-x^{i-j})}\,x^{\frac{j^2+j}{2}}\,a^j$$

which is right.

When $i=\infty$ each of these developments (like a multitude of others, including the Theta-functions) may be obtained intuitively by the graphical method of points given in my communication to the Johns Hopkins Scientific Association at its last meeting; it remains a desideratum to apply the same method to the above two developments, or either of them, for the case of i*.

In the Ferrers, Franklin, Durfee-Sylvester and other conjugate systems of partitions, the partible number is the same for the corresponding partitions; in this last example (and the like will be shown to be the case in the graphical development of the Theta-function, and its generalizations), the one-to-one correspondence is between partitions of two different numbers.

* Not so—the result derived springs from the immediate application of a general logical principle as will hereafter be shown.