182.

ON LAGRANGE'S SOLUTION OF THE PROBLEM OF TWO FIXED CENTRES.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 76—83.]

THE following variation of Lagrange's Solution of the Problem of Two Fixed Centres([1]), is, I think, interesting, as showing more distinctly the connection between the differential equations and the integrals. The problem referred to is as follows: viz. to determine the motion of a particle acted upon by forces tending to two fixed centres, such that r, q being the distances of the particle from the two centres respectively, and α, β, γ being constants, the forces are $\frac{\alpha}{r^2}+2\gamma r$ and $\frac{\beta}{q^2}+2\gamma q$.

Take the first centre as origin and the line joining the two centres as axis of x; and let h be the distance between the two centres, then writing for symmetry

$$x = x_1 = x_2 + h,$$

(so that x_1 is the coordinate corresponding to the first centre as origin, and x_2 the coordinate corresponding to the second centre as origin) the distances are given by the equations

$$r^2 = x_1^2 + y^2 + z^2, \qquad q^2 = x_2^2 + y^2 + z^2,$$

and the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{\alpha x_1}{r^3} - \frac{\beta x_2}{q^3} - 2\gamma (x_1 + x_2),$$

$$\frac{d^2y}{dt^2} = -\frac{\alpha y}{r^3} - \frac{\beta y}{q^3} - 4\gamma y,$$

$$\frac{d^2z}{dt^2} = -\frac{\alpha z}{r^3} - \frac{\gamma z}{q^3} - 4\gamma z,$$

[1] Lagrange's Solution was first published in the *Anciens Mémoires de Turin*, t. IV., [1766—69], and is reproduced in the *Mécanique Analytique*.

and we obtain at once the integral of Vis-viva, viz. multiplying the three equations by $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, adding and integrating $\left(\text{observing that } \frac{dx}{dt} = \frac{dx_1}{dt} = \frac{dx_2}{dt}\right)$, we have

$$\tfrac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \frac{\alpha}{r} + \frac{\beta}{q} - \gamma\,(r^2 + q^2) + 2H, \qquad (1,\ a)$$

and with equal facility, the equation of areas round the line joining the two centres, viz. multiplying the second and third equations by $-z$, y, adding and integrating, we have

$$y\,\frac{dz}{dt} - z\,\frac{dy}{dt} = B. \qquad (2,\ a)$$

So far Lagrange: to obtain a third integral I form the equation

$$\left[-2\,(y^2+z^2)\frac{dx}{dt} + (x_1+x_2)\left(y\,\frac{dy}{dt} + z\,\frac{dz}{dt}\right)\right] \times \left\{\frac{d^2x}{dt^2} + \frac{\alpha x_1}{r^3} + \frac{\beta x_2}{q^3} + 2\gamma\,(x_1+x_2)\right\}$$

$$+\left[\ (x_1+x_2)\,y\,\frac{dx}{dt} - \quad 2x_1x_2\,\frac{dy}{dt}\ \right] \times \left\{\frac{d^2y}{dt^2} + \frac{\alpha y}{r^3} + \frac{\beta y}{q^3} + 4\gamma y\right\}$$

$$+\left[\ (x_1+x_2)\,z\,\frac{dx}{dt} - \quad 2x_1x_2\,\frac{dz}{dt}\ \right] \times \left\{\frac{d^2z}{dt^2} + \frac{\alpha z}{r^3} + \frac{\beta z}{q^3} + 4\gamma z\right\} = 0.$$

The terms independent of the forces are

$$-2\,(y^2+z^2)\frac{dx}{dt}\frac{d^2x}{dt^2} + (x_1+x_2)\left(y\,\frac{dy}{dt} + z\,\frac{dz}{dt}\right)\frac{d^2x}{dt^2}$$

$$+ (x_1+x_2)\frac{dx}{dt}\left(y\,\frac{d^2y}{dt^2} + z\,\frac{d^2z}{dt^2}\right) - 2x_1x_2\left(\frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right),$$

which are equal to

$$\frac{d}{dt}\left[-(y^2+z^2)\left(\frac{dx}{dt}\right)^2 + (x_1+x_2)\frac{dx}{dt}\left(y\,\frac{dy}{dt} + z\,\frac{dz}{dt}\right) - x_1x_2\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}\right],$$

and the terms depending on the forces are readily reduced to the form

$$\frac{d}{dt}\left\{-\frac{h\alpha x_1}{r} + \frac{h\beta x_2}{q} + h^2\gamma\,(y^2+z^2)\right\},$$

in fact, considering first the terms multiplied by α, these are

$$\frac{x_1}{r^3}\left\{\quad -2\quad (y^2+z^2)\frac{dx}{dt} + (x_1+x_2)\left(y\,\frac{dy}{dt} + z\,\frac{dz}{dt}\right)\right\}$$

$$+\frac{1}{r^3}\left\{(x_1+x_2)(y^2+z^2)\frac{dx}{dt} - \quad 2x_1x_2\left(y\,\frac{dy}{dt} + z\,\frac{dz}{dt}\right)\right\},$$

which is equal to

$$\frac{1}{r^3}\left\{(x_2-x_1)(y^2+z^2)\frac{dx}{dt}+x_1(x_1-x_2)\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{1}{r^3}\left\{-h(y^2+z^2)\frac{dx}{dt}+hx_1\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{h}{r^3}\left\{-(x_1{}^2+y^2+z^2)\frac{dx_1}{dt}+x_1\left(x\frac{dx_1}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{h}{r^3}\left\{-r^2\frac{dx_1}{dt}+rx_1\frac{dr}{dt}\right\}$$

$$=-h\frac{d}{dt}\,\frac{x_1}{r},$$

and similarly the term multiplied by β is

$$=\quad h\frac{d}{dt}\,\frac{x_2}{q},$$

lastly, the term multiplied by γ is

$$-4(x_1+x_2)(y^2+z^2)\frac{dx}{dt}+2(x_1+x_2)^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$+4(x_1+x_2)(y^2+z^2)\frac{dx}{dt}-\;8x_1x_2\;\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\quad 2(x_1-x_2)^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\quad 2h^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\quad \frac{d}{dt}\,h^2(y^2+z^2).$$

The preceding combination of the differential equations gives therefore an equation integrable per se, and effecting the integration we have

$$-(y^2+z^2)\left(\frac{dx}{dt}\right)^2+(x_1+x_2)\frac{dx}{dt}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)-x_1x_2\left\{\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}$$

$$-\frac{h\alpha x_1}{r}+\frac{h\beta x_2}{q}+h^2\gamma(y^2+z^2)=K, \qquad (3, a)$$

which is the third integral equation. It may be convenient to mention here (what appears by the comparison of the formulæ obtained in the sequel with the corresponding formulæ of Lagrange) that the value of Lagrange's constant of integration C is

$$C=K-2Hh^2-B^2+\tfrac{1}{4}\gamma h^4.$$

Making use of the ordinary transformation

$$(y^2+z^2)\left\{\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2+\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)^2,$$

the integral equations may be written under the forms

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2+\frac{1}{2(y^2+z^2)}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2=\frac{\alpha}{r}+\frac{\beta}{q}-\gamma(r^2+q^2)+2H-\frac{B^2}{2(y^2+z^2)}, \qquad (1,\ b)$$

$$y\frac{dz}{dt}-z\frac{dy}{dt}=B, \qquad (2,\ b)$$

$$-(y^2+z^2)\left(\frac{dx}{dt}\right)^2+(x_1+x_2)\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\frac{dx}{dt}-\frac{x_1x_2}{y^2+z^2}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2$$
$$-\frac{h\alpha x_1}{r}+\frac{h\beta x_2}{q}+h^2\gamma(y^2+z^2)=K+\frac{B^2x_1x_2}{y^2+z^2}, \qquad (3,\ b)$$

and observing that y^2+z^2, x_1, x_2 are in fact functions of r, q, it is clear that the determination of r, q in terms of t depends upon the first and third equations alone. Moreover the form of the equations shows that we can at once eliminate dt and thus obtain a differential equation between r, q alone. It would be difficult to discover *à priori*, before actually obtaining the differential equation in question, that it would be possible to effect the separation of the variables, but we know that this can be done by taking instead of r, q the new variables $u=r+q$, $s=r-q$. In order to complete the solution the first step is to introduce the variables r, q into the first and third equations: for this purpose we have

$$x_1=\frac{h^2+r^2-q^2}{2h},\quad -x_2=\frac{h^2-r^2+q^2}{2h},\quad x_1+x_2=\frac{r^2-q^2}{h},$$

$$y^2+z^2=\frac{\nabla}{4h^2},$$

if for shortness

$$\nabla=2r^2q^2+2h^2r^2+2h^2q^2-h^4-r^4-q^4,$$

and consequently

$$\frac{dx}{dt}=\frac{1}{h}\left(r\frac{dr}{dt}-q\frac{dq}{dt}\right),$$

$$y\frac{dy}{dt}+z\frac{dz}{dt}=\frac{1}{2h^2}\left\{(h^2-r^2+q^2)\,r\frac{dr}{dt}+(h^2+r^2-q^2)\,q\frac{dq}{dt}\right\}.$$

Substituting these values in the two equations, we find for the first equation

$$\nabla(rdr-qdq)^2+\{(h^2-r^2+q^2)\,rdr+(h^2+r^2-q^2)\,qdq\}^2$$
$$=2h^2\left\{\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma(r^2+q^2)\nabla+2H\nabla-2h^2B^2\right\}dt^2, \qquad (1,\ c)$$

and for the second equation

$$\begin{aligned}&-\nabla^2(rdr-qdq)^2\\&+2\nabla(rdr-qdq)(r^2-q^2)\{(h^2-r^2+q^2)rdr+(h^2+r^2-q^2)qdq\}\\&+\{h^4-(r^2-q^2)^2\}\{(h^2-r^2+q^2)rdr+(h^2+r^2-q^2)qdq\}^2\end{aligned}$$

$$=h^4\left[\frac{2\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{2\beta\nabla}{q}(h^2-r^2+q^2)-\gamma\nabla^2+4K\nabla-4B^2\{h^4-(r^2-q^2)^2\}\right]dt^2. \quad (2,c)$$

The first equation is easily reduced to

$$4h^2\{r^2q^2(dr^2+dq^2)+(h^2-r^2-q^2)rq\,dr\,dq\}=2h^2\left\{\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma(r^2+q^2)\nabla+2H\nabla-2h^2B^2\right\}dt^2,$$

the second equation gives

$$h^4\{(h^2-r^2+q^2)rdr+(h^2+r^2-q^2)qdq\}^2-h^4\{(h^2-r^2-3q^2)rdr-(h^2-3r^2+q^2)qdq\}^2$$

$$=h^4\left[\frac{2\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{2\beta\nabla}{q}(h^2-r^2+q^2)-\gamma\nabla^2+4K\nabla-4B^2\{h^4-(r^2-q^2)^2\}\right]dt^2,$$

and the function on the left-hand side is

$$8h^4(h^2-r^2-q^2)q^2r^2(dr^2+dq^2)+4h^4\{(h^2-r^2-q^2)^2+4q^2r^2\}rq\,dr\,dq.$$

Hence putting for a moment

$$M=\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma\nabla(r^2+q^2)+2H\nabla-2h^2B^2,$$

$$N=\frac{\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{\beta\nabla}{q}(h^2-r^2+q^2)-\tfrac{1}{2}\gamma\nabla^2+2K\nabla-2B^2\{h^4-(r^2-q^2)^2\},$$

we have

$$2r^2q^2(dr^2+dq^2)+(h^2-r^2-q^2)2rq\,dr\,dq=Mdt^2,$$

$$2(h^2-r^2-q^2)2r^2q^2(dr^2+dq^2)+\{(h^2-r^2-q^2)^2+4q^2r^2\}2rq\,dr\,dq=Ndt^2,$$

and thence recollecting that

$$-(h^2-r^2-q^2)^2+4q^2r^2=\nabla,$$

we find

$$\nabla\,.\,2rq\,dr\,dq=\{(h^2-r^2-q^2)2M-N\}dt^2,$$

$$\nabla\,.\,2r^2q^2(dr^2+dq^2)=[-\{(h^2-r^2-q^2)^2+4q^2r^2\}M+(h^2-r^2-q^2)N]dt^2,$$

and substituting for M, N their values, the functions on the right-hand side contain ∇ as a factor, and dividing by ∇, we obtain

$$\begin{aligned}2rq\,dr\,dq=\Big[&\frac{\alpha}{r}(3r^2+q^2-h^2)+\frac{\beta}{q}(r^2+3q^2-h^2)\\&-\tfrac{1}{2}\gamma(3r^4+3q^4+10q^2r^2-2h^2r^2-2h^2q^2-h^4)\\&+4H(q^2+r^2-h^2)+2K-2B^2\Big]dt^2,\end{aligned} \quad (1,d)$$

$$2r^2q^2(dr^2+dq^2) = [2\alpha r(r^2+3q^2-h^2)+2\beta q(3r^2+q^2-h^2) - \tfrac{1}{2}\gamma\{r^6+q^6+15q^2r^2(r^2+q^2)-h^2(r^4+q^4+6r^2q^2)-h^4(r^2+q^2)+h^6\} + 2H\{r^4+q^4+6r^2q^2-2h^2(r^2+q^2)+h^4\} + 2K(r^2+q^2-h^2)-2(r^2+q^2)B^2]\,dt^2, \qquad (2,\ d)$$

and by comparing the first of these formulæ with the corresponding formulæ of Lagrange, we find, as already observed, that the relation between the constant K and Lagrange's constant C is $K = C + 2Hh^2 + B^2 - \tfrac{1}{4}\gamma h^4$. And substituting this value of K, the two equations become identical with those of Lagrange[1].

The equation $y\dfrac{dz}{dt} - z\dfrac{dy}{dt} = B$, (putting $y = \sqrt{(y^2+z^2)}\cos\phi$, $z = \sqrt{(y^2+z^2)}\sin\phi$) gives at once $(y^2+z^2)\,d\phi = Bdt$, and substituting for y^2+z^2 its value $= \dfrac{\nabla}{4h^2}$, we find

$$d\phi = \frac{4h^2B}{4q^2r^2-(h^2-r^2-q^2)^2}\,dt, \qquad (3,\ d)$$

which is the third of Lagrange's equations.

To complete the solution, the combination of the first and second equations gives

$$r^2q^2(dr\pm dq)^2 = [\alpha\{(r\pm q)^3-h^2(r\pm q)\}\pm\beta\{(r\pm q)^3-h^2(r\pm q)\} - \tfrac{1}{4}\gamma\{(r\pm q)^6-h^2(r\pm q)^4-h^4(r\pm q)^2+h^6\} + H\{(r\pm q)^4-2h^2(r\pm q)^2+h^4\} + 2K\{(r\pm q)^2-h^2\}-2B^2(r\pm q)^2]\,dt^2,$$

and thence putting $r+q=s$, $r-q=u$ and writing for shortness

$$S = (\alpha+\beta)(s^3-s^2h) - \tfrac{1}{4}\gamma(s^6-h^2s^4-h^4s^2+h^6) + H(s^4-2h^2s^2+h^4) + 2K(s^2-h^2)-2B^2s^2,$$

and

$$U = (\alpha-\beta)(u^3-u^2h) - \tfrac{1}{4}\gamma(u^6-h^2u^4-h^4s^2+h^6) + H(u^4-2h^2u^2-h^4) + 2K(u^2-h^2)-2B^2u^2,$$

[1] The formulæ referred to are the formulæ (*b*), (*c*), *Méc. Anal.* t. II. page 112 of the second edition and page 97 of the third edition, but there is an inaccuracy in the formulæ (*c*), B^2 ought to be changed into B^2h^2; the error is continued in the subsequent formulæ and moreover the constant term $-Ch^2$ is omitted on the right-hand side of the formulæ (*e*) and in the subsequent formulæ, i.e. in the functions of s, u, the term $-B^2$ should be $-B^2h^2-Ch^2$.

we have

$$\tfrac{1}{16}(s^2-u^2)^2\,ds^2 = S\,dt^2, \tag{1, e}$$

$$\tfrac{1}{16}(s^2-u^2)^2\,du^2 = U\,dt^2, \tag{2, e}$$

$$d\phi = -\frac{4h^2B}{(s^2-h^2)(u^2-h^2)}\,dt, \tag{3, e}$$

and thence finally

$$\frac{ds}{\sqrt{(S)}} = \frac{du}{\sqrt{(U)}}, \tag{1, f}$$

$$dt = \tfrac{1}{4}\left\{\frac{s^2ds}{\sqrt{(S)}} - \frac{u^2du}{\sqrt{(U)}}\right\}, \tag{2, f}$$

$$d\phi = \frac{Bh^2ds}{(s^2-h^2)\sqrt{(S)}} - \frac{Bu^2du}{(u^2-h^2)\sqrt{(U)}}, \tag{3, f}$$

so that the problem is reduced to quadratures, the functions to be integrated involving the square roots of two rational and integral functions of the sixth degree.

2, *Stone Buildings*, 10*th Nov*., 1856.