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ON SIR W. R. HAMILTON'S METHOD FOR THE PROBLEM OF THREE OR MORE BODIES.

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THE problem of three or more bodies is considered by Sir W. R. Hamilton in his two well-known memoirs on a general method in Dynamics, *Phil. Trans.* 1834 and 1835, and the differential equations for the relative motion, with respect to the central body, of all the other bodies are obtained in a form containing a single disturbing function only. Several methods of integration are given or indicated, among others, one which is in fact the method of the variation of the elements as applied to the particular form of the equations of motion. But the investigation shows (and Sir W. R. Hamilton notices this as a defect in his theory, as compared with the ordinary theory of the variation of the elements), that in the method in question, the elements are not osculating elements, i.e. that the positions only, and not the velocities of the bodies, can be calculated as if the elements remained constant during an element of time. The peculiar advantage of the method is of course the having a single disturbing function only, and this seems so important, that if I may venture to express an opinion, I cannot but think that the method will ultimately be employed for the purposes of Physical Astronomy. But, however this may be, it has appeared to me that it may be useful to present the method in a separate and distinct form, disengaged from the general theory as an illustration of which it was given by the author; and this is what I propose now to do.

Consider a central body M, and two other bodies M_1, M_2, and let the coordinates of M referred to a fixed origin be x, y, z, and the coordinates of M_1, M_2 referred to the body M as origin be x_1, y_1, z_1 and x_2, y_2, z_2 respectively. Then the coordinates of M_1, M_2 referred to the fixed origin, are $x+x_1$, $y+y_1$, $z+z_1$ and $x+x_2$, $y+y_2$, $z+z_2$

respectively, and if as usual T denotes the Vis-viva or half sum of each mass into the square of its velocity, and U denote the force function, then we have

$$\begin{aligned} T = {} & \tfrac{1}{2}M\ (x'^2 + y'^2 + z'^2), \\ & + \tfrac{1}{2}M_1 \{(x' + x_1')^2 + (y' + y_1')^2 + (z' + z_1')^2\}, \\ & + \tfrac{1}{2}M_2 \{(x' + x_2')^2 + (y' + y_2')^2 + (z' + z_2')^2\}, \end{aligned}$$

$$\begin{aligned} U = {} & \frac{MM_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}} \\ & + \frac{MM_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}} \\ & + \frac{M_1M_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}, \end{aligned}$$

and the equations of motion are as usual

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} = \frac{dU}{dx},$$

&c.

If we assume that the centre of gravity of the bodies is at rest, then we have

$$Mx' + M_1(x' + x_1') + M_2(x' + x_2') = 0, \text{ \&c.},$$

and consequently

$$x' = -\frac{M_1x_1' + M_2x_2'}{M + M_1 + M_2},\ y' = -\frac{M_1y_1' + M_2y_2'}{M + M_1 + M_2},\ z' = -\frac{M_1z_1' + M_2z_2'}{M + M_1 + M_2}.$$

Now the value of T is

$$\begin{aligned} T = {} & \tfrac{1}{2}(M + M_1 + M_2)(x'^2 + y'^2 + z'^2) \\ & + x'(M_1x_1' + M_2x_2') + y'(M_1y_1' + M_2y_2') + z'(M_1z_1' + M_2z_2') \\ & + \tfrac{1}{2}M_1(x_1'^2 + y_1'^2 + z_1'^2) \\ & + \tfrac{1}{2}M_2(x_2'^2 + y_2'^2 + z_2'^2), \end{aligned}$$

or, putting for x', y', z' their values,

$$\begin{aligned} T = {} & \tfrac{1}{2}M_1(x_1'^2 + y_1'^2 + z_1'^2) \\ & + \tfrac{1}{2}M_2(x_2'^2 + y_2'^2 + z_2'^2) \\ & - \tfrac{1}{2}\frac{1}{M + M_1 + M_2}\{(M_1x_1' + M_2x_2')^2 + (M_1y_1' + M_2y_2')^2 + (M_1z_1' + M_2z_2')^2\}, \end{aligned}$$

and with this new value of T the equations of motion still are

$$\frac{d}{dt}\frac{dT}{dx_1'} - \frac{dT}{dx_1} = \frac{dU}{dx_1}, \text{ \&c.}$$

Suppose now that the differential coefficients of T, with respect to x_1', y_1', z_1'; x_2', y_2', z_2', are respectively P_1, Q_1, R_1; P_2, Q_2, R_2, i.e. write

$$\frac{dT}{dx_1'} = P_1, \text{ \&c.},$$

and imagine T expressed as a function of P_1, Q_1, R_1; P_2, Q_2, R_2, and when this is done put $H = T - U$ (so that H stands for a function of P_1, Q_1, R_1; P_2, Q_2, R_2; x_1, y_1, z_1; x_2, y_2, z_2), then the equations of motion in Sir W. R. Hamilton's form are

$$\frac{dx_1}{dt} = \frac{dH}{dP_1}, \quad \frac{dP_1}{dt} = -\frac{dH}{dx_1}, \text{ \&c.}$$

Now from the last given value of T

$$P_1 = M_1 x_1' - \frac{M_1}{M + M_1 + M_2}(M_1 x_1' + M_2 x_2'),$$

$$P_2 = M_2 x_2' - \frac{M_2}{M + M_1 + M_2}(M_1 x_1' + M_2 x_2'),$$

and thence

$$P_1 + P_2 = \frac{M}{M + M_1 + M_2}(M_1 x_1' + M_2 x_2'),$$

and consequently

$$M_1 x_1' = P_1 + \frac{M_1}{M}(P_1 + P_2),$$

$$M_2 x_2' = P_2 + \frac{M_2}{M}(P_1 + P_2),$$

and we have

$$T = \frac{1}{2M_1}\left[\left\{P_1 + \frac{M_1}{M}(P_1 + P_2)\right\}^2 + \left\{Q_1 + \frac{M_1}{M}(Q_1 + Q_2)\right\}^2 + \left\{R_1 + \frac{M_1}{M}(R_1 + R_2)\right\}^2\right]$$

$$+ \frac{1}{2M_2}\left[\left\{P_2 + \frac{M_2}{M}(P_1 + P_2)\right\}^2 + \left\{Q_2 + \frac{M_2}{M}(Q_1 + Q_2)\right\}^2 + \left\{R_2 + \frac{M_2}{M}(R_1 + R_2)\right\}^2\right]$$

$$- \frac{1}{2M^2}(M + M_1 + M_2)\left[(P_1 + P_2)^2 + (Q_1 + Q_2)^2 + (R_1 + R_2)^2\right],$$

or, reducing,

$$T = \left(\frac{1}{2M_1} + \frac{1}{2M}\right)(P_1^2 + Q_1^2 + R_1^2)$$

$$+ \left(\frac{1}{2M_2} + \frac{1}{2M}\right)(P_2^2 + Q_2^2 + R_2^2)$$

$$+ \frac{1}{M}(P_1 P_2 + Q_1 Q_2 + R_1 R_2),$$

and consequently

$$H = \left(\frac{1}{2M_1} + \frac{1}{2M}\right)(P_1^2 + Q_1^2 + R_1^2)$$
$$+ \left(\frac{1}{2M_2} + \frac{1}{2M}\right)(P_2^2 + Q_2^2 + R_2^2)$$
$$+ \frac{1}{M}(P_1P_2 + Q_1Q_2 + R_1R_2)$$
$$- \frac{MM_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}$$
$$- \frac{MM_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}}$$
$$- \frac{M_1M_2}{\sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}}},$$

and H having this value, the equations of motion are as before mentioned

$$\frac{dx_1}{dt} = \frac{dH}{dP_1}, \quad \frac{dP_1}{dt} = -\frac{dH}{dx_1}, \text{ \&c.}$$

Instead of H write $H + \Upsilon$ where

$$H = \left(\frac{1}{2M_1} + \frac{1}{2M}\right)(P_1^2 + Q_1^2 + R_1^2)$$
$$+ \left(\frac{1}{2M_2} + \frac{1}{2M}\right)(P_2^2 + Q_2^2 + R_2^2)$$
$$- \frac{MM_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}$$
$$- \frac{MM_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}},$$

and

$$\Upsilon = \frac{1}{M}(P_1P_2 + Q_1Q_2 + R_1R_2)$$
$$- \frac{M_1M_2}{\sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}}},$$

and the function Υ is to be treated as a disturbing function. The equations of motion for the body M_1 become

$$\frac{dx_1}{dt} = \frac{M + M_1}{MM_1}P_1 + \frac{d\Upsilon}{dP_1}, \quad \frac{dP_1}{dt} = -\frac{MM_1x_1}{(x_1^2 + y_1^2 + z_1^2)^{\frac{3}{2}}} - \frac{d\Upsilon}{dx_1},$$
$$\frac{dy_1}{dt} = \frac{M + M_1}{MM_1}Q_1 + \frac{d\Upsilon}{dQ_1}, \quad \frac{dQ_1}{dt} = -\frac{MM_1y_1}{(x_1^2 + y_1^2 + z_1^2)^{\frac{3}{2}}} - \frac{d\Upsilon}{dy_1},$$
$$\frac{dz_1}{dt} = \frac{M + M_1}{MM_1}R_1 + \frac{d\Upsilon}{dR_1}, \quad \frac{dR_1}{dt} = -\frac{MM_1z_1}{(x_1^2 + y_1^2 + z_1^2)^{\frac{3}{2}}} - \frac{d\Upsilon}{dz_1},$$

and there is of course a precisely similar system of equations of motion for the body M_2.

If we neglect Υ, the left-hand equations show that P_1, Q_1, R_1 denote the velocities or differential coefficients $\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}$ multiplied by the constant factor $\frac{MM_1}{M+M_1}$, and substituting these values in the right-hand equations, we obtain the ordinary equations for the elliptic motion of the body M_1; and similarly for the body M_2. We may, if we please, complete the solution by the method of the variation of the arbitrary constants. Suppose for this purpose that $a_1, b_1, c_1, e_1, f_1, g_1$ are the elements for the elliptic motion of the body M_1, then treating these elements as variable we must have

$$\frac{dx_1}{da_1}\frac{da_1}{dt}+\frac{dx_1}{db_1}\frac{db_1}{dt}\ldots+\frac{dx_1}{dg_1}\frac{dg_1}{dt}=\frac{d\Upsilon}{dP_1}, \text{ \&c.},$$

$$\frac{dP_1}{da_1}\frac{da_1}{dt}+\frac{dP_1}{db_1}\frac{db_1}{dt}\ldots+\frac{dP_1}{dg_1}\frac{dg_1}{dt}=-\frac{d\Upsilon}{dx_1}, \text{ \&c.},$$

and it appears from these equations that as already noticed the disturbed values of the velocities are not (as they are in the ordinary theory) identical with the undisturbed values.

The disturbing function Υ may be considered as a function of the elements of the two orbits and of the time, and it is easy to obtain, as in the ordinary theory, the values of the differential coefficients $\frac{da_1}{dt}$, &c. in the form

$$\frac{da_1}{dt}=(a_1,\ b_1)\frac{d\Upsilon}{db_1}+(a_1,\ c_1)\frac{d\Upsilon}{dc_1}\ldots+(a_1,\ g_1)\frac{d\Upsilon}{dg_1},$$

where

$$(a_1,\ b_1)=\frac{\delta(a_1,\ b_1)}{\delta(x_1,\ P_1)}+\frac{\delta(a_1,\ b_1)}{\delta(y_1,\ Q_1)}+\frac{\delta(a_1,\ b_1)}{\delta(z_1,\ R_1)},$$

if for shortness

$$\frac{\delta(a_1,\ b_1)}{\delta(x_1,\ P_1)}=\frac{da_1}{dx_1}\frac{db_1}{dP_1}-\frac{da_1}{dP_1}\frac{db_1}{dx_1}.$$

It will be remembered that in the ordinary theory, if Ω denote Lagrange's disturbing function ($\Omega=-R$ if R is the disturbing function of the Mécanique Céleste) the corresponding formulæ are

$$\frac{da}{dt}=(a,\ b)\frac{d\Omega}{db}+(a,\ c)\frac{d\Omega}{dc}\ldots+(a,\ g)\frac{d\Omega}{dg},$$

where

$$(a,\ b)=\frac{\delta(a,\ b)}{\delta(x',\ x)}+\frac{\delta(a,\ b)}{\delta(y',\ y)}+\frac{\delta(a,\ b)}{\delta(z',\ z)},$$

if for shortness

$$\frac{\delta(a,\ b)}{\delta(x',\ x)}=\frac{da}{dx'}\frac{db}{dx}-\frac{da}{dx}\frac{db}{dx'},$$

or, what is the same thing, where

$$(a, b) = -\frac{\delta(a, b)}{\delta(x, x')} - \frac{\delta(a, b)}{\delta(y, y')} - \frac{\delta(a, c)}{\delta(z, z')},$$

and

$$\frac{\delta(a, b)}{\delta(x, x')} = \frac{da}{dx}\frac{db}{dx'} - \frac{da}{dx'}\frac{db}{dx}.$$

Now the values of the coefficients (a_1, b_1), &c. depend merely on the form of the expressions for a_1, b_1, &c. in terms of P_1, Q_1, R_1, x_1, y_1, z_1 and t; hence comparing the two systems of formulæ and observing P_1, Q_1, R_1 (which in the formulæ for the present theory correspond with x_1', y_1', z_1' in the other system of formulæ) are respectively equal to x_1', y_1', z_1', each of them multiplied by the constant factor $\frac{MM_1}{M+M_1}$, it is easy to see that the formulæ for the variations of any given system of elements in the present theory are at once deduced from the formulæ for the variations of the same system of elements in the ordinary theory by writing $-\Upsilon$ in the place of Ω and multiplying the values of the variations by the constant factor $\frac{MM_1}{M+M_1}$.

Take then as elements Jacobi's canonical system (¹), viz. if we put

a_1, the semiaxis major,

e_1, the eccentricity,

ϖ_1, the longitude in orbit of pericentre,

ϵ_1, the mean longitude in orbit at epoch,

θ_1, the longitude of node,

ϕ_1, the inclination,

and

n_1, the mean motion $\left\{= \sqrt{\left(\frac{M+M_1}{a_1^3}\right)}\right\}$,

then the canonical elements are

$$\mathfrak{A}_1 = -\tfrac{1}{2}n_1^2 a_1^2,$$

$$\mathfrak{B}_1 = n_1 a_1 \sqrt{(1-e_1^2)},$$

$$\mathfrak{C}_1 = n_1 a_1 \sqrt{(1-e_1^2)}\cos\phi_1,$$

$$\mathfrak{F}_1 = \frac{1}{n_1}(\epsilon_1 - \varpi_1),$$

$$\mathfrak{G}_1 = \varpi_1 - \theta_1,$$

$$\mathfrak{H}_1 = \theta_1,$$

[1] I have for uniformity adopted Jacobi's canonical system, see his paper "Neues Theorem der analytischen Mechanik," *Crelle*, t. xxx. pp. 117—120 (1846); but it is proper to remark that Sir W. R. Hamilton, in his Memoirs above referred to, employs a slightly different but equally elegant system of canonical elements, and that the discovery of such a system belongs to Sir W. R. Hamilton, and is part of his general theory.

(the signs of the two elements $\mathfrak{A}_1$, $\mathfrak{F}_1$ have been changed, but this makes no difference in the formulæ) then the equations for the variations of the elements are

$$\frac{d\mathfrak{A}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{F}_1},$$

$$\frac{d\mathfrak{B}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{G}_1},$$

$$\frac{d\mathfrak{C}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{H}_1},$$

$$\frac{d\mathfrak{F}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{A}_1},$$

$$\frac{d\mathfrak{G}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{B}_1},$$

$$\frac{d\mathfrak{H}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{C}_1},$$

and it is easy thence to deduce the formulæ for the variations of any system of elements which it may be thought proper to make use of, for instance the system $a_1, e_1, \varpi_1, \epsilon_1, \theta_1, \phi_1$.

It will be recollected that in the preceding system of formulæ the value of the disturbing function Υ is

$$\Upsilon = \frac{1}{M}(P_1P_2 + Q_1Q_2 + R_1R_2) - \frac{M_1M_2}{\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\}}},$$

and that as a first approximation P_1, Q_1, R_1 are respectively equal to the velocities x_1', y_1', z_1', each multiplied by the constant factor $\frac{MM_1}{M+M_1}$, and P_2, Q_2, R_2 are respectively equal to the velocities x_2', y_2', z_2', each multiplied by $\frac{MM_2}{M+M_2}$.

2, *Stone Buildings*, 18*th Oct.*, 1856.