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ON SOME INTEGRAL TRANSFORMATIONS.

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SUPPOSE that x, a, b, c and x', a', b', c' have the same anharmonic ratios, or what is the same thing, let these quantities satisfy the equation

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x, & a, & b, & c \\ x', & a', & b', & c' \\ xx', & aa', & bb', & cc' \end{vmatrix} = 0;$$

this equation may be represented under a variety of different forms, which are obtained without difficulty; thus, if for shortness

$$K \quad = \quad a(b'-c')(x'-a') + b\ (c'-a')(x'-b') + c\ (a'-b')(x'-c'),$$

then

$$Kx \quad = -\{bc(b'-c')(x'-a') + ca(c'-a')(x'-b') + ab(a'-b')(x'-c')\},$$

$$K(x-a) = \quad (c-a)(a-b)(b'-c')(x'-a'),$$

$$K(x-b) = \quad (a-b)(b-c)(c'-a')(x'-b'),$$

$$K(x-c) = \quad (b-c)(c-a)(a'-b')(x'-b').$$

Consider x, x' as variables; then

$$K^2\,dx \quad = \quad (b-c)\ (c-a)\ (a-b)(b'-c')(c'-a')(a'-b')\,dx';$$

let, d, d' be any corresponding values of x, x'; then

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ aa', & bb', & cc', & dd' \end{vmatrix} = 0$$

and we have

$$K(x-d)=D(x'-d');$$

where

$$D=(c'-a')(a'-b')(b'-c')\Lambda,$$

and

$$\Lambda=\frac{(a-d)(b-c)}{(a'-d')(b'-c')}=\frac{(b-d)(c-a)}{(b'-d')(c'-a')}=\frac{(c-d)(a-b)}{(c'-d')(a'-b')}.$$

Suppose $\alpha+\beta+\gamma+\delta=-2$; then

$$(x-a)^\alpha(x-b)^\beta(x-c)^\gamma(x-d)^\delta\,dx=J(x'-a')^\alpha(x'-b')^\beta(x'-c')^\gamma(x'-d')^\delta\,dx,$$

where

$$J=(b-c)^{\beta+\gamma+1}(c-a)^{\gamma+\alpha+1}(a-b)^{\alpha+\beta+1}(b'-c')^{\alpha+\delta+1}(c'-a')^{\beta+\delta+1}(a'-b')^{\gamma+\delta+1}D^\delta.$$

We may in particular take for a', b', c', d' the systems b, a, d, c : c, d, a, b and d, c, b, a respectively; this gives, writing successively y, z, w instead of x',

$$\begin{aligned}&(x-a)^\alpha(x-b)^\beta(x-c)^\gamma(x-d)^\delta\,dx\\&=M(y-a)^\beta(y-b)^\alpha(y-c)^\delta(y-d)^\gamma\,dy\\&=N(z-a)^\gamma(z-b)^\delta(z-c)^\alpha(z-d)^\beta\,dz\\&=P(w-a)^\delta(w-b)^\gamma(w-c)^\beta(w-d)^\alpha\,dw,\end{aligned}$$

where

$$\begin{aligned}M&=-(-)^{\gamma+\delta}(a-c)^{\alpha+\gamma+1}(a-d)^{\alpha+\delta+1}(b-c)^{\beta+\gamma+1}(b-d)^{\beta+\delta+1},\\N&=\quad(-)^{\gamma+\delta}(a-b)^{\alpha+\beta+1}(a-d)^{\alpha+\delta+1}(b-c)^{\beta+\gamma+1}(c-d)^{\gamma+\delta+1},\\P&=\qquad(a-b)^{\alpha+\beta+1}(a-c)^{\alpha+\gamma+1}(b-d)^{\beta+\delta+1}(c-d)^{\gamma+\delta+1};\end{aligned}$$

the relations between the variables x, y, z, w being

$$\begin{aligned}x&=\frac{(c+d)ab-(a+b)cd-(ab-cd)y}{ab-cd-(a+b-c-d)y}\\&=\frac{(b+d)ac-(a+c)bd-(ac-bd)z}{ac-bd-(a+c-b-d)z}\\&=\frac{(b+c)ad-(a+d)bc-(ad-bc)w}{ad-bc-(a+d-b-c)w}:\end{aligned}$$

these are, in fact, the formulæ in my note, "On an Integral Transformation," *Camb. and Dubl. Math. Jour.* t. III. (1848), p. 286 [62], which was suggested to me by Gudermann's transformation for elliptic functions, (*Crelle*, t. XXIII. (1846), p. 330).

Suppose now that the values of a', b', c', d' are 0, 1, ∞, ζ, we have in this case

$$x=\frac{a(b-c)+c(a-b)y}{(b-c)+(a-b)y},$$

and representing the denominator by K, then

$$\begin{aligned} K(x-a) &= -(a-b)(a-c)\,y, \\ K(x-b) &= (a-b)(b-c)(1-y), \\ K(x-c) &= (a-c)(b-c), \\ K(x-d) &= (a-b)(c-d)(y-\zeta), \end{aligned}$$

where

$$\zeta = \frac{(a-d)(b-c)}{(a-b)(c-d)},$$

and we have

$$K^2\,dx = -(a-b)(a-c)(b-c)\,dy,$$

whence

$$(x-a)^\alpha (x-b)^\beta (x-c)^\gamma (x-d)^\delta\,dx = \\ -(-)^\alpha (a-b)^{\alpha+\beta+\delta+1} (a-c)^{\alpha+\gamma+1} (b-c)^{\beta+\gamma+1} (c-d)^\delta\, y^\alpha (1-y)^\beta (y-\zeta)^\delta\,dy.$$

It is easy, by means of this equation, to generalise a remarkable formula given by M. Serret in his memoir, "Sur la Représentation géométrique des Fonctions elliptiques et ultrà-elliptiques," *Liouville*, t. XI. and XII. [1846 and 1847], and *Recueil des Savans étrangers*, t. XI. [1851].[1] In fact, suppose that the indices α, β, γ, δ are integers, and that two of these indices, e.g. γ, δ, are negative, the remaining two indices being positive, then writing $-\gamma$, $-\delta$ instead of γ, δ the integral

$$\int \frac{(x-a)^\alpha (x-b)^\beta\,dx}{(x-c)^\gamma (x-d)^\delta},$$

where $\gamma+\delta=\alpha+\beta+2$, depends on the integral

$$\int \frac{y^\alpha (1-y)^\beta\,dy}{(y-\zeta)^\delta}.$$

Suppose that the fraction under the integral sign is resolved into simple fractions, each of these fractions will be integrable algebraically, except the fraction having for its denominator the simple power $y-\zeta$, the integral of which is a logarithm. The coefficient of this fraction is at once found by writing in the numerator $\zeta+(y-\zeta)$ for y; and expanding in ascending powers of $y-\zeta$ and equating this coefficient to zero, we have

$$\left(\frac{d}{d\zeta}\right)^{\delta-1} \zeta^\alpha (1-\zeta)^\beta = 0,$$

which [observing that $(\gamma-1)+(\delta-1)=\alpha+\beta$] is easily seen to be equivalent to

$$\left(\frac{d}{d\zeta}\right)^{\gamma-1} \zeta^\beta (1-\zeta)^\alpha = 0.$$

[1] M. Serret has reproduced the theorem in his very interesting and instructive treatise, "Cours d'Algèbre supérieure," deuxième édition, Paris, 1854, [quatrième édition, Paris, 1877].

Hence if the function $\zeta = \dfrac{(a-d)(b-c)}{(a-b)(c-d)}$ satisfy this condition, the indefinite integral

$$\int \frac{(x-a)^\alpha (x-b)^\beta \, dx}{(x-c)^\gamma (x-d)^\delta},$$

where $\gamma + \delta = \alpha + \beta + 2$, will be expressible as a rational algebraical fraction.

It may be noticed, that in the general case, observing that $x = a$, $x = b$ give $y = 0$, $y = 1$, the integral

$$\int_a^b (x-a)^\alpha (x-b)^\beta (x-c)^\gamma (x-d)^\delta \, dx$$

depends on

$$\int_0^1 y^\alpha (1-y)^\beta (\zeta - y)^\delta \, dy,$$

or, putting $\zeta = \dfrac{1}{u}$, upon

$$\int_0^1 y^\alpha (1-y)^\beta (1-uy)^\delta \, dy,$$

which is expressible by means of a hypergeometric series having u for its argument or fourth element.

2, *Stone Buildings, Lincoln's Inn, Feb.* 1854.