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Insensitivity of materials to the exchange of deformation paths

Part II. Formal theory of viscosity of materials

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The aim of the paper is to define precisely a common notion of material viscosity in mechanics. It is proposed to understand by notion of viscosity the dependence of the response of a material on acceleration, retardation and freezing of the deformation processes. This conception is precisely defined by a notion of the inviscosity semigroup of a material. On the basis of this definition the corresponing classification of materials is obtained. As limit cases, the totally viscid materials and the completely inviscid materials can be considered. The general form of constitutive operators for different types of viscosity is obtained.

Praca stanowi propozycję uściślenia powszechnie używanego w mechanice pojęcia lepkości materiału. Przez lepkość proponujemy rozumieć zależność reakcji materiału od przyspieszania, opóźniania, zatrzymywania procesów odkształcenia. Myśl tę uściśla pojęcie półgrupy nielepkości materiału. Otrzymano podstawy odpowiedniej klasyfikacji materiałów według lepkości. Przypadkami skrajnymi są materiały całkowicie lepkie i materiały zupełnie nielepkie. Otrzymano ogólną postać operatorów konstytutywnych dla różnych typów lepkości.

Работа является предложением уточнения широко используемого в механике понятия вязкости материала. Под вязкостью предлагается понимать зависимость реакции материала от ускорения, замедления, задержек путей деформирования. Эта мысль получает оформление в виде понятия полугруппы невязкости материала. Дана соответствующая классификация материалов по вязкости. Крайними случаями являются полностью вязкие и полностью невязкие материалы. Получены общие формулы для определяющих операторов соответствующих различным типам вязкости.

Contents. Part II

6. Introduction

IN PART I of the paper we have defined the notion of *insensitivity semigroup of material*.

The present Part II contains an analysis of insensitivity semigroups of special type. In fact, this is a proposition to make more precise the concept of viscosity of a material which is so commonly used in mechanics.

By viscosity of a material we propose to understand the dependence of response of a material upon acceleration, retardation, arresting of deformation processes. This notion can be made more precise by the use of a notion of *inviscosity semigroup of material*. That is a semigroup combined of all the exchanges of time-realizations (in the sense precisely defined in § 7) of deformation paths which have no influence on the final value of stress.

The study of a semigroup of mappings of the half axis of the past time onto itself leads to the correct classification of materials as regards viscosity. We have obtained a spectrum of materials at the ends of which there appear totaly viscid and completely inviscid materials, respectively.

A general form of constitutive operators, thus individualized classes of materials, has been obtained (Theorems I–VII).

Finally, we have considered such exchanges of time-realization of deformation paths as preserve unchanged a whole sequence of stresses, and not only the final stress.

7. Inviscosity semigroups. Foundations of classification of viscous materials

One type of insensitivity of materials forces itself particularly strongly. A typical way of testing a material response for a mechanical macroexperiment is the change of a test process in time. We describe two, the most simple and well known, examples. The first is the classical relaxation test, consisting in arresting deformations (a detailed description is given in § 17). This test can be presented as the following exchange of deformation path

$$f(s) \to f[\sigma_q(s)], \tag{7.1}$$

where

$$\sigma_q(s) \equiv \begin{cases} 0 & \text{for} \quad s \in [0, q), \\ s-q & \text{for} \quad s \in [q, \infty). \end{cases} \tag{7.2}$$

The second example concerns the influence of stretching upon stresses. We are considering here a material response to exchanges of the type

$$f(s) \to f[\sigma(s)], \tag{7.3}$$

where

$$\sigma(s) \equiv as, \quad a > 0. \tag{7.4}$$

For $a < 1$ this corresponds to retardation, and for $a > 1$ to acceleration of the deformation process.

Thus, there arises a problem of response description, and, in particular, the description of insensitivity of a material to exchanges of the type:

$$f \to f \circ \sigma, \quad \sigma : \mathrm{R} \to \mathrm{R}. \tag{7.5}$$

This problem has been fragmentarily raised in many papers, among others in [3, 7, 24, 25, 26 and 27]. Materials have been studied which did not respond to *any* reasonable exchanges (7.5). Such materials have been called variously: scleronomic [1, 2], independent of natural time [24], rate-independent [26]. To examine systematically the whole question is the purpose of our paper.

Consider the entire set of mapping of R into itself,

$$\Psi \equiv \{\sigma\colon \mathrm{R} \to \mathrm{R} |\ \mathrm{Dom}\,\sigma = \mathrm{R}\}. \tag{7.6}$$

This represents a semigroup with respect to composition. This semigroup contains the right-hand side as well as the left-hand side identity ι and the zero o,

$$\iota(s) \equiv s, \quad o(s) \equiv 0. \tag{7.7}$$

To every $\sigma \in \Psi$, we assign an exchange of deformation paths $e(\sigma) \in \Gamma$ according to the rule $f{*}e(\sigma) \equiv f \circ \sigma$. It is obvious, that $e(\sigma \circ \mu) = e(\sigma) \circ e(\mu)$; thus $e\colon \Psi \to \Gamma$ is a homomorphic imbedding. In whose follows, to simplify language and notation, we shall identify σ and $e(\sigma)$, Ψ and $e(\Psi)$ — i.e., we shall consider Ψ as a subsemigroup of the semigroup Γ.

The studying of invariance with respect to the whole semigroup Ψ would not be realistic. Since we do not wish to restrict our considerations to particular or trivial processes, we reject pathological constitutive operators such as (2.24), and assume the consequent restrictions (cf. [7], § 99, [26]).

By not allowing the possibility of changing the order of deformation states $f(s)$ under the exchange $f \to f \circ \sigma$, we assume that

(7.8) σ is a monotone non-decreasing function.

By not allowing the possibility of removal from the deformation path of any of its parts — interior, initial, end part — we assume that

(7.9) σ is a continuous function,

$$\sigma(0) = 0, \tag{7.10}$$

$$\mathrm{Range}\,\sigma = \mathrm{Dom}\,\sigma = \mathrm{R}. \tag{7.11}$$

Recapitulating. We confine our considerations to exchanges of the form:

$$f \to f \circ \sigma, \quad \sigma \in \Sigma, \tag{7.12}$$

where Σ is a subsemigroup in Ψ defined by

$$\Sigma \equiv \{\sigma \in \Psi |\ \sigma \text{ satisfies (7.8)–(7.11)}\}. \tag{7.13}$$

Exchanges (7.13) are called, in what follows, exchanges of *time-realization* of a deformation path. Figure 1 illustrates the exchanges of time-realization for a one-dimensional process:

$$f(s) = l(s)\mathbf{P}, \quad \mathbf{P} = \mathrm{const} \in \mathcal{N}, \quad l\colon [0, \infty) \to (-\infty, +\infty). \tag{7.14}$$

Motivation of the subsequent procedure is as follows. Practice shows that non-rigorous language is indispensable in life and to the development of a scientific discipline. In a mature knowledge there takes place a translation of words from non-rigorous to a formalized language. Words like: *elasticity*, *viscosity*, *plasticity* are part of the permanent vocabulary of mechanics of materials. But only the first word — i.e., elasticity — has for long possess-ed a precise meaning — i.e., it is formalized by (two) definitions (cf. e.g. [7], §§ 43, 82). The remaining two words play their important, heuristic role, as part of non-rigorous language. The word viscosity is used with particular readiness. It is usually given an intuitive sense, referring to the Newtonian viscid fluid. We propose to formalize the word viscosity, taking as the point of departure the following non-rigorous description: viscosity is a dependence of material response to time-realization of a deformation path. The next part

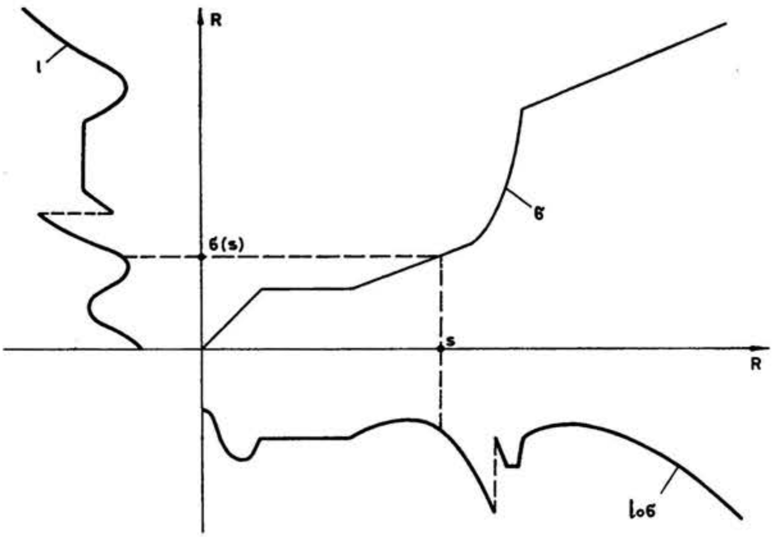

FIG. 1 An exchange of time-realisation of one-dimensional process (7.14).

of our paper is devoted to making the above formulation more rigorous and to deducing corresponding corollaries. Thus the main subject of our considerations will constitute insensitivity semigroups of the type

$$\Omega(\mathbf{K}, \mathscr{D}, \Lambda), \quad \Lambda \subset \Sigma_{\mathscr{D}} \subset \Sigma \subset \Psi. \tag{7.15}$$

Let us begin with the following two results.

THEOREM 5. *An insensitivity semigroup of the type $\Omega(\mathbf{K}, \mathscr{D}, \Lambda)$, $\Lambda \subset \Psi$, contains the zero element o iff the material behaves as an elastic one on $\mathscr{D}$.*

Proof. $(f \circ o)(s) = f(0)$ for every $f \in \mathscr{D}$, Q.E.D.

THEOREM 6. *An insensitivity semigroup of the type* (7.15) *is independent of a reference configuration* $\mathbf{K}$.

Proof. $(f \circ \sigma)\mathbf{P} = (f\mathbf{P}) \circ \sigma$ for every $f \in \mathscr{D}$ Q.E.D.

In order to emphasize the explicit dependence of a semigroup of a material, we introduce the notation:

$$\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}) \equiv \Omega(\mathbf{K}, \mathscr{D}, \Sigma_{\mathscr{D}}), \quad \mathscr{D} \subset \mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}}, \quad \Sigma_{\mathscr{D}} \subset \Sigma_{\mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}}}, \tag{7.16}$$

$$\Phi_{\mathfrak{F}_{\mathbf{K}}} \equiv \Phi(\mathfrak{F}_{\mathbf{K}}, \mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}}). \tag{7.17}$$

DEFINITION 4. *An insensitivity semigroup $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D})$ is said to be an* ***inviscosity semigroup of a material*** *$\mathfrak{F}_{\mathbf{K}}$ relative to a class of deformation paths $\mathscr{D}$. The insensitivity semigroup $\Phi_{\mathfrak{F}_{\mathbf{K}}}$ is called a* ***specific*** *inviscosity semigroup of the material.*

Explicitly: the inviscosity semigroup $\Phi(\mathfrak{F}_K, \mathscr{D})$ constitute all those exchanges of time-realization of deformation paths $\sigma \in \Sigma_{\mathrm{Dom}\,\mathfrak{F}}$, which do not lead out from $\mathscr{D}$,

$$\mathscr{D} \ni f \to (f \circ \sigma) \in \mathscr{D} \tag{7.18}$$

and do not change the material response

$$\mathfrak{F}_K[f \circ \sigma] = \mathfrak{F}_K[f] \tag{7.19}$$

in an arbitrary process $f \in \mathscr{D}$.

The definition assumed indicates that various types of viscosity are possible, corresponding to different subsemigroups in the semigroup Σ. We introduce, therefore, the following definition.

DEFINITION 5. *A relation on the set of all materials and all classes of deformation paths, defined by*

$$\Phi(\mathfrak{F}_K^1, \mathscr{D}^1) \subset \Phi(\mathfrak{F}_K^2, \mathscr{D}^2) \tag{7.20}$$

is called a ***quasi-ordering*** *(of materials and processes)* ***with respect to viscosity.***

It is obvious that a quasi-ordering with respect to viscosity is a relation of quasi-order [12]. Its extent is defined by the following inclusions:

$$\{\iota\} \subset \Phi_{\mathfrak{F}_K} \subset \Phi(\mathfrak{F}_K, \mathscr{D}) \subset \Sigma_{\mathscr{D}} \subset \Sigma_{\mathrm{Dom}\,\mathfrak{F}_K} \subset \Sigma. \tag{7.21}$$

Of course, it contains numerous comparisons which are not interesting from an experimental viewpoint. We shall not, however, deal with more extensive activity of the creation of definitions, the passing to factor relations, and so on. Let us mention the most reasonable cases only:

A. Restriction of the quasi-ordering with respect to viscosity to the set $\mathscr{M} \times \{\mathscr{D}\}$, where $\mathscr{M}$ is the set of all materials with the property $\mathrm{Dom}\,\mathfrak{F}_K \supset \mathscr{D}$. This corresponds to the comparison viscosities of different materials on a given class of deformation paths $\mathscr{D}$.

B. Restriction of the quasi-ordering to the set $\{\mathfrak{F}_K\} \times 2^{\mathrm{Dom}\,\mathfrak{F}_K}$. This corresponds to the comparison of viscosity of a given material $\mathfrak{F}_K$ on different classes of deformation paths.

C. Restriction of the quasi-ordering to the set of pairs $(\mathfrak{F}_K, \mathrm{Dom}\,\mathfrak{F}_K)$. This corresponds to the comparison of specific inviscosity subsemigroups $\Phi_{\mathfrak{F}_K}$ for different materials.

The following concepts are also useful:

DEFINITION 6. *We say that a material is*:

totally viscid on $\mathscr{D}$,	if	$\Phi(\mathfrak{F}_K, \mathscr{D}) = \{\iota\}$,
totally viscid,	if	$\Phi_{\mathfrak{F}_K} = \{\iota\}$,
completely inviscid on $\mathscr{D}$,	if	$\Phi(\mathfrak{F}_K, \mathscr{D}) = \Sigma_{\mathscr{D}}$,
completely inviscid,	if	$\Phi_{\mathfrak{F}_K} = \Sigma_{\mathrm{Dom}\,\mathfrak{F}_K}$.

The situation visualizes the following scheme:

t. v. on $\mathscr{D}$ — c. i. on $\mathscr{D}$

$$\{\iota\} \subset \Phi_{\mathfrak{F}_K} \subset \Phi(\mathfrak{F}_K, \mathscr{D}) \subset \Sigma_{\mathscr{D}} \subset \Sigma_{\mathrm{Dom}\,\mathfrak{F}_K} \subset \Sigma$$

t. v. — c. i.

A totally viscid material on a certain class $\mathscr{D}$ is a totally viscid material, but a totally viscid material must not necessarily be a totally viscid material on an arbitrarily chosen class $\mathscr{D}$. A completely inviscid material is a completely inviscid material on every class $\mathscr{D}$, but a completely inviscid material on a certain class $\mathscr{D}$ may not be a completely inviscid material.

The importance of the foregoing statements follows from the fact that in a physical experiment one can study inviscosity semigroups $\Phi(\mathfrak{F}_K, \mathscr{D})$ only.

To obtain an essential description of different types of inviscosity, we must have at our disposition the description of subsemigroups of the semigroup Σ. The author, unfortunately, was unsuccessful in finding any trace of any special study of the semigroup. By not undertaking such considerations in the present paper, we confine our studies to a selection of certain subsemigroups which seem to be of interest from the viscosity point of view. These are:

I. The group of uniform retardation or acceleration:

$$L \equiv \{\sigma \in \Sigma | \ \sigma(s) = as, \quad a > 0\}. \tag{7.22}$$

II. The semigroup of arrestings:

$$\mathrm{P} \equiv \{\sigma \in \Sigma | \ \sigma'(s) = 1 \text{ or } 0 \text{ almost everywhere}\}. \tag{7.23}$$

III. The semigroup:

$$\mathrm{S} \equiv \{\sigma \in \Sigma | \ \sigma'(s) = a \text{ or } 0 \text{ almost everywhere}\}. \tag{7.24}$$

IV. The semigroup Σ.

V. The group of bijections:

$$\mathrm{B} \equiv \{\sigma \in \Sigma)| \ \sigma(s_1) = \sigma(s_2) \quad \text{iff} \quad s_1 = s_2\}. \tag{7.25}$$

VI. The instantaneous semigroup:

$$\mathrm{I} \equiv \{\sigma \in \Sigma | \ \sigma'(0) = 1\}. \tag{7.26}$$

VII. The semigroup:

$$\mathrm{Q} \equiv \{\sigma \in \Sigma | \ 0 < \sigma'(0) < \infty\}. \tag{7.27}$$

It is obvious that every one of these subsets of Σ is a subsemigroup of Σ. The only reason for writing down the foregoing semigroups in the order given was the convenience of presenting proofs in the same order to the forthcoming theorems I–VII, on the invariance of constitutive operators. The reciprocal position of these semigroups with respect to the relation of set theory inclusion $\subset$ is shown in Fig. 2. Particularly interesting from a viscosity viewpoint are semigroups L, Σ, I. The semigroup P possesses an important auxiliary significance.

In addition to the subsemigroups mentiond above, we may indicate many others of Σ, for instance:

$$\{\sigma \in \Sigma | \ \sigma \text{ is a polynomial}\} \subset \mathrm{B}, \tag{7.28}$$

$$\{\sigma \in \Sigma | \ \sigma(s) = s^a, \ a > 0\} \subset \mathrm{B}, \tag{7.29}$$

$$\{\sigma \in \Sigma | \ \sigma(s) \leqslant s \text{ for every } s \in \mathrm{R}\} \subset \mathrm{P}, \tag{7.30}$$

etc. Note that all semigroups of the catalogue I–VII satisfy the following natural condition:

(7.31) for every $s_1 > s_2 > 0$ there exists, in any one of the semigroups, σ such that $\sigma(s_1) = s_2$, and if $s < s_1$ then $\sigma(s) < s_2$, if $s > s_1$ then $\sigma(s) > s_2$.

The semigroup

$$\{\sigma \in \Sigma \mid \quad \sigma(s) = s \text{ for } s \in [0, a)\}, \tag{7.32}$$

where a is a fixed positive number, does not satisfy that condition. We shall not study this and similar situations, although they are not to be rejected in advance.

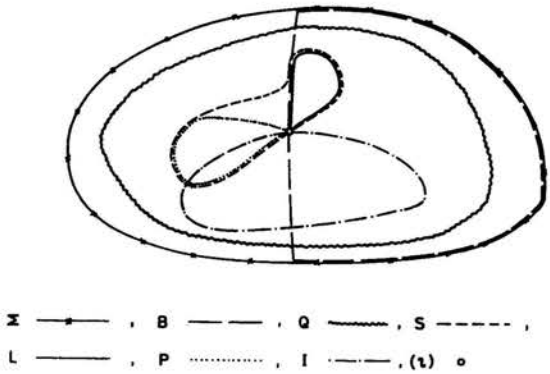

Fig. 2. Relations between the semigroups (7.22)–(7.27).

In the sequel, we shall be considering the semigroup $\Phi(\mathfrak{F}_K, \mathscr{D})$, and hence also the semigroups

$$L_{\mathscr{D}}, P_{\mathscr{D}}, S_{\mathscr{D}}, \Sigma_{\mathscr{D}}, B_{\mathscr{D}}, I_{\mathscr{D}}, Q_{\mathscr{D}}. \tag{7.33}$$

These semigroups are, in general, strongly restricted by the class $\mathscr{D} \subset \mathscr{A}$. In particular, they will not, as a rule, contain pathological mapping which enter into I–VII (such as, for instance, a mapping which duplicates many times the mapping of the interval [0, 1] onto itself with derivative equal to zero almost everywhere).

Finally, let us notice the following result.

THEOREM 7. *Every inviscosity semigroup $\Phi(\mathfrak{F}_K, \mathscr{D})$ is an exact one.*

Proof. The proof follows directly from the assumption $\sigma(0) = 0$; thus $(f * \pi) \circ \sigma = (f \circ \sigma) * \pi$ for every $f \in \mathscr{A}$, $\sigma \in \Sigma$. Q.E.D.

This result can be much strengthened. Introducing insensitivity semigroups, we took as a point of departure the operator $\mathfrak{F}_K$ from the formula:

$$\mathbf{T}(t) = \mathfrak{F}_K[\mathbf{F}^{(t)}(s)], \tag{7.34}$$

where $\mathbf{F}^{(t)}(s) \equiv \mathbf{F}(t-s)$. It is known from the principle of material objectivity (2.5) that the formula (7.34) can be written in many different ways, for instance;

$$\hat{\mathbf{T}}(t) = \mathfrak{G}_K[\mathbf{U}^{(t)}(s)], \tag{7.35}$$

where

$$\hat{\mathbf{T}}(t) \equiv \mathbf{R}^T(t)\mathbf{T}(t)\mathbf{R}(t), \quad \mathbf{U}^{(t)}(s) \equiv \{[\mathbf{F}^{(t)}(s)]^T[\mathbf{F}^{(t)}(s)]\}^{1/2},$$

or

$$\hat{\mathbf{T}}(t) = \mathfrak{R}_{\mathbf{K}}[\hat{\mathbf{U}}^{(t)}_{(t)}(s), \mathbf{U}(t)], \tag{7.36}$$

where

$$\hat{\mathbf{U}}^{(t)}_{(t)}(s) \equiv \mathbf{R}^T(t)\mathbf{U}^{(t)}_{(t)}(s)\mathbf{R}(t), \quad \mathbf{U}^{(t)}_{(t)}(s) \equiv \{[\mathbf{F}^{(t)}_{(t)}(s)]^T[\mathbf{F}^{(t)}_{(t)}(s)]\}^{1/2}, \quad \mathbf{F}^{(t)}_{(t)}(s) \equiv \mathbf{F}^{(t)}(s)\mathbf{F}^{-1}(t)$$

(cf. [7], § 29). It is obvious, that the *inviscosity semigroup is independent of the manner of writing the constitutive equation*. More precisely: the inviscosity semigroup of material is the insensitivity semigroup with respect to the exchanges of arguments corresponding to the transformation of the past time $s \to \sigma(s)$, $\sigma \in \Sigma$, of each of the constitutive operators describing this material, in particular — the operators (7.34). (7.35), (7.36).

For this reason, in the *entire subsequent text of the paper, by a constitutive operator we shall understand any one of the operators* $\mathfrak{F}_{\mathbf{K}}$, $\mathfrak{G}_{\mathbf{K}}$, $\mathfrak{R}_{\mathbf{K}}$, ..., *and by a deformation path f a corresponding tensor curve parametrized by s*, $\mathbf{F}^{(t)}(s)$, $\mathbf{U}^{(t)}(s)$, $\mathbf{U}^{(t)}_{(t)}(s)$, ...

Remark 3. As a matter of fact, the concept of an inviscosity semigroup refers, in the remainder of the paper, without any change to an arbitrary situation, be it physical, economical or biological, etc; in which a certain quantity **T** at the "instant" t depends on a "history" $f(t-s)$, $0 \leqslant s < \infty$, of some other quantity f up to the "instant" t. For instance, in the language of cybernetics, an operator will assume the term "system" (of special type), the quantity **T** will be an "output", and the quantity f will be an "input". Only for the sake of concreteness we use, in what follows, the language of mechanics, calling an operator a material, **T** — stress, and f — deformation path.

Remark 4. Taking a certain standard bijection $\varkappa:[0, \infty) \to [0, 1)$, we can pass to a semigroup of mappings of the interval $[0, 1)$ onto itself. This will not be very convenient in our case.

8. First introductory example: special classes of processes

For the simplest class of deformation paths, the phenomenon of inviscosity can be readily described, without referring to complicated general analysis.

Consider, for instance, the class $\mathscr{D}_1$ of the following deformation paths:

$$f(s) = \mathbf{F}+(\mathbf{1}-\mathbf{F})\mathrm{H}(s-q) \equiv \mathbf{F}^{h(s-q)}. \tag{8.1}$$

where $q > 0$, $\mathbf{F} = \mathrm{const} \in \mathscr{N}$, H is a step function

$$\mathrm{H}(x) \equiv \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad x \geqslant 0, \end{cases} \quad h(x) \equiv 1-\mathrm{H}(x). \tag{8.2}$$

A material remains in rest in a configuration **K** up to the instant $t-q$; at the instant $t-q$ it is suddenly deformed to another configuration **FK**, and again remains in rest up to the instant t, under consideration.

Every $f \in \mathscr{D}_1$ can be identified with a pair $(\mathbf{F}, q)$ and, therefore, for arbitrary material

$$\mathbf{T} = \mathfrak{F}_{\mathbf{K}}[f] = h_{\mathbf{K}}(\mathbf{F}, q) \tag{8.3}$$

on the class $\mathscr{D}_1$. The material respond to a single jump of deformation in a manner similar to an elastic material (for which $\mathbf{T} = l_{\mathbf{K}}(\mathbf{F})$), with the only difference that it remembers the time distance q This obvious but important fact was noted in [7].

This result can be strengthened in the following way.

THEOREM 8. *If the inviscosity semigroup of material $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}_1)$ satisfies the natural condition* (7.31), *then the material response on the class of jumps $\mathscr{D}_1$ is elastic — i.e.,*

$$\mathbf{T} = l_{\mathbf{K}}(\mathbf{F}). \tag{8.4}$$

Proof. In fact, let us take arbitrary $p > q > 0$. According to the condition (7.31), $\sigma(p) = q$ for some $\sigma \in \Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}_1)$, and if $s < p$, then $\sigma(s) < q$, if $s > p$, then $\sigma(s) > q$.

Hence

$$(f \circ \sigma)(s) = \mathbf{F} + (\mathbf{1} - \mathbf{F})\mathrm{H}[\sigma(s) - q] = \mathbf{F} + (\mathbf{1} - \mathbf{F})\mathrm{H}(s - p). \tag{8.5}$$

Since $\sigma \in \Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}_1)$, then, due to (7.6), (8.3), we obtain $h_{\mathbf{K}}(\mathbf{F}, q) = h_{\mathbf{K}}(\mathbf{F}, p)$ for every $p, q > 0$, and thus (8.4) follows. Q.E.D.

Thus in the class $\mathscr{D}_1$ it is impossible to verify either that the material is elastic or that it possesses only a non-trivial inviscosity semigroup.

Let us consider the class $\mathscr{D}_2$ of all jump-constant processes(¹):

$$f(s) = \mathbf{F} + \sum_{i=1}^{N} \mathbf{G}_i \mathrm{H}(s - s_i), \tag{8.6}$$

where $\mathbf{G}_i \neq \mathbf{0}$ is the magnitude of the jump of deformation gradient at the instant $t - s_i$, $0 < s_1 < \ldots < s_N < \infty$. Identifying $f \in \mathscr{D}_2$ with the aggregate $(\mathbf{F}; \mathbf{G}_1, \ldots, \mathbf{G}_N; s_1, \ldots, s_N)$, we have for an arbitrary material:

$$\mathbf{T} = h_{\mathbf{K}}(\mathbf{F}; \mathbf{G}_1, \ldots, \mathbf{G}_N; s_1, \ldots, s_N) \quad \text{on } \mathscr{D}_2. \tag{8.7}$$

Consider the inviscosity on $\mathscr{D}_2$. Since $\Sigma_{\mathscr{D}_2} = \mathrm{B}$, and the values $\sigma'(0)$ are unimportant, then in the frame work of our catalogue (cf. FIG. 2), it is reasonable to consider three cases only:

1. $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}_2) = \{\iota\}$, totally viscid materials on $\mathscr{D}_2$,
2. $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}_2) \supset \mathrm{L}$, materials unresponsive to uniform retardation-acceleration of processes from $\mathscr{D}_2$,
3. $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}_2) = \mathrm{B}$, completely inviscid materials on $\mathscr{D}_2$.

In the first case, the formula (8.7) cannot be simplified.

For the second case, taking into account $\sigma(s) = s_1 s$, we have $\mathrm{H}[\sigma(s) - s_i) = \mathrm{H}(s - s_i/s_1)$,

$$\mathbf{T} = h_{\mathbf{K}}(\mathbf{F}; \mathbf{G}_1, \ldots, \mathbf{G}_N; 1, s_2/s_1, \ldots, s_N/s_1). \tag{8.8}$$

It is not difficult to ascertain that the inverse holds true also — i.e., if (8.8) holds, then $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}_2) \supset \mathrm{L}$.

(¹) Introducing the notation $\mathbf{F}_i \equiv \mathbf{F} + \sum_{k=1}^{i} \mathbf{G}_k$, $\mathbf{F}_0 \equiv \mathbf{F}$, $\mathbf{H}_i \equiv \mathbf{F}_{i-1}\mathbf{F}^{-1}$, $\mathbf{H} \equiv \mathbf{F}_N$, we can write (8.6) in the form

$$f(s) = \mathbf{H}_1^{H(s-s_1)} \mathbf{H}_2^{H(s-s_2)} \ldots \mathbf{H}_N^{H(s-s_N)} \mathbf{H}$$

which reflects the succession of superposition of additional deformation upon a previous state.

In the third case we see that, since for every two sequences $0 < s_1 < \ldots < s_N$, $0 < \hat{s}_1 < \ldots \ldots < \hat{s}_N$ there exists $\sigma \in \mathrm{B}$ such that $\sigma(s_i) = \hat{s}_i$, then we have $h_{\mathbf{K}}(\mathbf{F}; \mathbf{G}_1, \ldots; s_1, \ldots) = h_{\mathbf{K}}(\mathbf{F}; \mathbf{G}_1, \ldots; \hat{s}_1, \ldots)$ and hence

$$\mathbf{T} = g_{\mathbf{K}}(\mathbf{F}; \mathbf{G}_1, \ldots, \mathbf{G}_N). \tag{8.9}$$

Thus we have obtained the following result.

THEOREM 9. 1. *A totally viscid material can remember the entire information of a jump-constant process, formula* (8.7);

2. *A material which is insensitive to uniform increase or decrease of stretching can remember at most the actual value of deformation* $\mathbf{F}$, *the magnitude and order of jumps* $\mathbf{G}_i$, *and the proportions of time distances* s_i/s_1, *formula* (8.8);

3. *A completely inviscid material can remember only* $\mathbf{F}$ *and the magnitude and order of jumps* $\mathbf{G}_i$, *formula* (8.9).

In particular, on rectangular impulses:

$$f(s) = \mathbf{1} + \mathbf{G}[\mathrm{H}(s-q) - \mathrm{H}(s-q+l)], \tag{8.10}$$

where $q, l > 0$, *we shall have*

1. $\mathbf{T} = k_{\mathbf{K}}(\mathbf{G}, q, l) \equiv h_{\mathbf{K}}(\mathbf{1}; \mathbf{G}, -\mathbf{G}; q, q+l)$,
2. $\mathbf{T} = l_{\mathbf{K}}(\mathbf{G}, l/q) \equiv h_{\mathbf{K}}(\mathbf{1}; \mathbf{G}, -\mathbf{G}; 1, 1+l/q)$,
3. $\mathbf{T} = m_{\mathbf{K}}(\mathbf{G}) \equiv g_{\mathbf{K}}(\mathbf{1}; \mathbf{G}, -\mathbf{G})$.

9. Second introductory example: of special class of materials

Previously, we considered a special class of processes for all materials. This time we shall consider all processes for a special class of materials.

Our considerations of insensitivity semigroups, and in particular, inviscosity semigroups can be considerably simplified for classes of materials individualized in a formal way by a special type of constitutive operators. Consider, for example, *materials of differential type*.

We assume then, (cf. [7], § 35), that a constitutive operator $\mathfrak{R}_{\mathbf{K}}$ in the formula:

$$\hat{\mathbf{T}} = \mathfrak{R}_{\mathbf{K}}[\hat{\mathbf{U}}_{(t)}^{(t)}(s), \mathbf{U}(t)] \tag{9.1}$$

is of the form:

$$\hat{\mathbf{T}} = q_{\mathbf{K}}(\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N; \mathbf{U}), \tag{9.2}$$

where

$$\mathbf{D}_i \equiv (-1)^i \frac{d^i}{ds^i} \mathbf{U}_{(t)}^{(t)}(s)_{s=0}, \quad \hat{\mathbf{D}}_i \equiv \mathbf{R}^T(t)\, \mathbf{D}_i \mathbf{R}(t) \tag{9.3}$$

— i.e., $\mathbf{D}_1 = \mathbf{D}$ is a stretching tensor, ..., $\mathbf{D}_n$ is the n-th stretching tensor. The class of materials under consideration is sufficiently large. All classical models of fluids (ideal, Newtonian viscid, non-Newtonian viscid) are included in this class, some models of solids and their various generalizations.

The exchanges $s \to \sigma(s)$ leave unchanged the actual quantities $\mathbf{R}(t)$, $\mathbf{U}(t)$. We agree to consider $\mathbf{R}, \mathbf{U}$ in formula (9.2) as fixed parametric tensors and have in mind an implicit supplement to all further statements: for every $\mathbf{R}, \mathbf{U}$. According to (9.2) and (2.6), the set C of tensor curves $f(s) = \mathbf{U}^{(t)}_{(t)}(s)$ N-times differentiable([2]) for each s, $f(0) = \mathbf{1}$, is the domain of the operator $\mathfrak{R}_\mathbf{K}$. Hence a set of permissible mappings Σ_C is the set of mappings σ N-times differentiable for every s.

The domain of tensor function $q_\mathbf{K}$ from the formula (9.2) is the Cartesian product $\mathcal{T}^N \equiv \mathcal{T} \times \ldots \times \mathcal{T}$ (N times). Let us take the mapping

$$d\colon \mathrm{C} \to \mathcal{T}^N, \quad d(\hat{\mathbf{U}}^{(t)}_{(t)}) \equiv (\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N). \tag{9.4}$$

To every $\sigma \in \Sigma_\mathrm{C}$ we assign a mapping $l(\sigma)$: $\mathcal{T}^N \to \mathcal{T}^N$ according to the formula:

$$l(\sigma)[d(f)] \equiv d(f \circ \sigma), \tag{9.5}$$

for every $f \in \mathrm{C}$. Using the definition of d and the formula (9.3), we have:

$$l(\sigma)[\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N] = [\hat{\mathbf{D}}_1^*, \ldots, \hat{\mathbf{D}}_N^*], \tag{9.6}$$

where

$$\begin{aligned} \hat{\mathbf{D}}_1^* &= a_1 \hat{\mathbf{D}}_1, \\ \hat{\mathbf{D}}_2^* &= a_1^2 \hat{\mathbf{D}}_2 - a_2 \hat{\mathbf{D}}_1, \\ \hat{\mathbf{D}}_3^* &= a_1^3 \hat{\mathbf{D}}_3 - 3a_1 a_2 \hat{\mathbf{D}}_2 + a_3 \hat{\mathbf{D}}_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{9.7}$$

$$a_i \equiv \frac{d^i}{ds^i} \sigma(s)|_{s=0}, \quad i = 1, \ldots, N. \tag{9.8}$$

Thus $l(\sigma)$ is a linear transformation of linear space $\mathcal{T}^N$ of dimension $6N$ into itself; this is a transformation of a very special form (9.7), uniquely defined by the N-tuple of numbers $(a_1, \ldots, a_N)$, here $a_1 \geqslant 0$ [cf. condition (7.10)]. It is not difficult to verify that

$$l(\sigma \circ \mu) = l(\mu) l(\sigma), \tag{9.9}$$

where, on the right-hand side, we have a composition of linear transformations $l(\mu)$, $l(\sigma)$ (corresponding to the product of matrices $6N \times 6N$, if some basis $\mathbf{e}_i$ is assumed) — i.e., l is a homomorphism of the semigroup Σ_C into the semigroup $\mathcal{L}(\mathcal{T}^N)$ of linear transformations $\mathcal{T}^N$ into itself.

It is obvious that for materials of differential type and complexity N, the mapping σ belongs to the inviscosity semigroup $\Phi_{\mathfrak{R}_\mathbf{K}}$, iff the linear transformation $l(\sigma)$ corresponding to it satisfies the condition:

$$q_\mathbf{K}(\hat{\mathbf{D}}_1^*, \ldots, \hat{\mathbf{D}}_N^*; \mathbf{U}) = q_\mathbf{K}(\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N; \mathbf{U}) \tag{9.10}$$

for every $(\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N) \in d(\mathrm{Dom}\, \mathfrak{R}_\mathbf{K})$, where $\hat{\mathbf{D}}_i^*$ are defined by (9.7).

In what follows we confine our considerations, for reasons of simplicity, to the case in which $\mathrm{Dom}\, q_\mathbf{K} \equiv d(\mathrm{Dom}\, \mathfrak{R}_\mathbf{K}) = \mathcal{T}^N$. First, notice the following result: singular transformations satisfy the condition (9.10) iff the material is elastic.

(In other words, every element $\sigma \in \Phi_{\mathfrak{R}_\mathbf{K}}$ such that $l(\sigma)$ is singular is equivalent to the element o). In fact, it is not difficult to verify that $l(\sigma)$ is singular iff $a_1 \equiv \sigma'(0) = 0$ [cf.

([2]) Strictly: N-times differentiable from the right-hand side for every s; we shall not use this requirement.

formulae (9.7)]. Now, setting in (9.10) $a_1 = 0$, we see that if this condition is satisfied on the entire space $\mathscr{T}^N$, then there exists a function $r_K: \mathscr{T}^M \to \mathscr{T}$, $M < N$ such that $q_K(\mathbf{D}_1, \ldots, \mathbf{D}_N; \mathbf{U}) = r_K(\mathbf{D}_1, \ldots, \mathbf{D}_M; \mathbf{U})$ — i.e., the material is of smaller complexity than N. Applying this procedure further on, we finally obtain $q_K(\mathbf{D}_1, \ldots, \mathbf{D}_N; \mathbf{U}) = h_K(\mathbf{U})$.

Recapitulating. The studying of inviscosity semigroups of material of the differential type and complexity N is equivalent to the studying of insensitivity of the constitutive tensor function $q_K: \mathscr{T}^N - \mathscr{T}$ from formula (9.2) with respect to the group $\Lambda \subset GL(\mathscr{T}^N) \subset \mathscr{L}(\mathscr{T}^N)$ of special non-singular linear transformations $\mathscr{T}^N$ onto itself, defined by the formulae (9.7) with $a_1 \neq 0$.

Reviewing the catalogue of semigroups in Fig. 2, we see that it is reasonable to consider the following three cases only:

1. Inviscosity with regard to the group

$$\Lambda_L \equiv \{l(\sigma) \in \Lambda| \quad (a_1, \ldots, a_N) = (a, 0, \ldots, 0), a > 0\}$$

2. Inviscosity with regard to the group Λ;
3. Inviscosity with regard to the group

$$\Lambda_I \equiv \{l(\sigma) \in \Lambda| \quad (a_1, \ldots, a_N) = (1, a_2, \ldots, a_N)\}.$$

THEOREM 10. *An inviscosity group of material of the differential type contains the group* Λ_L, *i.e.*,

$$q_K(a\hat{\mathbf{D}}_1, a^2\hat{\mathbf{D}}_2, \ldots, a^N\hat{\mathbf{D}}_N; \mathbf{U}) = q_K(\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N; \mathbf{U}) \tag{9.11}$$

for all $a > 0$ *and all* $(\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N) \in \mathscr{T}^N$ *iff*

$$\mathbf{T} = \begin{cases} q_K(\mathfrak{c}^{-1}\hat{\mathbf{D}}_1, \ldots, \mathfrak{c}^{-N}\hat{\mathbf{D}}_N; \mathbf{U}) & \textit{if there exists } 1 \leqslant i \leqslant \mathrm{N}, \textit{ such that } \mathbf{D}_i \neq \mathbf{0}, \\ h_K(\mathbf{U}) & \text{if} \quad \mathbf{D}_i = \mathbf{0} \quad \text{for every } 1 \leqslant i \leqslant N, \end{cases} \tag{9.12}$$

where $\mathfrak{c}$ *is a functional on* $\mathscr{T}^N$ *defined by*

$$\begin{gathered} \mathfrak{c}(\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N) \equiv |\hat{\mathbf{D}}_M|^{1/M} = (\operatorname{tr} \mathbf{D}_M \mathbf{D}_M)^{1/2M}, \\ M \equiv \min(k | \mathbf{D}_k \neq \mathbf{0}). \end{gathered} \tag{9.13}$$

P r o o f. If a constitutive equation is of the form (9.12) — i.e., $q_K(\hat{\mathbf{D}}_1, \ldots) \equiv q_K[(\mathfrak{c}(\hat{\mathbf{D}}_1, \ldots))^{-1}\hat{\mathbf{D}}_1, \ldots]$, then taking in lieu of $(\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N)$ the aggregate $(a\hat{\mathbf{D}}_1, \ldots, a^N\hat{\mathbf{D}}_N)$ and making use of

$$\mathfrak{c}(a\hat{\mathbf{D}}_1, \ldots, a^N\hat{\mathbf{D}}_N) = a\mathfrak{c}(\hat{\mathbf{D}}_1, \ldots, \hat{\mathbf{D}}_N), \tag{9.14}$$

we obtain the condition (9.11). Inversely, setting in (9.11) $a = (\mathfrak{c}(\ldots))^{-1}$, we obtain (9.12). Q.E.D.

This theorem is a particular case of Theorem I.

In particular, for materials of complexity one, $\mathrm{N} = 1$, $\Lambda_L = \Lambda$, thus:

A material of differential type and complexity one is completely inviscid iff

$$\hat{\mathbf{T}} = \begin{cases} q_K(\mathbf{D}/|\mathbf{D}|; \mathbf{U}) & \text{if} \quad \mathbf{D} \neq \mathbf{0}, \\ h_K(\mathbf{U}) & \text{if} \quad \mathbf{D} = \mathbf{0}. \end{cases} \tag{9.15}$$

This corollary is constantly used in the theory of ideal plasticity. When $\hat{\mathbf{T}}$ is independent of $\mathbf{U}$, then every semi-line $\{\mathbf{D} \in \mathscr{T}_D | \hat{\mathbf{D}} = a\hat{\mathbf{D}}^0, a > 0, |\mathbf{D}^0| = 1\}$ of the space of stretchings $\mathscr{T}_D$ maps itself into a single point $q_K(\mathbf{D}^0)$ in the space of stress $\mathscr{T}_T$; then the following condition takes place $\varphi_K(\mathbf{T}) = 0$, which is called a *yield condition*.

The following result, for reasons of simplicity only, we state for $N = 3$. It can be obtained, identically, for arbitrary N.

THEOREM 11. *A material of differential type and complexity three is completely inviscid — i.e., condition* (9.10) *is satisfied for all* $(a_1, \ldots, a_N)$, $a_1 > 0$, *iff*

(i) *for* $\mathbf{D}_1 \neq \mathbf{0}$

$$\hat{\mathbf{T}} = q_{\mathbf{K}}(\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2, \hat{\mathbf{H}}_3; \mathbf{U}), \tag{9.16}$$

where

$$\hat{\mathbf{H}}_1 \equiv \frac{\mathbf{D}_1}{|\mathbf{D}_1|}, \tag{9.17}$$

$$\hat{\mathbf{H}}_2 \equiv \frac{\hat{\mathbf{D}}_2}{|\mathbf{D}_1|^2} - \frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_2}{|\mathbf{D}_1|^4}\hat{\mathbf{D}}_1, \tag{9.18}$$

$$\hat{\mathbf{H}}_3 \equiv \frac{\hat{\mathbf{D}}_3}{|\mathbf{D}_1|^3} - 3\frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_2}{|\mathbf{D}_1|^5}\hat{\mathbf{D}}_2 + \left(4\frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_2}{|\mathbf{D}_1|^7} - \frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_3}{|\mathbf{D}_1|^5} - \frac{\operatorname{tr}\mathbf{D}_2\mathbf{D}_2}{|\mathbf{D}_1|^5}\right)\hat{\mathbf{D}}; \tag{9.19}$$

(ii) *for* $\mathbf{D}_1 = \mathbf{0}, \mathbf{D}_2 \neq \mathbf{0}$

$$\hat{\mathbf{T}} = h_{\mathbf{K}}\left(\frac{\hat{\mathbf{D}}_2}{|\mathbf{D}_2|}, \frac{\hat{\mathbf{D}}_3}{|\mathbf{D}_2|^{3/2}}; \mathbf{U}\right); \tag{9.20}$$

(iii) *for* $\mathbf{D}_1 = \mathbf{D}_2 = \mathbf{0}, \mathbf{D}_3 \neq \mathbf{0}$

$$\hat{\mathbf{T}} = l_{\mathbf{K}}\left(\frac{\hat{\mathbf{D}}_3}{|\mathbf{D}_3|}; \mathbf{U}\right); \tag{9.21}$$

(iv) *for* $\mathbf{D}_1 = \mathbf{D}_2 = \mathbf{D}_3 = \mathbf{0}$

$$\hat{\mathbf{T}} = k_{\mathbf{K}}(\mathbf{U}). \tag{9.22}$$

Proof. Sufficiency. We verify directly that

$$\hat{\mathbf{H}}_i(\hat{\mathbf{D}}_1, \hat{\mathbf{D}}_2, \hat{\mathbf{D}}_3) = \hat{\mathbf{H}}_i(\hat{\mathbf{D}}_1^*, \hat{\mathbf{D}}_2^*, \hat{\mathbf{D}}_3^*) \tag{9.23}$$

for every a_1, a_2, a_3. Similarly, we verify the other formulae. Necessity. Setting

$$\begin{gathered} a_1 = \frac{1}{|\mathbf{D}_1|}, \quad a_2 = \frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_2}{|\mathbf{D}_1|^4}, \\ a_3 = 4\frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_2}{|\mathbf{D}_1|^7} - \frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_3}{|\mathbf{D}_1|^5} - \frac{\operatorname{tr}\mathbf{D}_2\mathbf{D}_2}{|\mathbf{D}_1|^5} \end{gathered} \tag{9.24}$$

in (9.7), (9.10), we obtain (9.16). In the same way, we obtain the remaining formulae. Q.E.D.

A more constructive, but more complicated, proof can be given by making use of the theory of continuous groups of transformations(3) since Λ is an N-parametric group [28]. We shall not perform that since the form (9.16) follows from Theorem IV. Relations (9.16) constitute, of course, a modification of differential approximations given in [25].

In particular, using the formulae derived, it is possible to build *a variant of the theory of rigid-ideal plastic materials*, taking into account the influence of accelerations deforma-

(3) Such a proof was given by Z. KUROWSKI.

tions upon stresses. Since a plastic material is, by definition, a completely inviscid material, then we must use the following formulae:

$$\hat{\mathbf{T}} = \begin{cases} g_{\mathbf{K}}\left(\dfrac{\hat{\mathbf{D}}_1}{|\mathbf{D}_1|}, \dfrac{\hat{\mathbf{D}}_2}{|\mathbf{D}_1|^2} - \dfrac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_2}{|\mathbf{D}_1|^4}\hat{\mathbf{D}}_1\right) & \text{if} \quad \mathbf{D}_1 \neq \mathbf{0}, \\ h_{\mathbf{K}}\left(\dfrac{\hat{\mathbf{D}}_2}{|\mathbf{D}_2|}\right) & \text{if} \quad \mathbf{D}_1 = \mathbf{0}, \mathbf{D}_2 \neq \mathbf{0}. \end{cases} \tag{9.25}$$

The second argument of the function $g_{\mathbf{K}}$ is orthogonal to the first one, $\operatorname{tr}\mathbf{H}_1\mathbf{H}_2 = 0$; it is equal to zero if $\mathbf{D}_2$ is proportional to $\mathbf{D}_1$.

THEOREM 12. *An inviscosity group of material of differential type and complexity three contains the group Λ_{I} iff*

(i) *for* $\mathbf{D}_1 \neq \mathbf{0}$

$$\hat{\mathbf{T}} = g_{\mathbf{K}}(\hat{\mathbf{D}}_1, \hat{\mathbf{H}}_2, \hat{\mathbf{H}}_3; \mathbf{U}); \tag{9.26}$$

(ii) *for* $\mathbf{D}_1 = \mathbf{0}, \mathbf{D}_2 \neq \mathbf{0}$

$$\hat{\mathbf{T}} = h_{\mathbf{K}}\left(\hat{\mathbf{D}}_2, \frac{\hat{\mathbf{D}}_3}{|\mathbf{D}_2|^{3/2}}; \mathbf{U}\right); \tag{9.27}$$

(iii) *for* $\mathbf{D}_1 = \mathbf{D}_2 = \mathbf{0}, \mathbf{D}_3 \neq \mathbf{0}$

$$\hat{\mathbf{T}} = l_{\mathbf{K}}(\mathbf{D}_3; \mathbf{U}); \tag{9.28}$$

(iv) *for* $\mathbf{D}_1 = \mathbf{D}_2 = \mathbf{D}_3 = \mathbf{0}$

$$\hat{\mathbf{T}} = k_{\mathbf{K}}(\mathbf{U}). \tag{9.29}$$

Proof. The proof is exactly the same as that of Theorem 11; we only substitute $(1, a_2, a_3)$ for (a_1, a_2, a_3). Q.E.D.

10. Materials insensitive to uniform change of rate of deformation processes

We now approach the problem of systematic analysis of inviscosity of materials. The purpose of our present considerations is a derivation of a general constitutive equation for a material which contains in its inviscosity semigroup $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D})$ the semigroup $\Lambda \subset \Sigma_{\mathscr{D}}$, given in advance. In other words, we are willing to give an answer, in each case, to the following question: what elements of the deformation path influence, in reality, the final stress, if $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}) \supset \Lambda$? Briefly, what does a material remember for which $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D}) \supset \Lambda$? For the subsemigroup in Fig. 2, the answer is contained in formulae (I)–(VII).

Let us begin with a situation in which an inviscosity semigroup of materials, contains the semigroup L — i.e., when

$$\mathfrak{F}_{\mathbf{K}}[f(s)] = \mathfrak{F}_{\mathbf{K}}[f(as)] \tag{10.1}$$

for every $f \in \operatorname{Dom}\mathfrak{F}_{\mathbf{K}}$ and every $a \in (0, \infty)$; we assume, of course, that $(\operatorname{Dom}\mathfrak{F}_{\mathbf{K}}) * \mathrm{L} \subset \operatorname{Dom}\mathfrak{F}_{\mathbf{K}}$. The condition (10.1) reads that the material under consideration does not respond to any uniform acceleration ($a > 1$), or retardation of the process ($a < 1$).

A generalization of Theorem 8 to arbitrary materials and arbitrary processes can be obtained without difficulty. Let us agree to denote a mapping σ defined by the formula $\sigma(s) = as$ as follows $\langle a \rangle \equiv \sigma$:

$$(f \circ \langle a \rangle)(s) \equiv f(as). \tag{10.2}$$

It is obvious that $\langle a \rangle \circ \langle b \rangle = \langle ab \rangle$.

THEOREM I. *An inviscosity semigroup* $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{D})$ *of an arbitrary material relative to an arbitrary class of processes* $\mathscr{D} \subset \mathrm{Dom}\mathfrak{F}_{\mathbf{K}}$, $\mathscr{D} * \mathrm{L} \subset \mathscr{D}$, *contains the group of uniform accelerations-retardations* L, *iff for every positive functional*

$$\mathfrak{c} : \mathscr{D} \to (0, \infty) \tag{10.3}$$

such that

(i) *for all non-constant* $f \in \mathscr{D}$ *and all* $a \in (0, \infty)$

$$\mathfrak{c}[f \circ \langle a \rangle] = a\mathfrak{c}[f]; \tag{10.4}$$

(ii) *for all constant* $f \in \mathscr{D}$

$$\mathfrak{c}[f] = 1, \tag{10.5}$$

we have

$$\boxed{\mathbf{T} = \mathfrak{F}_{\mathbf{K}}\left[f \circ \left\langle \frac{1}{\mathfrak{c}[f]} \right\rangle\right]} \tag{I}$$

on $\mathscr{D}$.

P r o o f. According to the general *REPRESENTATION THEOREM* (3.22), it suffices to prove that the mapping

$$\vartheta : \mathscr{D} \to \mathscr{D}, \quad \vartheta(f) \equiv f \circ \left\langle \frac{1}{\mathfrak{c}[f]} \right\rangle \tag{10.6}$$

is L-separator for every $\mathfrak{c}$ satisfying the conditions (10.4)–(10.5). If $f\mathrm{L}g$ — i.e., $f = g \circ \langle a \rangle$ for some a, then

$$\vartheta(f) \equiv f \circ \left\langle \frac{1}{\mathfrak{c}[f]} \right\rangle = g \circ \langle a \rangle \circ \left\langle \frac{1}{\mathfrak{c}[g \circ \langle a \rangle]} \right\rangle = g \circ \langle a \rangle \circ \left\langle \frac{1}{a\mathfrak{c}[g]} \right\rangle = g \circ \left\langle \frac{1}{\mathfrak{c}[g]} \right\rangle \equiv \vartheta(g). \tag{10.7}$$

Conversely, if $\vartheta(f) = \vartheta(g)$, then

$$f = g \circ \langle a \rangle \quad \text{for} \quad a \equiv \frac{\mathfrak{c}[f]}{\mathfrak{c}[g]} \tag{10.8}$$

— i.e., $f\mathrm{L}g$. It is also obvious, that $\vartheta \circ \vartheta = \vartheta$ .Q.E.D.

An example of a material invariant with respect to uniform acceleration and retardation would be a non-linear modification of a linear visco-elastic material, defined by the following constitutive equation:

$$\hat{\mathbf{T}} = f_{\mathbf{K}}(\mathbf{C}) + \int_0^\infty K_{\mathbf{K}}(\mathbf{C}, s)\left[\hat{\mathbf{G}}\left(\frac{s}{\mathfrak{c}[\hat{\mathbf{G}}]}\right)\right] ds, \tag{10.9}$$

where

$$\hat{\mathbf{G}}(s) \equiv \hat{\mathbf{C}}_{(t)}^{(t)}(s) - \mathbf{1}, \quad \mathbf{C}_{(t)}^{(t)}(s) \equiv \overset{-1}{\mathbf{F}^T}(t)\mathbf{C}^{(t)}(s)\overset{-1}{\mathbf{F}}(t),$$
$$\mathbf{C}^{(t)}(s) \equiv \mathbf{U}^2(t-s), \quad \mathbf{C} \equiv \mathbf{C}^{(t)}(0) \quad \text{(cf. [7], Sec. 40)}.$$

This material behave in a manner entirely different from the linear visco-elastic material. The constitutive operator is non-linear with respect to the argument $\hat{\mathbf{G}}(s)$.

Effective use of the theorem proved requires a construction of, at least, one functional $\mathfrak{c}$ on $\mathscr{D}$. The construction will depend, of course, on the class of processes under consideration. We give a few examples.

If $\mathscr{D}$ is a set of analytic curves, then a good functional $\mathfrak{c}$ is

$$\mathfrak{c}[f] \equiv \begin{cases} |f^{(n)}(0)|^{1/n} & \text{for} \quad f \neq \text{const}, \\ 1 & \text{for} \quad f = \text{const}, \end{cases} \tag{10.10}$$

$$n \equiv \min(m | f^{(m)}(0) \neq \mathbf{0}). \tag{10.11}$$

If $\mathscr{D}$ is a set of curves N-times differentiable, then we can put

$$\mathfrak{c}[f] \equiv \begin{cases} |f^{(n)}(\bar{s})|^{1/n} & \text{for} \quad f \neq \text{const}, \\ 1 & \text{for} \quad f = \text{const}, \end{cases} \tag{10.12}$$

where the number $\bar{s} \in (0, \infty)$, and the integer n are defined for every $f \neq$ const as follows:

$$\bar{s} \equiv \min(s|\ f^{(i)}(s) \neq \mathbf{0} \quad \text{for some} \quad i = 1, \ldots, N), \tag{10.13}$$

$$n \equiv \min(m|\ f^{(m)}(\bar{s}) \neq \mathbf{0},\ 1 \leqslant m \leqslant N). \tag{10.14}$$

This functional coincides with the previous one on the set of analytic curves. In Theorem 8, we have used a similar functional.

On a set of curves which possess, at least, one point of discontinuity, and the first point of discontinuity $\bar{s} > 0$, then we can assume a completely different functional, simply setting

$$\mathfrak{c}[f] \equiv \bar{s}^{-1}. \tag{10.15}$$

This functional was already used in formula (8.8).

It is not difficult to verify that all the functionals mentioned satisfy the condition (10.4). For instance, if the first point of discontinuity of the function f is $\bar{s}$, then the first point of discontinuity of the function $f \circ \langle a \rangle$ will be $\bar{s}/a$ (cf. Fig. 1), and hence $\mathfrak{c}[f \circ \langle a \rangle] \equiv$ $\equiv a/\bar{s} = a\,\mathfrak{c}[f]$.

If we know, in advance, that a certain relation E decomposes Dom $\mathfrak{F}_{\mathbf{K}}$ into cosets that are contained in level sets, then it is necessary to assume $\mathfrak{c}$ such that it remains constant on E-cosets.

The theorem proved holds true for those classes of processes $\mathscr{D}$ which are stable with respect to the entire group L. In the case in which $\mathrm{L}_{\mathscr{D}} \neq \mathrm{L}$, L-separator may not be $\mathrm{L}_{\mathscr{D}}$-separator for two reasons:

1. it may lead outside $\mathscr{D}_a$,

2. it may cause too coarse a decomposition of $\mathscr{D}$ onto cosets — i.e., if $f \mathrm{L}_{\mathscr{D}} g$, then $\vartheta(f) = \vartheta(g)$ (this implication, of course, takes place for every $\mathrm{L}_{\mathscr{D}}$, $\mathfrak{c}$), but not the converse.

The first defect can be removed by imposing an extra condition on the functional:

$$f \circ \left\langle \frac{1}{\mathfrak{c}[f]} \right\rangle \in \mathscr{D} \quad \text{for every} \quad f \in \mathscr{D}, \quad \text{(iii)}. \tag{10.16}$$

The second one, in general, is impossible to remove([4]). Thus, by making use of the functional $\mathfrak{c}$, in a general situation, we can state only what follows:

if for a certain functional $\mathfrak{c}$, which satisfies (i), (ii), (iii), is satisfies (I), then

$$\Phi(\mathfrak{F}_K, \mathscr{D}) \supset L.$$

Strong theorems of the "iff" type should be formulated for every class $\mathscr{D}$ separately.

Consider an important example of materials with restricted stretchings,

$$\text{(10.17)} \qquad \operatorname{Dom} \mathfrak{F}_K = \{ f \in \mathscr{A} \mid \ |f'(s)| < d \quad \text{for every} \quad s \in \mathsf{R} \},$$

where $d > 0$ is a fixed number. It is obvious, that

$$\text{(10.18)} \qquad L_{\operatorname{Dom}\mathfrak{F}_K} = L_{(\leqslant)} \equiv \{ \langle a \rangle \in L \mid \ a \leqslant 1 \},$$

— i.e., only uniform retardations are permissible (they do not lead out *an arbitrary* $f \in \operatorname{Dom} \mathfrak{F}_K$ from $\operatorname{Dom} \mathfrak{F}_K$). Let us notice that the semigroup $L_{(\leqslant)}$ is not a group.

Take a fixed, but otherwise arbitrary, $k < d$ and set

$$\text{(10.19)} \qquad \mathfrak{c}[f] \equiv \begin{cases} \dfrac{1}{k} \sup\limits_s |f'(s)| & \text{for} \quad f \neq \text{const}, \\ 1 & \text{for} \quad f = \text{const}. \end{cases}$$

It is obvious that the functional is well defined on (10.17). It is also obvious, that condition (10.4) is fulfilled. The mapping ϑ from (10.7) satisfies the condition (10.16), since for every f from (10.17)

$$\text{(10.20)} \qquad |\vartheta(f)'(s)| = \frac{1}{\mathfrak{c}[f]} |f'(s)| = k \frac{|f'(s)|}{\sup\limits_s |f'(s)|} \leqslant k < d.$$

We show that, in this case, ϑ is $L_{(\leqslant)}$-separator. In fact, if $\vartheta(f) = \vartheta(g)$ — i.e.,

$$\text{(10.21)} \qquad f \circ \left\langle \frac{1}{\mathfrak{c}[f]} \right\rangle = g \circ \left\langle \frac{1}{\mathfrak{c}[g]} \right\rangle,$$

then for $\mathfrak{c}[f] \geqslant \mathfrak{c}[g]$ we have $f \circ \langle a \rangle = g_M$, $a \leqslant 1$, and for $\mathfrak{c}[f] \leqslant \mathfrak{c}[g]$ we have $f = g \circ \langle a \rangle$, $a \leqslant 1$ — i.e., f, g are $L_{(\leqslant)}$-equivalent. We obtain the following result.

THEOREM I′. *An inviscosity semigroup of a material with stretching restricted by the constant d contains a semigroup of uniform retardations* $L_{(\leqslant)}$ *iff*

$$\text{(I}'\text{)} \qquad \mathbf{T} = \mathfrak{F}_K \left[f \circ \left\langle \frac{k}{\sup\limits_s |f'(s)|} \right\rangle \right], \quad k < d.$$

11. Materials insensitive to arresting of deformation processes

We now approach the problem of describing invariance with respect to the next semigroup of our catalogue, the semigroup of arresting P. In other words, we wish to describe materials which do not respond to the arresting of an arbitrary process at any instant. In fact, we obtain results which are somewhat less general.

([4]) A formal counter-example

$$L_{\mathscr{I}} \equiv \{ \langle a \rangle \in L \mid \text{a is an integer} \}.$$

We introduce a convenient concept: a finite, closed interval $[a, b] \subset \mathrm{R}$, we shall call *a stop* of a function ζ defined on R, if $\zeta(s) = \mathrm{Z} = \text{const}$ for $s \in (a, b)$, and if there exists $\delta > 0$ such that $\zeta(s) \neq \mathrm{Z}$ for every $s \in (a-\delta, a)$ and every $s \in (b, b+\delta)$. The class of processes under consideration we define by

$$\mathscr{X} \equiv \{f \in \mathscr{A} | \ f \text{ satisfies (i), (ii)}\}, \tag{11.1}$$

where

(i) f is a continuous function or possesses a finite number of discontinuity points of the first kind, and is right-hand side continuous at each of the points.

(ii) f possesses a finite number of stops.

The class $\mathscr{X}$ satisfies the conditions (2.6), (2.7), (2.8). It comprises a remarkable number of processes which are interesting from the experimental viewpoint. The only essential loss constitutes periodic processes with discontinuities and (or) stops. But it is not difficult to consider them separately. The right-hand side continuity of processes can be interpreted physically as follows: at every instant $t-s$ the value of $f(s)$ is arbitrarily close to its prior values at the instant $t-s'$, $s' > s$, taken sufficiently close to $t-s$.

It is not difficult to verify that the subsemigroup $\mathrm{P}_{\mathscr{X}}$ consists of those and only those elements $\sigma \in \mathrm{P}$ which possess a finite number of stops.

Thus, in fact, we shall study the invariance of constitutive operators on the class $\mathscr{X}$ of processes with respect to exchanges from $\mathrm{P}_{\mathscr{X}}$.

Denote by $0 < s_1 < \ldots < s_N < \infty$ the sequence of discontinuity points of the function f, and by $[f]_i$, $i = 1, \ldots, \mathrm{N}$, $[f]_i \neq \mathbf{0}$ the sequence of the corresponding jumps. The formula

$$f(s) = f^c(s) + \sum_{i=1}^{\mathrm{N}} [f]_i \mathrm{H}(s-s_i) \tag{11.2}$$

defines a unique function f^c, for every function $f \in \mathscr{X}$. The function f^c is said to be *a continuous component* of f. It is obvious that $f^c \in \mathscr{X}$. Let $\mathscr{X}^c \subset \mathscr{X}$ denote a subset of continuous functions in $\mathscr{X}$. Let $\mathscr{P}$ be a set of sequences from $\mathscr{T} \times \mathsf{R}$ of the form $[([f]_i, s_i)$, $[f]_i \neq \mathbf{0}$ $0 < s_1 < \ldots < s_{\mathrm{N}} < \infty$, together with the void set $\varnothing$, included for convenience of notation.

Formula (11.2) defines a bijection

$$\mathscr{X} \ni f \overset{\beta}{\rightarrow} [f^c; ([f]_i, s_i)] \in \mathscr{X}^c \times \mathscr{P}. \tag{11.3}$$

To a continuous function $f \in \mathscr{X}$ we assign the pair $(f; \varnothing)$. Hence every constitutive equation can be written in the form

$$\mathbf{T} = \mathfrak{Z}_{\mathrm{K}}[f^c; ([f]_i, s_i)] \tag{11.4}$$

on the class of processes $\mathscr{X}$, where

$$\mathfrak{Z}_{\mathrm{K}} \equiv \mathfrak{F}_{\mathrm{K}} \circ \beta^{-1} : \mathscr{X}^c \times \mathscr{P} \rightarrow \mathscr{T}.$$

The stops of a continuous component f^c are either the stops of f or their unions. Let us separate the stops of f^c into three types:

(i) without discontinuity points s_i,

(ii) starting from discontinuity points,

(iii) containing discontinuity points but not starting from them(⁵).

It may happen that there exists $a \equiv \min(\hat{s}|f^c(s) = \text{const for } s \in [\hat{s}, \infty))$, and simultaneously f possesses discontinuity points on (a, ∞). We agree that the last stop of the continuous component f^c is in this case $[a, s_N]$.

Let us introduce a sequence $0 < q_1 < \ldots < q_M$ consisting of all origins of the second type stops and the third type stops of the continuous component f^c, and of all those discontinuity points of f which do not belong to stops of f^c. To every point q_α of this sequence, let us assign the sequence $[f]^1_\alpha, \ldots, [f]^L_\alpha$ in the following way. If $q_\alpha = s_k$ is the origin of the second type stop of f^c, then

$$[f]^1_\alpha \equiv [f]_k, \quad [f]^2_\alpha \equiv [f]_{k+1}, \ldots, \quad L_\alpha = r+1, \tag{11.5}$$

where $s_k < s_{k+1} < \ldots < s_{k+r}$, $r \geqslant 0$ constitute all discontinuity points in this stop. If q is the origin of the third type stop, then

$$[f]^1_\alpha \equiv \mathbf{0}, \quad [f]^2_\alpha \equiv [f]_m, \ldots, \quad L_\alpha = r+2, \tag{11.6}$$

where $s_m < s_{m+1} < \ldots < s_{m+r}$, $r \geqslant 0$, $q_\alpha < s_m$ constitute all discontinuity points in this stop. If, finally, $q_\alpha = s_l$ is a discontinuity point of f which belongs to none of the stops of f^c, then

$$[f]^1_\alpha \equiv [f]_l, \quad L_\alpha = 1. \tag{11.7}$$

It is obvious that every $[f]_i$ has been assigned to one of the sequences $[f]^i_\alpha$.

In this way, we have assigned to each discontinuous function $f \in \mathscr{X}$ a finite sequence of finite sequences

$$\begin{array}{c} ([f]^1_1, [f]^2_1, \ldots, [f]^{L_1}_1; q_1), \\ ([f]^1_2, [f]^2_2, \ldots, [f]^{L_2}_2; q_2), \\ \cdots\cdots\cdots\cdots \\ ([f]^1_M, [f]^2_M, \ldots, [f]^{L_M}_M; q_M), \end{array} \tag{11.8}$$

with the following properties:

(i) the sequence q_α, $\alpha = 1, \ldots, M \leqslant N$ is increasing, begininng with $q_1 > 0$,

(ii) $[f]^i_\alpha$ are different from zero with the exception of at most some $[f]^1_\alpha$ for $L_\alpha \geqslant 2$.

A set of sequences of this kind, $([f]^i_\alpha; q_\alpha)$, together with the void set $\varnothing$, we denote by Q. To a continuous function $f = f^c$ we assign the element $\varnothing$.

Let us assign to the continuous component f^c a mapping $\sigma(f^c) \in P_{\mathscr{X}}$, defined by the following prescription: the stops of $\sigma(f^c)$ are the only stops of f^c. It is not difficult to verify that the equation

$$f^c = \tilde{f}^c \circ \sigma(f^c) \tag{11.9}$$

(⁵) The necessity to distinguish the second from the third type of stops is evident from the following example. Let g be a continuous function without stops; consider the functions f, h

$$f(s) \equiv \begin{cases} g(s) & \text{for} \quad s \in [0, a], \\ g(a) & \text{for} \quad s \in [a, b], \\ g(s-b+a)+\mathbf{G} & \text{for} \quad s \in [b, \infty), \end{cases} \qquad h(s) \equiv \begin{cases} f(s) & \text{for} \quad s \in [0, a), \\ g(a)+\mathbf{G} & \text{for} \quad s \in [a, b], \\ g(s-b+a)+\mathbf{G} & \text{for} \quad s \in [b, \infty), \end{cases}$$

where $\mathbf{G} \neq \mathbf{0}$, $b > a > 0$. It is not difficult to verify that the functions f, h are not $P_{\mathscr{X}}$-equivalent.

possesses, for every $f^c \in \mathscr{X}^c$, one and only one solution $\tilde{f}^c \in \mathscr{X}^c$ which is a function without stops. (Of course, it may happen that $\tilde{f}^c(s) = \text{const}$ on $[c, \infty)$). The subset of all continuous functions without stops we denote by $\tilde{\mathscr{X}}^c$. (Constant functions belong to $\tilde{\mathscr{X}}^c$). The operation $\mathscr{X}^c \ni f \to \tilde{f}^c \in \tilde{\mathscr{X}}^c$ may be called a "telescopic shrinking" of continuous functions that liquidates stops.

Introduce, finally, a sequence

$$\tilde{q}_\alpha \equiv \sigma(f^c)(q_\alpha), \quad \alpha = 1, \dots, M \tag{11.10}$$

— i.e., $\tilde{q}_\alpha$ is translated with respect to q_α to the left-hand side by a distance equal to the sum of stop lengths of f^c (i.e., also of f) prior to q_α.

Recapitulating. To every function $f \in \mathscr{X}$ we have assigned a pair from $\tilde{\mathscr{X}}^c \times Q$,

$$f \to [\tilde{f}^c; ([f]^i_\alpha, \tilde{q}_\alpha)]. \tag{11.11}$$

The pair $[\tilde{f}^c; \varnothing]$ corresponds to a continuous function. This operation is not, of course, invertible. We may call it the "telescopic shrinking" of the function f. This construction leads to the following result.

THEOREM II. *An inviscosity semigroup of a material* $\Phi(\mathfrak{F}_K, \mathscr{X})$ *contains a semigroup of arrestings* $P_{\mathscr{X}}$, *iff there exists an operator* $\mathfrak{R}_K: \tilde{\mathscr{X}}^c \times Q \to \mathscr{T}$ *such that*

$$\boxed{\mathbf{T} = \mathfrak{R}_K[\tilde{f}^c; ([f]^k_\alpha, \tilde{q}_\alpha)]} \tag{II}$$

for every deformation path $f \in \mathscr{X}$.

P r o o f. Take any function $f \in \mathscr{X}$, and apply to it first the bounded operation of "telescopic shrinking" ϑ_1, which will liquidate stops without discontinuity points only. This is an operation defined as follows. Assign to f a mapping $\mu_1 = \mu_1(f) \in P_{\mathscr{X}}$ defined by the following prescription: stops of μ_1 are stops without discontinuity points of the function f and only these. It is not difficult to verify that the equation

$$f = \vartheta_1(f) \circ \mu_1 \tag{11.12}$$

possesses in $\mathscr{X}$ one and only one solution $\vartheta_1(f)$. The function $\vartheta_1(f)$ possesses no more stops without discontinuity points.

If $f = f^c \in \mathscr{X}^c$, then, of course, $\vartheta_1(f) = \tilde{f}^c$. We show now that ϑ_1 restricted to $\mathscr{X}^c$ is $P_{\mathscr{X}}$-separator. It is obvious that for every $\sigma \in P_{\mathscr{X}}$ and every function $h \in \mathscr{X}^c$ we have $\vartheta_1(h \circ \sigma) = \vartheta_1(h)$. Take now two arbitrary $P_{\mathscr{X}}$-equivalent continuous functions $f, g \in \mathscr{X}^c$. This means that there exists a $P_{\mathscr{X}}$-connecting sequence $c_1 = f, \dots, c_k = g$, $c_i = c_{i+1} \circ \sigma_i$ or $c_i \circ \sigma_i = c_{i+1}$, $\sigma_i \in P_{\mathscr{X}}$, for every $i = 1, \dots, k$. We have $\vartheta_1(f) \equiv \vartheta_1(c_1) = \vartheta_1(c_2) = \dots = \vartheta_1(g)$. Inversely, if $\vartheta_1(f) = \vartheta_1(g)$, then $c_1 \equiv f$, $c_2 \equiv \vartheta_1(f) = \vartheta_1(g)$, $c_3 \equiv g$ is a $P_{\mathscr{X}}$-connecting sequence, since $\mu_1(f) \in P_{\mathscr{X}}$ and $\mu_1(g) \in P_{\mathscr{X}}$. Applying the representative formula (3.22), we have $\mathfrak{F}_K = \mathfrak{F}_K \circ \vartheta_1$ on $\mathscr{X}^c$, and this proves the formula (II) on $\mathscr{X}^c$. (We identify here $\tilde{f}^c$ with the pair $[f^c; \varnothing]$, only for convenience in writing).

It remains to consider $\mathscr{X}^d \equiv \mathscr{X} — \mathscr{X}^c$. It is obvious that if $f \in \mathscr{X}^d$, then $\vartheta_1(f) \in \mathscr{X}^d$ also. Let us denote by d_f the maximal length of a stop of the function $\vartheta_1(f)$; the existence of such a number is implied by the condition (ii), concerning the class $\mathscr{X}$. We introduce a sequence $0 < p_1 < \dots < p_s$ which has been formed of all origins of stops of the function and all its discontinuity points; by assumptions (i), (ii), concerning the class $\mathscr{X}$, such a sequence p_α is finite. We wish to use the sequence to construct a function $\vartheta_2[\vartheta_1(f)] \in \mathscr{X}^d$ with the following features: every discontinuity point of this function is to be the origin of its stop, every stop is to begin and/or end with a discontinuity point, all stops are to be of the same length d_f, and, finally, the function is to be $P_{\mathscr{X}}$-equivalent to the function $\vartheta_1(f)$. Denote by $l_\beta \in [0, d_f]$ the length of that stop of $\vartheta_1(f)$ which has its origin at the point p_β (we assume $l_\beta = 0$ when p_β is not an origin of a stop). Denote by $\Delta_\alpha \equiv d_f - l_\alpha$, $r_\alpha \equiv p_\alpha + \sum_{\beta=1}^{\alpha-1} \Delta_\beta$. We introduce a mapping $\mu_2 \equiv \mu_2[\vartheta_1(f)] \in \in \mathrm{P}_{\mathscr{X}}$ defined by the prescription: stops of μ_2 are the intervals $[r_\alpha, r_\alpha + \Delta_\alpha]$ and only such. The function sought for is equal to

$$\vartheta_2[\vartheta_1(f)] \equiv \vartheta_1(f) \circ \mu_2. \tag{11.13}$$

Points r_α are the origins of stops of this function. The last step of our construction is a normalization of stop length. Take a standard length $\varepsilon > 0$ and a sequence $t_\alpha \equiv r_\alpha + (\alpha - 1)(d_f - \varepsilon)$. Let us introduce $\mu_3 \equiv \equiv \mu_3[\vartheta_2(\vartheta_1(f))] \in \mathrm{P}_{\mathscr{X}}$ by means of the prescription: intervals $[t_\alpha, t_\alpha + |d_f - \varepsilon|]$ are the only stops of μ_3. If $d_f < \varepsilon$, then we assume

$$\vartheta_3[\vartheta_2(\vartheta_1(f))] \equiv \vartheta_2[\vartheta_1(f)] \circ \mu_3. \tag{11.14}$$

If $d_f > \varepsilon$, then the function $\vartheta_3[\vartheta_2(\vartheta_1(f))]$ is accepted as a solution of the equation

$$\vartheta_2[\vartheta_1(f)] = \vartheta_3[\vartheta_2(\vartheta_1(f))] \circ \mu_3. \tag{11.15}$$

It is obvious that this function belongs to $\mathscr{X}^d$; its stops are of length ε and they have their origins at the points t_α.

We now show that the operation

$$\vartheta^\varepsilon \equiv \vartheta_3 \circ \vartheta_2 \circ \vartheta_1 : \; \mathscr{X}^d \to \mathscr{X}^d \tag{11.16}$$

is $\mathrm{P}_{\mathscr{X}}$-separator on $\mathscr{X}^d$. Let us first notice that if $c_i, c_{i+1} \in \mathscr{X}^d$ are $\mathrm{P}_{\mathscr{X}}$-connected, then $\vartheta^\varepsilon(c_i) = \vartheta^\varepsilon(c_{i+1})$. In fact, if $c_i = c_{i+1} \circ \sigma_i$ or $c_i \circ \sigma_i = c_{i+1}$ for some $\sigma_i \in \mathrm{P}_{\mathscr{X}}$, then the functions c_i, c_{i+1} differ one from the other by length and the number of stops. If we cut out the stops without discontinuity points (operation ϑ_1), introduce the stops over all discontinuity points and equate the length of stops with the maximal length (operation ϑ_2), and, finally, by normalizing the length of stops of both functions to the standard length ε (operation ϑ_3), we obtain the same result. Now, if f, g are $\mathrm{P}_{\mathscr{X}}$-equivalent, then taking a $\mathrm{P}_{\mathscr{X}}$-connecting sequence $c_1 = f, c_2, \ldots, c_N = g$, we obtain:

$$\vartheta^\varepsilon(f) \equiv \vartheta^\varepsilon(c_1) = \vartheta^\varepsilon(c_2) = \ldots = \vartheta^\varepsilon(c_N) \equiv \vartheta^\varepsilon(g). \tag{11.17}$$

Inversely, if $\vartheta^\varepsilon(f) = \vartheta^\varepsilon(g)$, then the sequence

$$\begin{aligned} c_1 \equiv g, \quad c_2 \equiv \vartheta_1(f), \quad c_3 \equiv \vartheta_2[\vartheta_1(f)], \quad c_4 \equiv \vartheta^\varepsilon(f) = \vartheta^\varepsilon(g), \\ c_5 \equiv \vartheta_2[\vartheta_1(g)], \quad c_6 \equiv \vartheta_1(g), \quad c_7 \equiv g \end{aligned} \tag{11.18}$$

is a $\mathrm{P}_{\mathscr{X}}$-connecting sequence, since from (11.12), (11.13). (11.14), (11.15) it follows that every pair c_i, c_{i+1}, $i = 1, \ldots, 6$ is $\mathrm{P}_{\mathscr{X}}$-connecting; thus f, g are $\mathrm{P}_{\mathscr{X}}$-equivalent. The condition $\vartheta^\varepsilon \circ \vartheta^\varepsilon = \vartheta^\varepsilon$ is, of course, fulfilled.

Thus ϑ^ε is a $\mathrm{P}_{\mathscr{X}}$-separator on $\mathscr{X}^d$, and $\vartheta^\varepsilon(\mathscr{X}^d)$ is the set of representative for $\mathrm{P}_{\mathscr{X}}$-orbits. The representatives of $\mathrm{P}_{\mathscr{X}}$-orbits are functions with standard length of stops $\varepsilon > 0$; here, every stop starts or(and) ends with a discontinuity point, and every discontinuity point is the origin of a stop.

It remains to free ourselves of the parameter ε. We achieve this by constructing the following bijection

$$\vartheta^\varepsilon(\mathscr{X}) \leftrightarrow \tilde{\mathscr{X}}^d \times \mathrm{Q}. \tag{11.19}$$

Let us, before we state the next theorem, apply to the function $\vartheta^\varepsilon(f)$ the procedure described in connection with function f. The formula

$$\vartheta^\varepsilon(f)(s) = [\vartheta^\varepsilon(f)]^c(s) + \sum_{i \in \mathscr{K}} [f]_i \mathrm{H}(s - t_i) \tag{11.20}$$

corresponds to the decomposition (11.2), where $\mathscr{K}$ is the subset of numbers for which the origin of a stop t_i is simultaneously a discontinuity point. The sequence $'q_\alpha$, comprised of those t_i which are the origins of stops of the continuous component $[\vartheta^{\varepsilon}(f)]^c$, corresponds to the sequence q_α. A counterpart to the sequences $[f]^i_\alpha$, $i = 1, \ldots, L_\alpha$, will be the sequences $'[f]^i_\alpha$, $i = 1, \ldots, 'L_\alpha$. It is easy to verify that

(11.21) $$'L_\alpha = L_\alpha, \quad '[f]^i_\alpha = [f]^i_\alpha,$$

(11.22) $$'q_\alpha = q_\alpha + \Big(\sum_{\beta=1}^{\alpha-1} L_\beta\Big)\varepsilon, \quad \alpha = 1, \ldots, M,$$

and also

(11.23) $$\widetilde{[\vartheta^{\varepsilon}(f)]^c} \equiv \vartheta_1([\vartheta^{\varepsilon}(f)]^c) = \tilde{f}^c.$$

From the last formula it follows that

(11.24) $$[\vartheta^{\varepsilon}(f)]^c = \tilde{f}^c \circ \mu,$$

where $\mu \in P_{\mathscr{X}}$ remains constant on the stops of $[\vartheta^{\varepsilon}(f)]^c$ — i.e., over the intervals $['q_\alpha, 'q_\alpha + L_\alpha \varepsilon]$ only.

The formula (11.20) can now be rewritten in the following equivalent form:

(11.25) $$\vartheta^{\varepsilon}(f)(s) = (\tilde{f}^c \circ \mu)(s) + \sum_{\alpha=1}^{M} \sum_{k=1}^{L_\alpha} [f]^i_\alpha H\Big\{s - \Big[q_\alpha + \Big(\sum_{\beta=1}^{\alpha-1} L_\beta + k - 1\Big)\varepsilon\Big]\Big\}.$$

This is the bijection sought for

(11.26) $$\vartheta^{\varepsilon}(f) \leftrightarrow [f^c; ([f]^i_\alpha, \tilde{q}_\alpha)].$$

By applying the *REPRESENTATION THEOREM* (3.21), we obtain (II). Q.E.D.([6]).

To this end, let us notice that the assumptions (i), (ii) concerning the class of processes $\mathscr{X}$ form the basis for the whole procedure. Take the stopless function g: $[0, a) \to \mathscr{T}$. Let us introduce the sequence:

(11.27) $$r_1 = 0, \quad r_2 = \frac{a}{2}, \ldots, \quad r_k = \frac{(k-1)a}{k}, \ldots$$

and a mapping ν: $[0, \infty) \to [0, a)$, continuous, constant at the intervals $[r_i + (i-2)l, r_i - (i-1)l]$, $l > 0$, and only at them, and equal to $\nu(s) = s + \text{const}$, for the remaining $s \in R$. It is obvious that $f \equiv g \circ \nu \in \mathscr{A}$; here, the deformation procedure f possesses an infinite number of stops. The operation of a "telescopic shrinking" of f does not give a deformation process. Similar difficulties are encountered when dealing with processes which have an infinite number of discontinuity points.

In § 9 we need a certain modified version of the theorem proved. Let us take, instead of $P_{\mathscr{X}}$, a narrower semigroup

(11.28) $$P'_{\mathscr{X}} \equiv P_{\mathscr{X}} \cap I.$$

This semigroup contains, according to the definition of an instantaneous semigroup I, all the $\sigma \in P_{\mathscr{X}}$ which satisfy the condition: there exists $a > 0$ such that $\sigma(s) = s$ in $[0, a]$.

Let us introduce

(11.29) $$\mathscr{X}_1 \equiv \{f \in \mathscr{X} \mid \text{for every } a > 0 \text{ the interval } [0, a] \text{ is not a stop of } f\}$$

(11.30) $$\mathscr{X}_2 \equiv \mathscr{X} \setminus \mathscr{X}_1.$$

The result beginning holds true.

([6]) The foregoing proof is a toilsome technical procedure of the idea of "telescopic shrinking" of a function. The "telescopic shrinking" of the function $\vartheta^{\varepsilon}(f)$ is equivalent to the limiting passage, $\varepsilon \to 0$. A limit of a function sequence $\vartheta^{\varepsilon}(f)$ is not a function but an aggregate $[\tilde{f}^c; ([f]^i_\alpha, q_\alpha)]$. And inversely, every such aggregate can be "telescopically stretched" according to the rule (11.25), and we obtain a function from $\vartheta^{\varepsilon}(\mathscr{X}^d)$.

THEOREM II'. $\Phi(\mathfrak{F}_K, \mathscr{X}) \supset P'_{\mathscr{X}}$ iff

$$\text{(II')} \qquad \mathbf{T} = \mathfrak{N}'_K[\tilde{f}^c; ([f]^i_\alpha, q_\alpha); \varepsilon(f)]$$

for every $f \in \mathscr{X}$, where ε is the characteristic function of $\mathscr{X}_1$, i.e.,

$$(11.31) \qquad \varepsilon(f) \equiv \begin{cases} 1 & \text{if} \quad f \in \mathscr{X}_1, \\ 0 & \text{if} \quad f \notin \mathscr{X}_1. \end{cases}$$

Proof. The necessity to distinguish $\mathscr{X}_1$ and $\mathscr{X}_2$ follows from the fact that both sets are stable with respect to $P'_{\mathscr{X}}$; thus no $f \in \mathscr{X}_1$ is $P'_{\mathscr{X}}$-equivalent to any $f \in \mathscr{X}_2$.

Consider $\mathscr{X}_1$. It is obvious that for $f \in \mathscr{X}_1$ we have $\mu_1, \mu_2, \mu_3 \in P'_{\mathscr{X}}$. Hence $\vartheta^\varepsilon(f)$ will be $P'_{\mathscr{X}}$ -equivalent to f and ϑ^ε will be $P'_{\mathscr{X}}$-separator on $\mathscr{X}_1$. Carrying out the identification (11.16), we obtain the formula (II') on $\mathscr{X}_1$.

Let us pass to $\mathscr{X}_2$. Let us repeat the algorithm of the proof of Theorem (II) for one variable: point $s = 0$ is formally included into the set of discontinuity points of $f \in \mathscr{X}_2$. It is obvious, that μ'_1, μ'_2, μ'_3 so modified will belong to $P'_{\mathscr{X}}$, and hence $\vartheta^{\varepsilon'}$ will be $P'_{\mathscr{X}}$-separator on $\mathscr{X}_2$. Identification (11.26) now gives formula (II') on $\mathscr{X}_2$ .Q.E.D.

12. Materials insensitive to uniform change of rate and arresting of a deformation process

Intuition, being only an abstract of experiment, leads us to hope that if a material is irresponsive to arbitrary arresting of a deformation process, then it will not, in general, respond to a series of other disturbances of the time-realization of the process. In other words, a semigroup P and its subsemigroups will constitute, in general, proper parts of an inviscosity semigroup.

The first such case follows from Theorems I and II. Consider the invariance of constitutive operators with respect to the third semigroup of our catalogue. We confine our considerations to the class of processes $\mathscr{X}$, thus also to the semigroup $S_{\mathscr{X}}$.

It is obvious that every $\sigma \in S_{\mathscr{X}}$ we can write, uniquely, in the form:

$$(12.1) \qquad \sigma = \langle a \rangle \circ \mu, \quad \langle a \rangle \in L, \quad \mu \in P_{\mathscr{X}},$$

From this observation follows a solution to the problem.

THEOREM III. *An inviscosity semigroup of a material $\Phi(\mathfrak{F}_K, \mathscr{X})$ contains the semigroup* $S_{\mathscr{X}}$ iff

$$\text{(III)} \qquad \boxed{\mathbf{T} = \mathfrak{N}_K\left[\tilde{f}^c \circ \left\langle \frac{1}{\mathfrak{c}[\tilde{f}^c]} \right\rangle; ([f]^i_\alpha, \mathfrak{c}[\tilde{f}^c]\tilde{q}_\alpha)\right]}$$

for every deformation path $f \in \mathscr{X}$, where $\mathfrak{c}$ is the restriction of the functional $\mathfrak{c}$ to the class $\tilde{\mathscr{X}}^c$.

Proof. Since $S_{\mathscr{X}} \supset P_{\mathscr{X}}$, then a necessary condition for $\Phi(\mathfrak{F}_K, \mathscr{X}) \supset S_{\mathscr{X}}$ is the fulfillment of formula (II). To every exchange $f \to f \circ \sigma$, $\sigma \in S_{\mathscr{X}}$ correspond the following exchanges:

$$(12.2) \qquad \tilde{f}^c \to \widetilde{(f \circ \sigma)^c}, \quad ([f]^i_\alpha, q_\alpha(f)) \to ([f]^i_\alpha, q_\alpha(f \circ \sigma)).$$

But due to (12.1), and the definitions of $\tilde{f}^c$ and q_α, we have:

$$\widetilde{(f\circ\sigma)^c} \equiv \widetilde{(f\circ\langle a\rangle\circ\mu)^c} = \widetilde{(f\circ\langle a\rangle)^c} = \tilde{f}^c\circ\langle a\rangle, \tag{12.3}$$

$$\dot{q}_\alpha(f\circ\sigma) \equiv \dot{q}_\alpha(f\circ\langle a\rangle\circ\mu) = \tilde{q}_\alpha(f\circ\langle a\rangle) = a^{-1}q_\alpha(f). \tag{12.4}$$

Hence to the exchange $\sigma\in S_{\mathfrak{X}}$ corresponds the exchange

$$[\tilde{f}^c; ([f]^i_\alpha, \tilde{q}_\alpha)] \to (\tilde{f}^c\circ\langle a\rangle; ([f]^i_\alpha, a^{-1}\dot{q}_\alpha)). \tag{12.5}$$

The condition $\Phi(\mathfrak{F}_K, \mathscr{X}) \supset S_{\mathfrak{X}}$ is thus equivalent to the invariance of the operator $\mathfrak{N}_K$ with respect to the group L, acting in its domain $\tilde{\mathscr{X}}^c\times Q$, according to the rule (12.5). It is easy now to show that $\vartheta: \tilde{\mathscr{X}}^c\times Q \to \tilde{\mathscr{X}}^c\times Q$ defined by

$$\vartheta[\tilde{f}^c; ([f]^i_\alpha, \tilde{q}_\alpha)] \equiv [\tilde{f}^c\circ\left\langle\frac{1}{\mathfrak{c}[\tilde{f}^c]}\right\rangle; ([f]^i_\alpha, \mathfrak{c}[\tilde{f}^c]q_\alpha)], \tag{12.6}$$

where $\mathfrak{c}$ is a positive functional, from Theorem I, on $\tilde{\mathscr{X}}^c$, is an L-separator on $\tilde{\mathscr{X}}^c\times Q$. Q.E.D.

13. Materials completely inviscid

In the study of viscosity, of great use is an idea presented by PIPKIN and RIVLIN [25] for writing constitutive equations. For simplicity, we begin with continuous processes.

Deformation paths are curves in the space of Euclidean tensors $\mathscr{T}$, which is a metric space. This fact enables us to introduce the concept of an arc length. Consider the following class of deformation paths,

(13.1) $C \equiv \{f\in\mathscr{X} \mid f$ is continuous function of bounded variation over every interval $[0, a], a\in R\}$(⁷).

In accordance with a Jordan theorem, there exists *a natural parametrization* for every function $f\in C$. This means that there exist, uniquely defined by f, the functions:

$$l: R\to R, \quad \bar{f}: \text{Range}\, l\to\mathscr{T} \tag{13.2}$$

such that

1. $f = \bar{f}\circ l$,
2. $l(s)$ is the length of the arc $\{f(\tau) \mid \tau\in[0, s]\}$.

Let us collect the well known properties of natural parametrization:

(i) l is a monotonically non-decreasing function;
(ii) l is a continuous function;
(iii) $l(0) = 0$;
(iv) $\text{Dom}\, l = R$,
(v) the only stops of l are those of f; hence $\bar{f}$ is a function without stops;
(vi) $f = g\circ\sigma$ for some $\sigma\in\Sigma$ iff $\bar{f} = \bar{g}$.

Denote by S_L the set of functions $l: R\to R$ which satisfy the conditions (i)–(iv), and the condition $\text{Range}\, l = [0, L]$, $L\in R$; for $L<\infty$; S_L is not a subset of Σ_C. By $\tilde{C}_L$ we denote a set of tensor curves in natural parametrization of the length L; for $L<\infty$, $\tilde{C}_L$ is not a subset of C.

(⁷) In a fixed base e_i, $i = 1, 2, 3$, every real valued function $f^{ik}(s)$ defined by the formula $f(s) = f^{ik}(s)e_i\otimes e_k$ is a continuous function of bounded variation in every interval $[0, a]$, $a\in R$.

The formula $f = \bar{f} \circ l$ defines the bijection

$$C \ni f \to (\bar{f},\ l) \in \bigcup_{L} \tilde{C}_L \times S_L. \tag{13.3}$$

Thus on the class of deformation paths C every constitutive equation can be written in the form:

$$\mathbf{T} = \mathfrak{D}_K[\bar{f}, l], \tag{13.4}$$

where $\mathfrak{D}_K : \bigcup\limits_{L} \tilde{C}_L \times S_L \to \mathscr{T}$. The first argument $\bar{f}$ may be called an "ordered sequence of successive states", and the second argument may be called a "time-realization of that sequence". The formula (13.4) is a modification of the formula (2.8) from the paper [25]. ILYUSHIN [1] has for long paid attention to the importance of the notion of natural parametrization of deformation processes.

Let us pass to the description of exchanges $f \to f \circ \sigma$, $\sigma \in \Sigma$. It is obvious, that $\Sigma_C = \Sigma$, since the composition $f \circ \sigma$ of a continuous function f of bounded variation with a monotonically non-decreasing continuous function σ is a continuous function of bounded variation. From the properties of natural parametrization we see that to every exchange $f \to f \circ \sigma$, $\sigma \in \Sigma$ corresponds the exchange

$$(\bar{f}, l) \to (\bar{f}, l \circ \sigma); \tag{13.5}$$

here, both pairs belong to the same set $\tilde{C}_L \times S_L$. Hence the condition defining insensitivity of a material to the exchanges $f \to f \circ \sigma$ is written in the form:

$$\mathfrak{D}_K[\bar{f}, l \circ \sigma] = \mathfrak{D}_K[\bar{f}, l] \tag{13.6}$$

for all $(\bar{f}, l) \in \tilde{C}_L \times S_L$ and all L. This equation clearly reflects the basic notion of the study of inviscosity: without disturbing the sequence of successive deformation states, $\bar{f} \to \bar{f}$, we are seeking such exchanges of time-realization, $l \to l \circ \sigma$, as are not felt by the material([8]).

Since we do not want to lose the possibility of describing discontinuous processes, we must modify the formula (13.4). Thus we consider the following class of deformation paths:

$$\mathscr{Y} \equiv \{f \in \mathscr{X} \mid f^c \in C\}. \tag{13.7}$$

It is obvious, that $\mathscr{Y}$ satisfies the conditions (2.6), (2.7), (2.8). Writing each $f \in \mathscr{Y}$ in the form

$$f = \bar{f}^c \circ l^c + \sum_{i=1}^{N} [f]_i H(s - s_i), \tag{13.8}$$

we obtain a bijection:

$$\mathscr{Y} \ni f \ \leftrightarrow\ [\bar{f}^c, l^c; ([f]_i, s_i)] \in (\bigcup_{L} \tilde{C}_L \times S_L) \times \mathscr{P} \tag{13.9}$$

([8]) The study of general insensitivity semigroup of material based on the formula (13.6) is not, in general, convenient.

(with the previous convention concerning continuous functions). Every constitutive equation can be written in the form:

$$\mathbf{T} = \mathfrak{P}_{\mathrm{K}}[\bar{f}^c, l;\ ([f]_i, s_i)] \tag{13.10}$$

on the class of deformation paths $\mathscr{Y}$.

Note that for discontinuous processes, we could not achieve the decomposition onto "a sequence of successive deformation states" and "time-realization of that sequence". In fact, the sequence s_i may not be replaced, in a general case, by the sequence $l^c(s_i)$ since it may happen that $l^c(s_k) = l^c(s_{k+1}) = \ldots$. This defect can easily be removed for materials insensitive to arresting of the deformation process.

It is obvious that $\Sigma_{\mathscr{Y}} = \Sigma_{\mathscr{X}}$.

Consider materials insensitive to the arresting of the deformation process $\Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{Y}) \supset \supset \mathrm{P}_{\mathscr{Y}}$. Since $\mathscr{Y} \subset \mathscr{X}$, then we may apply Theorem II, according to which

$$\mathbf{T} = \mathfrak{N}_{\mathrm{K}}[\tilde{f}^c;\ ([f]^i_\alpha, \tilde{q}_\alpha]). \tag{13.11}$$

Let us denote by $\tilde{\mathrm{S}}_{\mathrm{L}}$ the set of bijections $[0, \infty) \to [0, \mathrm{L})$. Applying to f^c the passage to natural parametrization, we obtain:

$$\tilde{f}^c \equiv \vartheta_1(f^c) = \vartheta_1(\bar{f}^c \circ l^c) = \bar{f}^c \circ \vartheta_1(l^c) \equiv \bar{f}^c \circ \tilde{l}^c; \tag{13.12}$$

here, $\bar{f}^c \in \tilde{\mathrm{C}}_{\mathrm{L}}$, $\tilde{l}^c \in \tilde{\mathrm{S}}_{\mathrm{L}}$, where $\mathrm{Range}\,\tilde{l}^c = \mathrm{Dom}\bar{f}^c = [0, \mathrm{L})$. Denote by Q_{L} the set of sequences $([f]^i_\alpha, p_\alpha) \in \mathrm{Q}$ which satisfy the condition $p_\alpha < \mathrm{L}$, $\alpha = 1, \ldots, \mathrm{M}$. Let us introduce the notation $\bar{q}_\alpha \equiv \tilde{l}^c(\tilde{q}_\alpha)$. It is obvious, that $\bar{q}_1 < \bar{q}_2 < \ldots < \bar{q}_{\mathrm{M}}$; thus the sequences $\tilde{q}_\alpha$ and $\bar{q}_\alpha$ form a one-to-one correspondence.

Recapitulating(9). We see that there exists a bijection

$$\tilde{\mathscr{X}}^c \times \mathrm{Q} \ni [\tilde{f}^c;\ ([f]^i_\alpha, \tilde{q}_\alpha)] \ \leftrightarrow\ [\bar{f}^c, \tilde{l}^c;\ ([f]^i_\alpha, \bar{q}_\alpha)] \in \bigcup_{\mathrm{L}} \tilde{\mathrm{C}}_{\mathrm{L}} \times \tilde{\mathrm{S}}_{\mathrm{L}} \times \mathrm{Q}_{\mathrm{L}}. \tag{13.13}$$

In other words, constitutive equations of each material insensitive to arresting of a deformation process can be written, for processes included in $\mathscr{Y}$, in the form:

$$\mathbf{T} = \mathfrak{D}_{\mathrm{K}}[\bar{f}^c, \tilde{l}^c;\ ([f]^i_\alpha, \bar{q}_\alpha)]. \tag{II''}$$

This is the form of Theorem II for the class $\mathscr{Y}$.

Now, it is not difficult to consider inviscid materials on $\mathscr{Y}$ — i.e., such that $\Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{Y}) = = \Sigma_{\mathscr{Y}} \equiv \Sigma_{\mathscr{X}}$. Let us notice, first of all, that for every $\sigma \in \Sigma_{\mathscr{X}}$ there exists a pair β, μ (and only one pair) such that

$$\sigma = \beta \circ \mu, \quad \beta \in \mathrm{B}, \quad \mu \in \mathrm{P}_{\mathscr{X}} \tag{13.14}$$

In fact, it sufficess to define μ as follows: the stops of μ are the stops of σ, and only these.

(9) The method assumed of length parametrization of discontinuous processes is entirely different from that presented in [27], where moduli of jumps are added to lengths. That seems to be entirely improper terminology.

THEOREM IV. *A material is completely inviscid relative to the class of deformation processes* $\mathscr{Y}$ — i.e., $\Phi(\mathfrak{F}_K, \mathscr{Y}) = \Sigma_{\mathscr{Y}}$ — *iff there exists an operator* $\mathfrak{R}_K: \bigcup_L \tilde{C}_L \times Q_L \to \mathscr{T}$ *such that*

$$\text{(IV)} \qquad \boxed{\mathbf{T} = \mathfrak{R}_K[\bar{f}^c; ([f]^i_\alpha, \bar{q}_\alpha)]}$$

for every $f \in \mathscr{Y}$.

P r o o f. Since $\Sigma_{\mathscr{Y}} \supset P_{\mathscr{Y}}$, then a necessary condition for $\Phi(\mathfrak{F}_K, \mathscr{Y}) = \Sigma_{\mathscr{Y}}$ is the fulfillment of formula (II). Thus it suffices to consider the invariance of the operator $\mathfrak{D}_K$. To every exchange $f \to f \circ \sigma$, $f \in \mathscr{Y}$, $\sigma \in \Sigma_{\mathscr{Y}}$ corresponds to the following exchange of arguments in this operator:

$$\text{(13.15)} \qquad \bar{f}^c \to \overline{(f \circ \sigma)}^c, \quad \tilde{l}^c \to \widetilde{l^c \circ \sigma}, \quad ([f]^i_\alpha, \bar{q}_\alpha(f)) \to ([f]^i_\alpha, \bar{q}_\alpha(f \circ \sigma)).$$

By making use of the decomposition (13.14), and bearing in mind the definition of $\bar{f}^c, \tilde{l}^c, \bar{q}_\alpha$, we have:

$$\text{(13.16)} \qquad \overline{(f \circ \sigma)}^c = \bar{f}^c, \quad \widetilde{l^c \circ \sigma} = \widetilde{l^c \circ \beta}, \quad \bar{q}_\alpha(f \circ \sigma) = \bar{q}_\alpha(f).$$

The second equation can be written in the form:

$$\text{(13.17)} \qquad \widetilde{l^c \circ \sigma} = \tilde{l}^c \circ \gamma, \quad \gamma = \gamma(\beta, l^c) \equiv (\tilde{l}^c)^{-1} \circ \widetilde{l^c \circ \beta} \in B.$$

To the exchange $f \to f \circ \sigma$ corresponds then the exchange:

$$\text{(13.18)} \qquad [\bar{f}^c, \tilde{l}^c, ([f]^i_\alpha, \bar{q}_\alpha)] \quad \to \quad [\bar{f}^c, \tilde{l}^c \circ \gamma, ([f]^i_\alpha, \bar{q}_\alpha)];$$

here, both aggregates belong to the same set $\tilde{C}_L \times \tilde{S}_L \times Q_L$. By not involving ourselves in the analysis of the mapping $B \ni \beta \to \gamma(\beta, l^c) \in B$ it suffices to state that for every l^c this is a mapping of B onto itself — i.e., $\gamma(B, l^c) = B$. The condition $\Phi(\mathfrak{F}_K, \mathscr{Y}) = \Sigma_{\mathscr{Y}}$ is thus equivalent to the invariance of the operator $\mathfrak{D}_K$ from (II'') with respect to the group of bijections B which acts in its domain $\bigcup_L \tilde{C}_L \times \tilde{S}_L \times Q_L$ according to the prescription (13.18). It suffices to consider the mapping $\tilde{l}^c \to \tilde{l}^c \circ \gamma$, $\gamma \in B$ in the sets $\tilde{S}_L$. We notice at once that B is transitive in $\tilde{S}_L$. In fact, for every $\tilde{l}, \tilde{m} \in \tilde{S}_L$ there exists $\gamma \equiv \tilde{m}^{-1} \circ \tilde{l}: [0, \infty) \to [0, L] \to [0, \infty)$, $\gamma \in B$, such that $\tilde{l} = \tilde{m} \circ \gamma$. In other words, every $\tilde{l}, \tilde{m} \in \tilde{S}_L$ are B-equivalent and, therefore, $\tilde{S}_L$ is a single B-orbit. B-orbits in $\bigcup_L \tilde{C}_L \times \tilde{S}_L \times Q_L$ may, therefore, be identified with the aggregates $[\bar{f}^c; ([f]^i_\alpha, \bar{q}_\alpha)]$. Q.E.D.

The theorem proved belongs to the category of "intuitively obvious" theorems. It is close to a result of paper [26](10).

To this end, we shall show how, from Theorem IV, follows, the form of formulae appearing in Theorem 11, concerning completely inviscid materials of differential type.

In applications to the operator (7.17) and to the subclass of continuous and continuously differentiable N-times deformation paths, Theorem IV takes the form:

$$\text{(13.19)} \qquad \hat{\mathbf{T}} = \mathfrak{S}_K[\widehat{\mathbf{U}^{(i)}_{(i)}}(l), \mathbf{U}];$$

(10) The essential difference are as follows. Firstly, our definition of a completely inviscid material differs from the definition of time-independent material assumed in [26], for we allow only those transformations $s \to \sigma(s)$ which satisfy the condition $\mathrm{Dom}\,\sigma = \mathrm{Range}\,\sigma = R$. Secondly, the result concerns another class of deformation paths, for in [26] were considered absolutely continuous functions only, and at the same time no conditions were imposed on the set of stops. For these reasons our manner of the proof is entirely different and, it seems to us, much simpler.

Here,

(13.20) $$l(s) = \int\limits_0^s \left(\operatorname{tr} \mathbf{U}^{(t)\prime}_{(t)}(\tau)\mathbf{U}^{(t)\prime}_{(t)}(\tau)\right)^{1/2} d\tau, \quad \mathbf{U}^{(t)\prime}_{(t)}(\tau) \equiv \frac{d}{ds}\mathbf{U}^{(t)}_{(t)}(s)|_{s=\tau}$$

(13.21) $$\mathbf{U}^{(t)}_{(t)}(s) \equiv \overline{\mathbf{U}^{(t)}_{(t)}}[l(s)].$$

For a differential material of complexity $\mathbf{N}$, we have

(13.22) $$\mathfrak{S}_{\mathbf{K}}[\overline{\mathbf{U}^{(t)}_{(t)}}, \mathbf{U}] = g_{\mathbf{K}}(\hat{\mathbf{H}}_1, \ldots, \hat{\mathbf{H}}_{\mathbf{N}}; \mathbf{U}),$$

where

(13.23) $$\mathbf{H}_i \equiv (-1)^i \frac{d^i}{dl^i} \overline{\mathbf{U}^{(t)}_{(t)}}(l)|_{l=0}.$$

Since for $\mathbf{D}_1 \equiv -\mathbf{U}^{(t)\prime}_{(t)}(0) \neq \mathbf{0}$, and taking into account (13.20), we have

(13.24) $$l'(0) = |\mathbf{D}_1|, \quad l''(0) = -\frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_2}{|\mathbf{D}_1|}, \ldots$$

Then applying the well known formulae for a derivative of an inverse function l^{-1}, we obtain:

(13.25) $$\hat{\mathbf{H}}_1 = \frac{1}{l'(0)}\hat{\mathbf{D}}_1 = \frac{\hat{\mathbf{D}}_1}{|\mathbf{D}_1|},$$

(13.26) $$\hat{\mathbf{H}}_2 = \frac{1}{[l'(0)]^2}\hat{\mathbf{D}}_2 - \frac{l''(0)}{[l'(0)]^2}\hat{\mathbf{D}}_1 = \frac{1}{|\mathbf{D}_1|}\hat{\mathbf{D}}_2 - \frac{\operatorname{tr}\mathbf{D}_1\mathbf{D}_2}{|\mathbf{D}_1|^4}\hat{\mathbf{D}}_1,$$

. . . .

In this way, we obtain the formulae (9.26). By a suitable modification of the procedure for $\mathbf{D}_1 \neq \mathbf{0}$, we obtain (9.20), etc.

14. Materials insensitive to stopless exchanges of time-realization of deformation paths

Consider the invariance of a material with respect to the fifth semigroup of our catalogue — groups of bijections B.

We confine our considerations to the class of processes $\mathscr{Y}$. It is obvious that $\mathrm{B}_{\mathscr{Y}} = \mathrm{B}$. We wish to find a general form of a constitutive equation for $\Phi(\mathfrak{F}_{\mathbf{K}}, \mathscr{Y}) \supset \mathrm{B}$.

The situation under consideration is different from that in Section 13, since any stopless process f cannot be B-equivalent to a process with stops. It may be hoped that stresses will depend on the variables appearing in (IV), and on additional arguments which preserve some residual information about each stop of the process. It is obvious that the length of stops has no influence upon stress.

Guided by these suspicions, let us first make the following concretization of the general form (13.10) of constitutive equations on $\mathscr{Y}$. We preserve the notation $\bar{f}^c$, $\tilde{l}^c$, $[f]^i_\alpha$, $\bar{q}_\alpha$, $l^c = \tilde{l}^c \circ \mu$. In place of the aggregates $([f]^i_\alpha, \bar{q}_\alpha)$, it is necessary to introduce the aggregates $([f]^i_\alpha, \bar{q}_\alpha, \varepsilon_\alpha)$, where $\varepsilon_\alpha = 1$ if a point s, at which f suffers a jump $[f]^{L\alpha}_\tau$, is the origin of a stop of f and $\varepsilon_\alpha = 0$ if that is not so. Denote by $\bar{x}_i \equiv l^c(x_i)$, $i = 1, \ldots, \mathrm{P}$, where x_i is a sequence of origins of all stops of the function f which do not contain discontinuity points. Denote by d_p, $p = 1, \ldots, \mathrm{Q} \geqslant \mathrm{P}$ the length of all stops of f, with and without discontinuity points.

We show that there exists a bijection:

$$f \leftrightarrow [\bar{f}^c;([f]^i_\alpha, \bar{q}_\alpha, \varepsilon_\alpha), \bar{x}_i; \tilde{l}^c, d_p]. \tag{14.1}$$

In other words, we wish to state that the aggregate written out on the right-hand side contains the entire information about the process f; here, every element of the process is independent of the remaining ones. In fact, a procedure which reconstitutes f is as follows. We determine $\mu(x_i) \equiv (\tilde{l}^c)^{-1}(\bar{x}_i)$, $\mu(q_\alpha) = (\tilde{l}^c)^{-1}(\bar{q}_\alpha)$. Having these numbers and L_α, ε_α as well as the length of all stops d_p, we can, starting from $s = 0$, construct a function μ, and also find the discontinuity points s_i. Knowing μ, $\tilde{l}^c$, we obtain $l^c = \tilde{l}^c \circ \mu$ and $f^c = \bar{f}^c \circ l^c$. In this way, we obtain the aggregate $[\bar{f}^c, l^c, ([f]_i, s_i)]$ equivalent to f, in accordance with (13.8).

Thus we have shown that every constitutive equation can be written in the form:

$$\mathbf{T} = \mathfrak{S}_{\mathrm{K}}[\bar{f}^c,([f]^i_\alpha, \bar{q}_\alpha, \varepsilon_\alpha), \bar{x}^i, \tilde{l}^c, d_p] \tag{14.2}$$

on the class $\mathscr{Y}$, where

$$\mathfrak{S}_{\mathrm{K}}: \bigcup_{\mathrm{L}} \tilde{\mathrm{C}}_{\mathrm{L}} \times \mathrm{Q}_{\mathrm{L}} \times \mathscr{K}_{\mathrm{L}} \times \mathrm{S}_{\mathrm{L}} \times \mathscr{M} \to \mathscr{T};$$

here. $\mathscr{K}_{\mathrm{L}}$ is the set of sequences of the form $0 \leqslant x_1 < \ldots < x_{\mathrm{M}} < \mathrm{L}$, and $\mathscr{M}$ is the set of finite sequences of positive real numbers. This is a further concretization of the formula (13.10). It is now not difficult to state the result sought for.

THEOREM V. *An inviscosity semigroup of material $\Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{Y})$ contains the group of bijections* B *iff there exists an operator* $\mathfrak{T}_{\mathrm{K}}: \bigcup_{\mathrm{L}} \tilde{\mathrm{C}}_{\mathrm{L}} \times \mathrm{Q}_{\mathrm{L}} \times \mathscr{K}_{\mathrm{L}} \to \mathscr{T}$ *such that*

$$\boxed{\mathbf{T} = \mathfrak{T}_{\mathrm{K}}[\bar{f}^c,([f]^i_\alpha, \bar{q}_\alpha, \varepsilon_\alpha), \bar{x}_i]} \tag{V}$$

for every $f \in \mathscr{Y}$.

Proof. To the exchange $f \to f \circ \sigma$, $f \in \mathscr{Y}$, $\sigma \in \mathrm{B}$ corresponds the following exchanges of arguments of the operator $\mathfrak{S}_{\mathrm{K}}$:

$$\begin{aligned} \bar{f}^c &\to \overline{(f \circ \sigma)^c} = \bar{f}^c, \\ \bar{q}_\alpha(f) &\to \bar{q}_\alpha(f \circ \sigma) = \bar{q}_\alpha(f), \\ \bar{x}_i(f) &\to \bar{x}_i(f \circ \sigma) = \bar{x}_i(f) \end{aligned} \tag{14.3}$$

and

$$\tilde{l}^c \to \widetilde{l^c \circ \sigma}, \quad d_p(f) \to d_p(f \circ \sigma); \tag{14.4}$$

here, Range $\widetilde{l^c \circ \sigma}$ = Range $\tilde{l}^c = [0, \mathrm{L})$, $d_p(f \circ \sigma) > 0$ for every $p = 1, \ldots, \mathrm{Q}$. According to the formula (13.17), we have:

$$\widetilde{l^c \circ \sigma} = l^c \circ \gamma, \quad \gamma = \gamma(\sigma, f) \in \mathrm{B}. \tag{14.5}$$

If $d_p(f)$ is the length of a stop of the function f with the origin $r_p(f)$, then $d_p(f \circ \sigma)$ is the length of a stop originating at $\sigma(r_p(f))$, of the function $f \circ \sigma$. We can write

$$d_p(f \circ \sigma) = a_p d_p(f), \tag{14.6}$$

where

$$a_p = a_p(\sigma, f) \equiv \frac{\sigma[r_p(f)+d_p(f)]-\sigma[r_p(f)]}{d_p(f)} \in \mathrm{R}. \tag{14.7}$$

Passing over details of the mappings $\gamma(\sigma, f)$, $a_p(\sigma, f)$ it suffices to state that for each function f different from a constant and possessing, at least, one stop, and for every $\delta \in \mathrm{B}$, $b \in \mathrm{R}$ we can find $\sigma \in \mathrm{B}$ such that $\gamma(\sigma, f) = \delta$ and $a_p(\sigma, f) = b$. Hence it follows that the condition $\Phi(\mathfrak{F}_\mathrm{K}, \mathscr{Y}) \supset \mathrm{B}$ is equivalent to the invariance of the operator $\mathfrak{S}_\mathrm{K}$ with respect to the group of bijections B and the group of positive real numbers R, which act in their domain according to the rule:

$$[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha, \varepsilon_\alpha), \bar{x}_i; \tilde{l}^c, d_p] \rightarrow [\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha, \varepsilon_\alpha), \bar{x}_i; \tilde{l}^c \circ \gamma, a_p d_p]. \tag{14.8}$$

It is obvious that this is a transitive acting in every set $\tilde{\mathrm{S}}_\mathrm{L} \times \mathscr{M}$. Hence orbits in the domain of $\mathfrak{S}_\mathrm{K}$ can be identified with the aggregates $[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha, \varepsilon_\alpha), \bar{x}_i]$. Q.E.D.

15. Materials with instantaneous viscosity

The word "viscosity" originated in mechanics from Newton's viscid fluid. In agreement with this genealogy, by viscosity is understood, usually, a stress dependence of actual stretching, or otherwise, speaking more rigorously, dependence $\mathbf{T}(t)$ of $|\mathbf{D}(t)|$. This is the situation for materials of differential type of complexity one,

$$\mathbf{T} = f_\mathrm{K}(\mathbf{D}, \mathbf{U}) \tag{15.1}$$

if

$$f_\mathrm{K}(a\mathbf{D}, \mathbf{U}) \not\equiv f_\mathrm{K}(\mathbf{D}, \mathbf{U}), \quad a \in \mathrm{R}, \tag{15.2}$$

in particular, for viscid Newtonian and non-Newtonian fluids.

We obtain a generalization and a more precise meaning of this important kind of viscosity, considering the invariance of constitutive operators with respect to the sixth semigroup of our catalogue — the instantaneous semigroup I. We wish to examine then materials which cannot distinguish the deformation path $f \circ \sigma$ from the path f, in the case in which $\sigma'(0) = 1$. The heart of the matter lies in the fact that the actual rate of both processes is the same:

$$(f \circ \sigma)'(0) = f'(0)\sigma'(0) = f'(0). \tag{15.3}$$

We confine our considerations to the following processes

$$\mathscr{X} \equiv \{f \in \mathscr{Y} \mid f \text{ satisfies } (15.5)\}. \tag{15.4}$$

The condition is of the form:

(15.5) there exists a sequence $0 < a_1 < a_2 < \ldots$,
such that f is analytic in every interval $[0, a_1)$, $[a_1, a_2)$, ...

It is obvious that $\mathscr{X}$ satisfies the conditions (2.6), (2.7), (2.8). The set of all restrictions imposed upon the function f belonging to $\mathscr{X}$ follows from (11.1), (13.7). The class of deformation paths under consideration is sufficiently large and, at the same time, it makes it possible to obtain an effective representation formula (VI).

It is obvious that a set of admissible exchanges $\Sigma_\mathscr{X} \subset \Sigma_\mathscr{Y}$ consists of all the $\sigma \in \Sigma_\mathscr{Y}$ which satisfy the additional condition (15.5), where we have first substituted σ in place of f.

Our aim is to consider the invariance of constitutive operators with respect to the semigroup:

$$\mathrm{I}_{\mathscr{Z}} \equiv \mathrm{I} \cap \Sigma_{\mathscr{Z}}, \tag{15.6}$$

combining all these $\sigma \in \Sigma_{\mathscr{Z}}$, for which $\sigma'(0) = 1$. It is easy to ascertain that every $\sigma \in \mathrm{I}_{\mathscr{Z}}$ can be represented (in a unique way) in the form:

$$\sigma = \beta \circ \pi, \quad \beta \in \mathrm{B} \cap \mathrm{I}_{\mathscr{Z}}, \quad \pi \in \mathrm{P} \cap \mathrm{I}_{\mathscr{Z}}. \tag{15.7}$$

For every $\pi \in \mathrm{P} \cap \mathrm{I}_{\mathscr{Z}}$, there exists $a > 0$ such that $\sigma(s) = s$ for $s \in [0, a]$.

Let us introduce

$$\mathscr{Z}_1 \equiv \mathscr{Z} \cap \mathscr{X}_1, \quad \mathscr{Z}_2 \equiv \mathscr{Z} \cap \mathscr{X}_2, \tag{15.8}$$

where $\mathscr{X}_1, \mathscr{X}_2$ are defined by (11.29), (11.30).

THEOREM VI. *An inviscosity semigroup of material $\Phi(\mathfrak{F}_K, \mathscr{Z})$ contains the instantaneous semigroup $\mathrm{I}_{\mathscr{Z}}$ iff there exists an operator $\mathfrak{U}_K: (\bigcup_L \tilde{\mathrm{C}}_L \times \mathrm{Q}_L) \times \mathrm{R} \to \mathscr{T}$ such that*

$$\boxed{\mathbf{T} = \mathfrak{U}_K[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha), \mathfrak{p}(f)]} \tag{VI}$$

for every $f \in \mathscr{Z}$, where $\mathfrak{p}: \mathscr{Z} \to \mathrm{R}$ is a non-negative functional defined by the restriction

$$\mathfrak{p}(f) \equiv \begin{cases} l^{c(n)}(0) & \text{if} \quad f \in \mathscr{Z}_1, \\ 0 & \text{if} \quad f \in \mathscr{Z}_2, \end{cases} \tag{15.9}$$

(15.10) *l^c is a natural parameter of the continuous component f^c,*

$$n \equiv \max\{m \mid l^{c\prime}(0) = l^{c\prime\prime}(0) = \dots = l^{c(m-1)}(0) = 0\}. \tag{15.11}$$

Proof. A necessary condition for $\Phi(\mathfrak{F}_K, \mathscr{Z}) \supset \mathrm{I}_{\mathscr{Z}}$ is $\Phi(\mathfrak{F}_K, \mathscr{Z}) \supset \mathrm{I}_{\mathscr{Z}} \cap \mathrm{P}_{\mathscr{Z}}$. Let us bear in mind Theorem II′. It is obvious that it remains true after we restrict $\mathscr{X}$ to $\mathscr{Z}$, if the semigroup $\mathrm{P}'_{\mathscr{X}}$ is replaced by the semigroup $\mathrm{I}_{\mathscr{Z}} \cap \mathrm{P}_{\mathscr{Z}}$. Using the representation (II′) in a form similar to (II″), we see that a necessary condition for $\Phi(\mathfrak{F}_K, \mathscr{Z}) \supset \mathrm{I}_{\mathscr{Z}}$ is the fulfillment of the condition

$$\mathbf{T} = \mathfrak{Q}'_K[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha), \tilde{l}^c, \varepsilon(f)] \tag{II‴}$$

on $\mathscr{Z}$.

Consider an arbitrary exchange $f \to f \circ \sigma$, $f \in \mathscr{Z}$, $\sigma \in \mathrm{I}_{\mathscr{Z}}$. We have:

$$\overline{(f \circ \sigma)^c} = \bar{f}^c, \quad \bar{q}_\alpha(f \circ \sigma) = \bar{q}_\alpha(f), \quad \varepsilon(f \circ \sigma) = \varepsilon(f), \tag{15.12}$$

and, due to the condition (15.7):

$$\widetilde{l^c \circ \sigma} = \widetilde{l^c \circ \beta}, \quad \beta'(0) = \sigma'(0) = 1. \tag{15.13}$$

The formula

$$\widetilde{l^c \circ \sigma} = \tilde{l}^c \circ \gamma \tag{15.14}$$

defines $\gamma = \gamma(\beta, l)$. We shall not analyze in detail the mapping $\gamma(\beta, l)$, but only state the following characteristic properties.

If $f \in \mathscr{Z}_1$, then l^c does not possess a stop of the type $[0, a]$, $a > 0$. Differentiating the formula (15.14) at $s = 0$, as many times as it is necessary, to obtain a result different form the type $0 = 0$, we arrive at $\gamma'(0) = \beta'(0) = 1$ — i.e., $\gamma(\beta, f) \in \mathrm{I}_{\mathscr{Z}} \cap \mathrm{P}_{\mathscr{Z}}$. It is not difficult to ascertain that $\gamma(\mathrm{I}_{\mathscr{Z}} \cap \mathrm{P}_{\mathscr{Z}}, f) = \mathrm{I}_{\mathscr{Z}} \cap \mathrm{P}_{\mathscr{Z}}$.

If $f \in \mathscr{X}_2$, then l^c possesses a stop $[0, a]$, $a > 0$. Hence,

$$\widetilde{l^c \circ \beta} = \widetilde{\hat{l}^c \circ \hat{\beta}}, \tag{15.15}$$

where

$$\hat{l}^c(s) \equiv l^c(s+a), \quad \hat{\beta}(s) \equiv [s+\beta^{-1}(a)]-a. \tag{15.16}$$

Now, $\gamma'(0) = \hat{\beta}'(0)$ and, in general, $\gamma'(0) \neq 1$. It is not difficult to show that $\gamma(\mathrm{I}_{\mathscr{X}} \cap \mathrm{B}_{\mathscr{X}}, f) = \mathrm{I}_{\mathscr{X}} \cap \mathrm{B}_{\mathscr{X}}$.

Recapitulating, we see that the condition $\Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{X}) \supset \mathrm{I}_{\mathscr{X}}$ is equivalent to the invariance of the operator $\mathfrak{Q}'_{\mathrm{K}}$ with respect to $\mathrm{I}_{\mathscr{X}} \cap \mathrm{B}_{\mathscr{X}}$, for $\varepsilon = 1$, and with respect to $\mathrm{B}_{\mathscr{X}}$, for $\varepsilon = 0$; here, each group acts in its domain according to the prescription:

$$[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha), \tilde{l}^c, \varepsilon] \to [\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha), \tilde{l}^c \circ \gamma, \varepsilon]. \tag{15.17}$$

It suffices to consider how the groups act in the sets $\tilde{\mathrm{S}}^{\mathscr{X}}_{\mathrm{L}}$ ($\tilde{\mathrm{S}}^{\mathscr{X}}_{\mathrm{L}}$ consists of those $\tilde{l} \in \tilde{\mathrm{S}}_{\mathrm{L}}$, which correspond to $f \in \mathscr{X}$).

Every two curves $\tilde{l}, \tilde{k} \in \tilde{\mathrm{S}}^{\mathscr{X}}_{\mathrm{L}}$ are $\mathrm{B}_{\mathscr{X}}$-equivalent, since

$$\tilde{l} = \tilde{k} \circ \sigma, \quad \sigma \equiv \tilde{l} \circ \tilde{k}^{-1} \in \mathrm{B}_{\mathscr{X}}. \tag{15.18}$$

We shall show that two curves $\tilde{l}, \tilde{k} \in \tilde{\mathrm{S}}^{\mathscr{X}}_{\mathrm{L}}$ are $\mathrm{I}_{\mathscr{X}} \cap \mathrm{B}_{\mathscr{X}}$-equivalent iff

$$n(\tilde{l}) = n(\tilde{k}) \quad \text{and} \quad \tilde{l}^{(n)}(0) = \tilde{k}^{(n)}(0), \tag{15.19}$$

where the integer-valued functional n on $\tilde{\mathrm{S}}^{\mathscr{X}}_{\mathrm{L}}$ given by the formula

$$n(\tilde{a}) \equiv \max\{m | \quad \tilde{a}'(0) = \dot{a}''(0) = \ldots = a^{(m-1)}(0) = 0\} \tag{15.20}$$

is well defined, since each $a \in \tilde{\mathrm{S}}^{\mathscr{X}}_{\mathrm{L}}$ is an analytic function at $s = 0$, and different from a constant in the arbitrary neighbourhood of $s = 0$. Differentiating (15.18), we obtain:

$$\begin{aligned} &\tilde{l}'(0) = \tilde{k}'(0)\sigma'(0), \\ &\tilde{l}''(0) = \tilde{k}''(0)[\sigma'(0)]^2 + k'(0)\sigma''(0), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{15.21}$$

Now, if the formulae (15.19) are fulfilled, then

$$\tilde{l}^{(n)}(0) = \tilde{k}^{(n)}(0)[\sigma'(0)]^{(n)} = \tilde{l}^{(n)}(0)[\sigma'(0)]^{(n)}; \tag{15.22}$$

thus $\sigma'(0) = 1$ — i.e., $\sigma \in \mathrm{I}_{\mathscr{X}} \cap \mathrm{B}_{\mathscr{X}}$. Inversely, if $\sigma'(0) = 1$, then from (15.21) we obtain also (15.19).

In this way, we have shown that $\mathrm{B}_{\mathscr{X}}$-orbits in the domain of the operator $\mathfrak{Q}'_{\mathrm{K}}$ can be identified with the aggregates $[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha)]$, and $\mathrm{I}_{\mathscr{X}} \cap \mathrm{B}_{\mathscr{X}}$-orbits with the aggregates $[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha), \tilde{l}^{c(n)}(0)]$. Noting now that

$$\tilde{l}^{c(n)}(0) = l^{c(n)}(0) \equiv \mathfrak{p}(f) \tag{15.23}$$

for every $f \in \mathscr{X}_1$, we obtain (VI). Q.E.D.

An important particular case of (VI) are constitutive equations of the form

$$\mathbf{T} = \mathfrak{U}_{\mathrm{K}}[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha), |f'(0)|]. \tag{15.24}$$

Materials of this kind behave as completely inviscid on processes of the set $\{f \in \mathscr{X} \mid l'(0) \equiv \equiv |f'(0)| = 0\}$. A general material (VI) behaves as completely inviscous on a narrower set $\mathscr{X}_2$, when all derivatives of f at $s = 0$ are equal to zero. Taking as a point of departure the constitutive operator (7.17), and confining our considerations to continuous processes only, we write (15.24) in the form:

$$\hat{\mathbf{T}}(t) = \mathfrak{B}_{\mathrm{K}}[\hat{\bar{\mathbf{U}}}^{(t)}_{(t)}(l), \mathbf{D}(t), \mathbf{U}(t)]. \tag{15.25}$$

A particular case of such a material will be, for instance, an incompressible fluid

$$\mathbf{T}(t) = -p\mathbf{1}+2\mu[\overline{\mathrm{U}\{^{(t)}_{(t)}\}}(l)]\,\mathbf{D}(t) \tag{15.26}$$

with viscosity coefficient μ being a functional of the history of a relative stretch in natural parametrization. The simplest of all particular cases is:

$$\mathbf{T} = -p\mathbf{1}+2\mu(\mathrm{L})\,\mathbf{D}, \tag{15.27}$$

where L is the length of the tensor curve $\mathrm{U}\{^{(t)}_{(t)}\}$. The parameter L is analogical to one of the hardening parameters used in the theory of plasticity (strain-hardening).

16. Materials almost completely inviscous

The last semigroup of our catalogue corresponds to materials of the weakest form of viscosity — almost completely inviscous. Those materials do not distinguish the process $f\circ\sigma$ from the process f, if only $0 < \sigma'(0) < \infty$; the heart of the matter lies in the fact that the actual rate $(f\circ\sigma)'(0)$ of the process is neither zero nor infinity if the actual rate $f'(0)$ of the process f is neither zero nor infinity.

We shall examine the invariance of constitutive operators defined on the deformation class $\mathscr{X}$ with respect to the semigroup $\mathrm{Q}_{\mathscr{X}}$,

$$\mathrm{Q}_{\mathscr{X}} \equiv \mathrm{Q} \cap \Sigma_{\mathscr{X}}. \tag{16.1}$$

Since, according to Fig. 2,

$$\mathrm{I}_{\mathscr{X}} \subset \mathrm{Q}_{\mathscr{X}} \subset \Sigma_{\mathscr{X}}, \tag{16.2}$$

then we may expect that the representative formula sought for will be something between (IV) and (VI). And so it is in fact.

THEOREM VII. *An inviscosity semigroup of material $\Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{X})$ contains the semigroup $\mathrm{Q}_{\mathscr{X}}$ iff there exists an operator $\mathfrak{B}_{\mathrm{K}}: (\bigcup_{\mathrm{L}}\tilde{\mathrm{C}}_{\mathrm{L}}\times \mathrm{Q}_{\mathrm{L}})\times \mathrm{R} \to \mathscr{T}$ such that*

$$\boxed{\mathbf{T} = \mathfrak{B}_{\mathrm{K}}[\bar{f}^c, ([f]^l_\alpha, q_\alpha), \mathfrak{n}(f)]} \tag{VII}$$

for every $f\in\mathscr{X}$, where the functional $\mathfrak{n}$ is defined on $\mathscr{X}$ by the formula:

$$\mathfrak{n}(f) \equiv \begin{cases} n(l^c) & \text{if} \quad f\in\mathscr{X}_1, \\ 0 & \text{if} \quad f\in\mathscr{X}_2, \end{cases} \tag{16.3}$$

and n is defined by (15.11).

P r o o f. Proof of the previous theorem can be repeated here, with the only change that a necessary and sufficient condition $\mathrm{Q}_{\mathscr{X}} \cap \mathrm{B}_{\mathscr{X}}$-equivalence for the curves $\tilde{l}, \tilde{k} \in \tilde{\mathrm{S}}^{\mathscr{X}}_{\mathrm{L}}$ will be now $n(\tilde{l}) = n(\tilde{k})$. Q.E.D.

Materials almost completely inviscid, like materials with instantaneous viscosity, behave as materials completely inviscid only on the subset of processes $\mathscr{X}_2$. Confrontation of formulas (IV), (VI), (VII) is a good illustration of the how delicacy of the problem of viscosity of materials.

17. Comparison of responses of discovered classes of materials in a relaxation test and in a test of uniform retardation-acceleration

Tests mentioned in the title are the simplest experimental ways of changing a time-realization of a deformation process.

Let us consider an exchange

$$f \to f_q \equiv f \circ \sigma_q, \quad q > 0, \tag{17.1}$$

where σ_q is defined by (7.2), thus

$$f_q(s) \equiv \begin{cases} f(0) & \text{for} \quad s \in [0, q), \\ f(s-q) & \text{for} \quad s \in [q, \infty); \end{cases} \tag{17.2}$$

here, f is a deformation path from the set

$$\mathscr{A}_1 \equiv \{f \in \mathscr{A} | f \text{ does not possess a stop of the type } [0, a], a > 0\}. \tag{17.3}$$

Thus in the process $f \circ \sigma_q$ we cease to change deformation from a fixed instant t_0,

$$t_0 \equiv t-q, \quad t = t_0+q. \tag{17.4}$$

The process $f \circ \sigma_q, f \in \mathscr{A}_1, q > 0$ described is called *a relaxation test*; its main purpose lies in the examination of a response of material

$$\mathbf{T}(t_0+q) \equiv \mathfrak{F}_{\mathbf{K}}[f \circ \sigma_q], \tag{17.5}$$

since $q \to \infty$. Let us denote

$$\mathbf{T}(t_0) \equiv \mathfrak{F}_{\mathbf{K}}[f]. \tag{17.6}$$

In the papers [8, 9, 10, 11] it was shown that for what are called *materials with fading memory* the following theorem on relaxation of stresses is true([11]):

$$\mathbf{T}(t_0+q) \to \mathbf{T}(t_0), \tag{17.7}$$

since $q \to \infty$.

The property of materials under study — viscosity — is by nature of a type different from the fading memory mentioned above. In reference to completely inviscous materials (according to our terminology), attention has been paid to this in [26, § V]. A somewhat broader illustration of the difference between viscosity and fading memory gives the following result.

THEOREM 13. *Materials of the types II, III, and* IV *"do not respond to a relaxation test", i.e.,*

$$\mathbf{T}(t_0+q) = \mathbf{T}(t_0) = \text{const} \tag{17.8}$$

for every $q > 0$.

Materials of the types V, VI *and* VII *"do respond to a relaxation test but at the first instant only"* — i.e.,

$$\mathbf{T}(t_0+q) = \mathbf{T}(t_0+\varepsilon) = \text{const}, \tag{17.9}$$

for every $q > \varepsilon > 0$.

([11]) More precisely, it has been shown that one can select a set of processes $\mathscr{H} \subset \mathscr{A}$, and a topology on $\mathscr{H}$, so that the formula (17.7) would arise, in accordance with experiments, for a number of materials.

Materials of type I *respond in a relaxation test according to the formula*:

$$\mathbf{T}(t_0+q) = \mathfrak{F}_{\mathbf{K}}[f\circ\langle q^{-1}\rangle\circ\sigma_1]. \tag{17.10}$$

P r o o f. For materials of types II, III, and IV, σ_q belongs by definition to an inviscosity semigroup. For materials of types V, VI, VII, σ_q does not belong to an inviscosity semigroup, since $\sigma_q\notin \mathrm{B}$, $\sigma_q\notin \mathrm{I}$. For V, according to (V), (17.2),

$$\mathbf{T}(t_0+q) = \mathfrak{T}_{\mathbf{K}}[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha, \varepsilon_\alpha); 0, \bar{x}_1, \ldots]; \tag{17.11}$$

for VI, according to (VI), (17.2),

$$\mathbf{T}(t_0+q) = \mathfrak{U}_{\mathbf{K}}[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha); 0]; \tag{17.12}$$

for VII, according to (VII), (17.2),

$$\mathbf{T}(t_0+q) = \mathfrak{V}_{\mathbf{K}}[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha); 0]. \tag{17.13}$$

Hence follows (17.9). In general, $\mathbf{T}(t_0+q)\neq \mathbf{T}(t_0)$ (for instance, for VI: $\mathfrak{p}(f)\neq 0$, since by assumption $f\in\mathscr{X}_1$).

Finally, for materials of the type I, we define a functional $\mathfrak{c}$ by means of the formula:

$$\mathfrak{c}[f\circ\sigma_q]\equiv q^{-1}. \tag{17.14}$$

on the set of relaxation tests.

This satisfies the condition (10.4), for

$$f\circ\sigma_q\circ\langle a\rangle = f\circ\langle a\rangle\circ\sigma_{q/a} \tag{17.15}$$

and hence

$$\mathfrak{c}[f\circ\sigma_q\circ\langle a\rangle] = \frac{a}{q} = a\,\mathfrak{c}[f\circ\sigma_q]. \tag{17.16}$$

Now,

$$(f\circ\sigma_q)\circ\left\langle\frac{1}{\mathfrak{c}[f\circ\sigma_q]}\right\rangle = f\circ\langle q^{-1}\rangle\circ\sigma_1 \tag{17.17}$$

and from (I) follows (17.10). Q.E.D.

Let us now pass to a uniform retardation-acceleration test.

$$f\to f\circ\langle a\rangle,\qquad a>0. \tag{17.18}$$

THEOREM 14. *Materials of types* I, III, IV, V *and* VII "*do not respond to a retardation-acceleration test*".

A material of type II *responds to this test according to the following formula*:

$$\mathbf{T} = \mathfrak{R}_{\mathbf{K}}[\tilde{f}^c\circ\langle a\rangle, ([f]^i_\alpha, a^{-1}\tilde{q}_\alpha)]. \tag{17.19}$$

And a material of the type VI *responds according to the formula*:

$$\mathbf{T} = \mathfrak{U}_{\mathbf{K}}[\bar{f}^c, ([f]^i_\alpha, \bar{q}_\alpha), a^n\mathfrak{p}(f)], \tag{17.20}$$

where n is defined by (15.11).

P r o o f. For materials I, III, IV, V, VII, $\langle a\rangle$ belongs to an inviscosity semigroup. For the material II, (17.19) follows from (II), since $\widetilde{(f\circ\langle a\rangle)^c} = \tilde{f}^c\circ\langle a\rangle$, $\tilde{q}_\alpha(f\circ\langle a\rangle) = a^{-1}\tilde{q}_\alpha(f)$. For the material VI, (17.19) is implied by (VI), since $\mathfrak{p}(f\circ\langle a\rangle) = a^n\mathfrak{p}(f)$. Q.E.D.

18. Strong inviscosity semigroups

Let us write the constitutive equation of a simple material in its primitive form:

$$\mathbf{T}(t) = \mathfrak{F}_{\mathbf{K}}[\mathbf{F}(t-s)]. \tag{18.1}$$

We have been concerned so far with exchanges

$$\mathbf{F}^{(t)} \to \mathbf{F}^{(t)} \circ \sigma, \quad \mathbf{F}^{(t)}(s) \equiv \mathbf{F}(t-s), \tag{18.2}$$

such that for a fixed t and every $\mathbf{F}^{(t)} \in \operatorname{Dom} \mathfrak{F}_{\mathbf{K}}$

$$\mathfrak{F}_{\mathbf{K}}[\mathbf{F}(t-\sigma(s)] = \mathfrak{F}_{\mathbf{K}}[\mathbf{F}(t-s)]. \tag{18.3}$$

We have been interested in such exchanges of time-realization of deformation history up to the instant t, which did not change the *final* value of the stress $\mathbf{T}(t)$.

Now we ask the second natural question: what exchanges of time-realization of deformation history do not change *the entire sequence* of successive stress states? Let us make this question more precise.

Consider a pair $(\mathbf{F}, \mathbf{T})$

$$\mathbf{F}\colon (-\infty, t] \to \mathscr{T}, \quad \mathbf{T}\colon (-\infty, t] \to \mathscr{T}. \tag{18.4}$$

We may interpret it as a pair parametrized by time $\tau \in (-\infty, t]$ of tensor curves — the first in the deformation space (deformation gradients), the second in the stress space. Another good geometrical interpretation was introduced by ILYUSHIN [1]; the pair $(\mathbf{F}, \mathbf{T})$

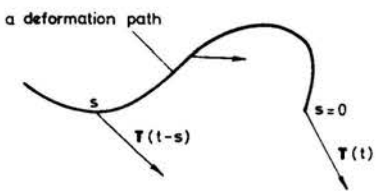

Fig. 3. Ilyushin's geometric representation of a pair (F, T)

is considered as a tensor curve $\mathbf{F}$ in the deformation space brestled at each point $\mathbf{F}(\tau)$, $\tau \in (-\infty, t]$, by the stress tensor $\mathbf{T}(\tau)$, Fig. 3.

A pair $(\mathbf{F}, \mathbf{T})$ will be called *a realizable process* for a material $\mathfrak{F}_{\mathbf{K}}$, if for every $\tau \in (-\infty, t]$

$$\mathbf{T}(\tau) = \mathfrak{F}_{\mathbf{K}}[\mathbf{F}(\tau-s)]. \tag{18.5}$$

Let us introduce a semigroup S of the following continuous transformations

$$\varkappa\colon (-\infty, t] \to (-\infty, t], \tag{18.6}$$

which satisfy the natural conditions

(i) $\varkappa(t) = t$,

(ii) $\varkappa(\tau) \to -\infty$ if $\tau \to -\infty$,

(iii) $\varkappa$ is a monotically non-decreasing continuous function.

Every transformation $\varkappa \in \mathrm{S}$ generates an exchange of pairs

$$(\mathbf{F}, \mathbf{T}) \to (\mathbf{F} \circ \varkappa, \mathbf{T} \circ \varkappa). \tag{18.7}$$

A new property of invariance, hidden in the question asked, we formulate by means of the following definition.

DEFINITION 7. *A transformation $\varkappa \in \mathrm{S}$ is said to be* ***admissible*** *for a given material if for every realizable process* $(\mathbf{F}, \mathbf{T})$ *the pair* $(\mathbf{F} \circ \varkappa, \mathbf{T} \circ \varkappa)$ *is a realizable process. The subsemigroup* $\mathrm{D} \subset \mathrm{S}$ *of all admissible transformations is said to be* ***a global inviscosity semigroup of material.***

Under admissible transformation, Ilyushin's geometrical image of a process does not change; only its time-parametrization changes.

We show that consideration of an insensitivity global semigroup is equivalent to consideration of a certain part of an inviscosity semigroup, in the previous sense. For this purpose, to every $\varkappa \in \mathrm{S}$ let us assign $\sigma_\varkappa \in \Sigma$, defined by the formula

$$\sigma_\varkappa(s) \equiv t - \varkappa(t-s). \tag{18.8}$$

A mapping

$$h\colon\ \mathrm{S} \to \Sigma, \quad h(\varkappa) \equiv \sigma_\varkappa \tag{18.9}$$

is an isomorphism S on Σ. In fact, it is easy to show that this mapping is *on*, invertible and preserves the semigroup structure

$$\sigma_{\varkappa \circ \mu} = \sigma_\varkappa \circ \sigma_\mu. \tag{18.10}$$

The image $h(\mathrm{D})$ of the inviscosity global semigroup D is then a subsemigroup in Σ; $h(\mathrm{D})$ will also be called a *global inviscosity semigroup of material.*

THEOREM 15. *A mapping $\sigma \in \Sigma$ belongs to the global inviscosity semigroup $h(\mathrm{D})$, if the mapping σ_τ defined by the formula*

$$\sigma_\tau(s) \equiv \sigma(t-\tau+s) - \sigma(t-\tau), \quad s \in \mathrm{R}, \tag{18.11}$$

for every $\tau \in (-\infty, t]$ belongs to the inviscosity semigroup Φ; hence,

$$h(\mathrm{D}) \subset \Phi. \tag{18.12}$$

Proof. Let us take an arbitrary $\varkappa \in \mathrm{S}$ and an arbitrary realizable process $(\mathbf{F}, \mathbf{T})$. Then

$$\mathbf{T}(\varkappa(\tau)) = \mathfrak{F}_{\mathbf{K}}[\mathbf{F}(\varkappa(\tau)-s)] \tag{18.13}$$

for every $\tau \in (-\infty, t]$. According to definition 7, $\varkappa \in \mathrm{D}$ iff

$$(\mathbf{T} \circ \varkappa)(\tau) = \mathfrak{F}_{\mathbf{K}}[(\mathbf{F} \circ \varkappa)(\tau - s)] \tag{18.14}$$

for every pair $(\mathbf{F}, \mathbf{T})$ satisfying the condition (18.13), and every $\tau \in (-\infty, t]$.

Let us make use of the identity

$$\varkappa(\tau-s) \equiv \varkappa(\tau) - [\varkappa(\tau) - \varkappa(\tau-s)] \equiv \varkappa(\tau) - [\sigma_\varkappa(t-\tau+s) - \sigma_\varkappa(t-\tau)] \equiv \varkappa(\tau) - \sigma_{\varkappa,\tau}(s), \tag{18.15}$$

in accordance with which

$$(\mathbf{F} \circ \varkappa)(\tau - s) = \mathbf{F}[\varkappa(\tau) - \sigma_{\varkappa,\tau}(s)]. \tag{18.16}$$

Comparing now (18.13) and (18.14), we obtain the following result: $\varkappa \in \mathrm{D}$, iff

$$\mathfrak{F}_{\mathbf{K}}[\mathbf{F}(\varkappa(\tau)-s)] = \mathfrak{F}_{\mathbf{E}}[\mathbf{F}(\varkappa(\tau) - \sigma_{\varkappa,\tau}(s))], \tag{18.17}$$

for every $\mathbf{F}$, and every $\tau \in (-\infty, t]$.

Let us take $\mathbf{G}^{(t)}(s) \equiv \mathbf{F}^{\varkappa(\tau)}(s)$; if $\mathbf{F}^{\varkappa(\tau)}$ runs over the whole set Dom $\mathfrak{F}_{\mathbf{K}}$, then also $\mathbf{G}^{(t)}$ runs over the whole set Dom $\mathfrak{F}_{\mathbf{K}}$. Hence $\varkappa \in \mathrm{D}$, iff

$$\mathfrak{F}_{\mathbf{K}}[\mathbf{G}(t-s)] = \mathfrak{F}_{\mathbf{K}}[\mathbf{G}(t - \sigma_{\varkappa,\tau}(s))] \tag{18.18}$$

for every $G^{(t)} \in \operatorname{Dom} \mathfrak{Q}_{\mathbf{K}}$, and every $\tau \in (-\infty, t)$. This means that $\sigma_{\varkappa,\tau}$ belongs to the inviscosity semigroup Φ, for every $\tau \in (-\infty, t]$. Q.E.D.

The theorem obtained is an effective description of global inviscosity semigroups $h(\mathrm{D})$. It is obvious that every $h(\mathrm{D})$ contains an identity $\iota \in \Sigma$, which satisfies trivially the condition $\iota_\tau \in \Phi$, since $\iota_\tau = \iota$.

DEFINITION 8. *An inviscosity semigroup of material Φ is said to be* **strong** *if it coincides with the global insensitivity semigroup*

$$h(\mathrm{D}) = \Phi \tag{18.19}$$

and **weak**, *if $\Phi \neq \{\iota\}$ and simultaneously*

$$h(\mathrm{D}) \equiv h(\mathrm{D}) \cap \Phi = \{\iota\}. \tag{18.20}$$

Particularly important are strong inviscosity semigroups. They comprise such transformations of time as, acting on an arbitrary deformation path, not only leave the final stress values unchanged but also a whole sequence of stress states, corresponding to that path.

THEOREM 16. *In all the following cases*

$$\begin{array}{ll} \text{I. } \Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{D}) = \mathrm{L}, & \text{IV. } \Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{Y}) = \Sigma_{\mathscr{Y}}, \\ \text{II. } \Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{X}) = \mathrm{P}_{\mathscr{X}}, & \text{V. } \Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{Y}) = \mathrm{B}, \\ \text{III. } \Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{X}) = \mathrm{S}_{\mathscr{X}}, & \end{array} \tag{18.21}$$

the inviscosity semigroup is strong.

In the case

$$\text{VI. } \Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{Z}) = \mathrm{I}_{\mathscr{Z}}, \tag{18.22}$$

the inviscosity semigroup is weak.

In the case

$$\text{VII. } \Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{Z}) = \mathrm{Q}_{\mathscr{Z}} \tag{18.23}$$

the inviscosity semigroup is neither strong nor weak.

Proof. Let us consider the case I. Take an arbitrary $\sigma \in \mathrm{L}$, $\sigma(s) = as$, $a > 0$. We have

$$\sigma_\tau(s) \equiv \sigma(t-\tau+s) - \sigma(t-s) = as = \sigma(s); \tag{18.24}$$

thus $\sigma_\tau = \sigma \in \mathrm{L}$, for every τ. According to Theorem 15, $\sigma \in h(\mathrm{D})$ and hence $\Phi(\mathfrak{F}_{\mathrm{K}}, \mathscr{D}) = h(\mathrm{D})$. In a similar way we prove the first statement of all semigroups indicated.

Let us consider case VI. Take $\sigma \in h(\mathrm{D}) \subset \mathrm{I}_{\mathscr{Z}}$. According to (18.11), we obtain the condition

$$\sigma_\tau'(s) = \sigma'(t-\tau) = 1 \tag{18.25}$$

for every $\tau \in (-\infty, t]$. This equation possesses a unique solution $\sigma = \iota$ in the class $\mathrm{I}_{\mathscr{Z}}$, and hence $h(\mathrm{D}) = \{\iota\}$.

In case VII, the condition $\sigma \in h(\mathrm{D})$ we write in the form:

$$0 < \sigma'(t-r) < \infty \tag{18.26}$$

for every $\tau \in (-\infty, t]$, and thus the global inviscosity semigroup is here a group of bijections without stationary and bend points. This group is a proper part of $\mathrm{Q}_{\mathscr{Z}}$. Q.E.D.

To this end, we only point out the possibility of formulating the problem more generally — namely, to seek pairs of transformations $(\varkappa_1, \varkappa_2) \in \mathrm{S} \times \mathrm{S}$ mapping every realizable process from a certain class, $(\mathbf{F}, \mathbf{T}) \in \mathscr{P}$, into another realizable process from the same class $(\mathbf{F} \circ \varkappa_1, \mathbf{T} \circ \varkappa_2) \in \mathscr{P}$. This problem is significant, for instance, for cyclic processes.

By introducing above the concept of realizable processes, we have ignored completely the equation of motion. It is obvious that if $(\mathbf{F}, \mathbf{T})$ is a realizable process satisfying the motion equations in the presence of a body force $\mathbf{b}$, then the realizable process $(\mathbf{F} \circ \varkappa, \mathbf{T} \circ \varkappa)$, $\varkappa \in \mathrm{D}$ will satisfy the motion equations only then when the body force $\mathbf{b}$ is replaced by a suitably chosen body force $\mathbf{b}^*$. In a quasistatic approximation $\mathbf{b} = \mathbf{b}^*$.

19. Final remarks

Remark 5. In practice, we attempt to work with materials which possess a certain smoothness — continuity and differentiability. This smoothness is chosen according to a motivation which generalizes experimental data. A proper procedure involves the construction of smoothness of a material *after we have made the most complete use possible of information concerning an insensitivity semigroup of material.*

Let us illustrate this notion by an example of simple fluids. Information (5.5) concerning an insensitivity semigroup of simple fluids leads to the representation formula (5.8). We introduce, therefore, continuity and differentiability for the operators $\mathfrak{T}_K$ (cf. [8, 7], and others). If we assume additionally that the simple fluid possesses a non-trivial inviscosity semigroup, then this procedure ceases to be the correct one. If, for instance, $\mathrm{Dom}\, \mathfrak{T}_K \subset \mathscr{X}$ and $\Phi_{\mathfrak{T}_K} = \mathrm{I}_{\mathscr{X}}$, then it is necessary first to apply theorem VI, obtaining a new general form of a constitutive operator of type (VI). The proper procedure now is to construct smoothness of the operator $\mathfrak{U}_K$ by introducing into its domain a suitable topology.

Remark 6. The notion of an inviscosity semigroup introduced led us to the individualization of new classes of materials. At this point, the role of theory temporarily ends — what follows depends on experiments. It is the matter for experiment — and only experiment can answer the question as to if, when — and which among — the *logical possibilities* discovered will find their counterparts, with a sufficient degree of accuracy, as regards the behaviour of real media, which are or will be in use. In the present paper, we have sought to fulfill the first of the requirements set out by HERTZ: "………; in der gereiften Erkenntnis ist die logische Reinheit in erster Linie zu berücksichtigen; nur logisch reine Bilder sind zu prüfen auf ihre Richtigkeit, nur richtige Bilder zu vergleichen nach ihre Zweckmäßigkeit", ([29], p. 11). It is not excluded that some of the possibilities discovered will find their real counterparts outside the field of mechanics (cf. Remark 3).

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