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REPORT ON THE RECENT PROGRESS OF THEORETICAL DYNAMICS.

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THE object of the *Mécanique Analytique* of Lagrange is described by the author in the "Avertissement" to the first edition as follows:—"On a déjà plusieurs traités de mécanique, mais le plan de celui-ci est entièrement neuf. Je me suis proposé de réduire la théorie de cette science et l'art de résoudre tous les problêmes qui s'y rapportent à des formules générales dont le simple développement donne toutes les équations nécessaires pour la solution de chaque problême." And the intention is carried out; the principle of virtual velocities furnishes the general formulæ for the solution of statical problems, and d'Alembert's principle then leads to the general formulæ for the solution of dynamical problems. The general theory of statics would seem to admit of less ulterior development; but as regards dynamics, the formulæ of the first edition of the *Mécanique Analytique* have been the foundation of a series of profound and interesting researches constituting the science of analytical dynamics. The present report is designed to give, so far as I am able, a survey of these researches; there will be found at the end a list, in chronological order, of the works and memoirs referred to, and I shall in the course of the report preserve as far as possible the like chronological order. It is proper to remark that I confine myself to the general theories of dynamics. There are various *special problems* of great generality, and susceptible of the most varied and extensive developments, such for instance as the problem of the motion of a single particle (which includes as particular cases the problem of central forces, that of two fixed centres, and that of the motion of a conical pendulum, either with or without regard to the motion of the earth round its axis), the problem of three bodies, and the problem of the rotation of a solid body about a fixed point. But a detailed account of the researches of geometers in relation to these special problems would properly form the subject of a separate report, and it is not my intention to enter upon them otherwise than incidentally, so far as it may appear desirable to do so. One problem, however, included in the first of the above-mentioned special problems, I shall have frequent occasion to allude to: I mean the problem of the variation of the elements of a planet's orbit, which has a close historical connexion with the general theories which form the subject of this report. The so-called ideal coordinates of Hansen, and the principles of his method of integration in the planetary and lunar theories, have a bearing on the general subject, and might have been considered in the present report; but on the whole I have considered it better not to do so.

1. Lagrange, *Mécanique Analytique*, 1788.—The equations of motion are obtained, as before mentioned, by means of the principle of virtual velocities and d'Alembert's principle. In their original forms they involve the coordinates x, y, z of the different particles m or dm of the system, quantities which in general are not independent. But Lagrange introduces, in place of the coordinates x, y, z of the different particles, any variables or (using the term in a general sense) coordinates $\xi, \psi, \phi, \ldots$ whatever, determining the position of the system at the time t: these may be taken to be independent, and then if $\xi', \psi', \phi', \ldots$ denote as usual the differential coefficients of $\xi, \psi, \phi, \ldots$ with respect to the time, the equations of motion assume the form

$$\frac{d}{dt}\frac{dT}{d\xi'} - \frac{dT}{d\xi} + \Xi = 0;$$
$$\vdots$$

or when $\Xi, \Psi, \Phi, \ldots$ are the partial differential coefficients with respect to $\xi, \psi, \phi, \ldots$ of one and the same function V, then the form

$$\frac{d}{dt}\frac{dT}{d\xi'} - \frac{dT}{d\xi} + \frac{dV}{d\xi} = 0.$$
$$\vdots$$

In these equations, T, or the *vis viva* function, is the *vis viva* of the system, or sum of all the elements each into the half square of its velocity, expressed by means of the coordinates $\xi, \psi, \phi, \ldots$; and (when such function exists) V, or the force function[1], is a function depending on the impressed forces and expressed in like manner by means of the coordinates $\xi, \psi, \phi, \ldots$; the two functions T and V are given functions, by means of which the equations of motion for the particular problem in hand are completely expressed. In any dynamical problem whatever, the *vis viva* function T is a given function of the coordinates $\xi, \psi, \phi, \ldots$, of their differential coefficients $\xi', \psi', \phi', \ldots$ and of the time t; and it is of the second order in regard to the differential coefficients $\xi', \psi', \phi', \ldots$; and (when such function exists) the force function V is a given function of the coordinates $\xi, \psi, \phi, \ldots$ and of the time t. This is the most general form of the functions T, V, as they occur in dynamical problems, but in an extensive class of such problems the forms are less general, viz. T and V are each of them independent of the time, and T is a homogeneous function of the second order in regard to the differential coefficients $\xi', \psi', \phi', \ldots$; the equations of motion

[1] The sign attributed to V is that of the *Mécanique Analytique*, but it would be better to write $V = -U$, and to call U (instead of V) the force function.

have in this case an integral $T+V=h$, which is the equation of *vis viva*, and the problems are distinguished as those in which the principle of *vis viva* holds good. It is to be noticed also that in this case since t does not enter into the differential equations, the integral equations will contain t in the form $t+c$, that is, in connexion with an arbitrary constant c attached to it by addition.

2. The above-mentioned form is *par excellence* the Lagrangian form of the equations of motion, and the one which has given rise to almost all the ulterior developments of the theory; but it is proper just to refer to the form in which the equations are in the first instance obtained, and which may be called the unreduced form, viz. the equations for the motion of a particle whose rectangular coordinates are x, y, z, are

$$m\frac{d^2x}{dt^2}=X+\lambda\frac{dL}{\delta x}+\mu\frac{dM}{\delta x}+\dots$$

$$\vdots$$

where $L=0$, $M=0, \dots$ are the equations of condition connecting the coordinates of the different points of the system, and λ, $\mu, \dots$ are indeterminate multipliers.

3. The idea of a force function seems to have originated in the problems of physical astronomy. Lagrange, in a memoir "On the Secular Equation of the Moon," crowned by the French Academy of Sciences in the year 1774, expressed the attractive forces, decomposed in the directions of the axes of coordinates, by the partial differential coefficients of one and the same function with respect to these coordinates. And it was in these problems natural to distinguish the forces into principal and disturbing forces, and thence to separate the force function into two parts, a principal force function and a disturbing function. The problems of physical astronomy led also to the idea of the variation of the arbitrary constants of a mechanical problem. For as a fact of observation the planets move in ellipses the elements of which are slowly varying; the motion in a fixed ellipse was accounted for by the principal force, the attraction of the sun; the effect of the disturbing force is to produce a continual variation of the elements of such elliptic orbit. Euler, in a memoir published in 1749 in the *Memoirs of the Academy of Berlin* for that year, obtained differential equations of the first order for two of the elements, viz. the inclination and the longitude of the node, by making the arbitrary constants which express these elements in the fixed orbit to vary: this seems to be the first attempt at the method of the variation of the arbitrary constants. Euler afterwards treated the subject in a more complete manner, and the method is also made use of by Lagrange in his "Memoir on the Perturbations of the Planets" in the Berlin Memoirs for 1781, 1782, 1783, and by Laplace in the *Mécanique Céleste,* t. I. 1799. The method in its original form seeks for the expressions of the variations of the elements in terms of the differential coefficients of the disturbing function *with respect to the coordinates.* As regards one element, the longitude of the epoch, such expression (at least in a finite form) was first obtained by Poisson in his memoir of 1808, to be spoken of presently; but I am not able to refer to any place where such expressions in their best form are even now to be found; the question seems to have been unduly passed over in consequence of the new form immediately afterwards assumed by the method. It was very early

observed that the variation of one of the elements, viz. the mean distance, was expressible in a remarkable form by means of the differential coefficients of the disturbing function *taken with respect to the time t, in so far as it entered into the function through the coordinates of the disturbed planet.* I am not able to say at what time, or whether by Euler, Lagrange, or Laplace, it was observed that such differential coefficient with respect to the time was equivalent to the differential coefficient of the disturbing function with respect to one of the elements. But however this may be, the notion of the representation of the variations of the elements by means of the differential coefficients of the disturbing function *with respect to the elements* had presented itself *à posteriori,* and was made use of in an irregular manner prior to the year 1800, and therefore some eight years at any rate before the establishment by Lagrange of the general theory to which these forms belong.

4. Poisson's memoir of the 20th of June, 1808, "On the Secular Inequalities of the Mean Motion of the Planets," was presented by him to the Academy at the age of twenty-seven years. It contains, as already remarked, an expression in finite terms for the variation of the longitude of the epoch. But the memoir is to be considered rather as an application of known methods to an important problem of physical astronomy, than as a completion or extension of the theory of the variation of the planetary elements. The formulæ made use of are those involving the differential coefficients of the disturbing function with respect to the *coordinates;* and there is nothing which can be considered an anticipation of Lagrange's idea of the investigation, *à priori,* of expressions involving the differential coefficients with respect to the elements. But, as well for its own sake as historically, the memoir is a very important one. Lagrange, in his memoir of the 17th of August, 1808, speaks of it as having recalled his attention to a subject with which he had previously occupied himself, but which he had quite lost sight of; and Arago records that, on the death of Lagrange, a copy in his own handwriting of Poisson's memoir was found among his papers; and the memoir is referred to in, and was probably the occasion of, Laplace's memoir also of the 17th of August, 1808.

5. With respect to Laplace's memoir of the 17th of August, 1808, it will be sufficient to quote a sentence from the introduction to Lagrange's memoir:—"Ayant montré à M. Laplace mes formules et mon analyse, il me montra de son côté en même temps des formules analogues qui donnent les variations des élémens elliptiques par les différences partielles d'une même fonction, relatives à ces élémens. J'ignore comment il y est parvenu; mais je présume qu'il les a trouvées par une combinaison adroite des formules qu'il avait données dans la *Mécanique Céleste.*" This is, in fact, the character of Laplace's analysis for the demonstration of the formulæ.

6. In Lagrange's memoir of the 17th of August, 1808, "On the Theory of the Variations of the Elements of the Planets, and in particular on the Variations of the Major Axes of their Orbits," the question treated of appears from the title. The author obtains formulæ for the variations of the elements of the orbit of a planet in terms of the differential coefficients of the disturbing function with respect to the elements; but the method is a general one, quite independent of the particular form

of the integrals, and the memoir may be considered as the foundation of the general theory. The equations of motion are considered under the form,

$$\frac{d^2x}{dt^2} - \frac{1+m}{r^3}x = \frac{d\Omega}{dx},$$

$$\frac{d^2y}{dt^2} - \frac{1+m}{r^3}y = \frac{d\Omega}{dy},$$

$$\frac{d^2z}{dt^2} - \frac{1+m}{r^3}z = \frac{d\Omega}{dz},$$

and it is assumed that the terms in Ω being neglected, the problem is completely solved, viz., that the three coordinates, x, y, z, and their differential coefficients, x', y', z', are each of them given as functions of t, and of the constants of integration a, b, c, f, g, h; the disturbing function Ω is consequently also given as a function of t, and of the arbitrary constants. The velocities are assumed to be the same as in the undisturbed orbit. This gives the conditions

$$\delta x = 0, \qquad \delta y = 0, \qquad \delta z = 0;$$

and then the equations of motion give

$$\delta\,\frac{dx}{dt} = \frac{d\Omega}{dx}, \quad \delta\,\frac{dy}{dt} = \frac{d\Omega}{dy}, \quad \delta\,\frac{dz}{dt} = \frac{d\Omega}{dz},$$

equations in which δx, &c. denote the variations of x, &c., arising from the variations of the arbitrary constants, viz., $\delta x = \frac{dx}{da}\delta a + \frac{dx}{db}\delta b +$, &c. The differential coefficients $\frac{d\Omega}{\delta x}$, &c., can of course be expressed by means of $\frac{d\Omega}{da}$, &c.; and, by a simple combination of the several equations, Lagrange deduces expressions for $\frac{d\Omega}{da}$, &c., in terms of $\frac{da}{dt}$, &c.; viz.

$$\frac{d\Omega}{da} = (a,\ b)\frac{db}{dt} + (a,\ c)\frac{dc}{dt} + (a,\ f)\frac{df}{dt} + (a,\ g)\frac{dg}{dt} + (a,\ h)\frac{dh}{dt},$$

$$\vdots$$

where (1)

$$(a,\ b) = \frac{\partial\,(x,\ x')}{\partial\,(a,\ b\ )} + \frac{\partial\,(y,\ y')}{\partial\,(a,\ b\ )} + \frac{\partial\,(x,\ z')}{\partial\,(a,\ b)},$$

in which, for shortness,

$$\frac{\partial\,(x,\ x')}{\partial\,(a,\ b\ )} \text{ stands for } \frac{dx}{da}\frac{dx'}{db} - \frac{dx'}{da}\frac{dx}{db}.$$

[1] These are substantially the formulæ of Lagrange; but I have introduced here and elsewhere the very convenient abbreviation, due, I think, to Prof. Donkin, of the symbols $\frac{\partial\,(x,\ x')}{\partial\,(a,\ b)}$.

The form of the expressions shows at once that $(a, b) = -(b, a)$, so that the number of the symbols (a, b) is in fact fifteen.

Lagrange proceeds to show, that the differential coefficient with respect to t of the expression represented by the symbol (a, b) vanishes identically; and it follows, that the coefficients (a, b) are *functions of the elements only, without the time t.*

The general formulæ are applied to the problem in hand; and, in consequence of the vanishing of several of the coefficients (a, b), it is easy in the particular problem to pass from the expressions for $\frac{d\Omega}{da}$, &c. in terms of $\frac{da}{dt}$, &c. to those for $\frac{da}{dt}$, &c. in terms of $\frac{d\Omega}{da}$, &c. The author thus obtains an elegant system of formulæ for the variations of the elements of a planet's orbit, in terms of the differential coefficients of the disturbing function with respect to the elements; but it is not for the present purpose necessary to consider the form of the system, or the astronomical consequences deduced by means of it.

7. Lagrange's memoir of the 13th of March, 1809, "On the General Theory of the Variation of the Arbitrary Constants in all the Problems of Mechanics."—The method of the preceding memoir is here applied to the general problem; the equations of motion are considered under the form

$$\frac{d}{dt}\frac{dT}{dr'} - \frac{dT}{dr} + \frac{dV}{dr} = \frac{d\Omega}{dr},$$
$$\vdots$$

where T and V are of the degree of generality considered in the *Mécanique Analytique*, viz., T is a function of $r, s \ldots r', s', \ldots$ homogeneous of the second order as regards the differential coefficients $r', s', \ldots$, and V is a function of $r, s, \ldots$ only; or, rather, the equations are considered in a form obtained from the above, by writing $T - V = R$, viz., in the form

$$\frac{d}{dt}\frac{dR}{dr'} - \frac{dR}{dr} = \frac{d\Omega}{dr},$$

and, as in the preceding memoir, expressions are investigated for the differential coefficients $\frac{d\Omega}{dt}$, &c., in terms of $\frac{db}{dt}$, &c.: these are, as before, of the form

$$\frac{d\Omega}{da} = (a, b)\frac{db}{dt} + , \text{ \&c.}$$

where (a, b), &c., are in the body of the memoir obtained under a somewhat complicated form, and this complicates also the demonstration which is there given of the theorem that (a, b), &c. *are functions of the elements only, without the time t*; but in the Addition (published as part of the memoir, and without a separate date) and in the Supplement the investigation is simplified, and the true form of the functions (a, b) obtained viz., writing $\frac{dT}{dr} = \rho, \ldots$ then

$$(a, b) = \frac{\partial(r, \rho)}{\partial(a, b)} + \frac{\partial(s, \sigma)}{\partial(a, b)} + \ldots$$

if, for shortness,

$$\frac{\partial(r, \rho)}{\partial(a, b)} = \frac{dr}{da}\frac{d\rho}{db} - \frac{d\rho}{da}\frac{dr}{db}, \text{ \&c.}$$

The representation of $\frac{dT}{dr'}$, $\frac{dT}{ds'}$, &c. by single letters *is* made by Lagrange in the Addition, No. 26 (Lagrange writes $\frac{dT}{dr'} = T'$, $\frac{dT}{ds'} = T''$, &c.), but quite incidentally in that number only, for the sake of the formula just stated: I have noticed this, as the step is an important one.

8. It is proper to remark that, in order to prove that the expressions (a, b), &c. are independent of the time, Lagrange, instead of considering the differential coefficients of each of these functions separately, establishes a general equation (see Nos. 25, 34, 35 of the Addition, and also the Supplement)

$$\frac{d}{dt}\left(\Delta r\delta\frac{dR}{dr'} - \delta r'\Delta\frac{dR}{dr'} + \ldots\right) = 0,$$

where, if Δa, $\Delta b, \ldots$ denote any arbitrary increments whatever of the constants of integration $a, b \ldots$ then Δr, &c. are the corresponding increments of the coordinates r, &c.; this is, in fact, a grouping together of several distinct equations by means of arbitrary multipliers, and it is extremely elegant as a method of demonstration, and has been employed as well by Lagrange, here and elsewhere, as by others who have written on the subject; but I think the meaning of the formulæ is best seen when the component equations of the group are separately exhibited, and in the citation of formulæ I have therefore usually followed this course. Lagrange gives also an equation which is in fact a condensed form of the preceding expression for $\frac{d\Omega}{da}$, but which it is proper to mention, viz.:

$$\frac{d\Omega}{da}dt = \frac{dr}{da}\,\delta\frac{dR}{dr'} + \ldots - \delta r\frac{d}{da}\frac{dR}{dr'} - \ldots.$$

In fact, in the formula $\delta\frac{dR}{dr}$ stands for $\left(\frac{d}{da}\frac{dR}{dr'}\frac{da}{dt} + \frac{d}{db}\frac{dR}{dr}\frac{db}{dt} + \ldots\right)dt$, and δr for $\left(\frac{dr}{da}\frac{da}{dt} + \frac{dr}{dt}\frac{db}{dt} + \right)dt$; and, on substituting these values, the identity of the two expressions is seen without difficulty.

9. Lagrange remarks, that in the case where the condition of *vis viva* holds good, then if a be the constant of *vis viva* $(T + V = a)$, and c the constant attached by addition to the time, then $\frac{da}{dt} = \frac{d\Omega}{dc}$, which, he observes, is an equation remarkable as

well from its simplicity and generality as because it can be obtained *à priori*, independently of the variations of the other arbitrary constants: this is obviously the generalisation of the expression for the variation of the mean distance of a planet.

10. The consideration of Lagrange's function (a, b) originated, as appears from what has preceded, in the theory of the variation of the elements; but it is to be noticed, that the function (a, b) is altogether independent of the disturbing function, and the fundamental theorem that (a, b) is a function of the elements only, without the time, is a property of the undisturbed equations of mótion. The like remark applies to Poisson's function (a, b), in the memoir next spoken of.

11. Poisson's memoir of the 16th of October, 1809.—The formulæ of this memoir are, so to speak, the reciprocals of those of Lagrange. The relations between the differential coefficients $\frac{d\Omega}{da}$, &c. of the disturbing function and the variations $\frac{da}{dt}$, &c. of the elements, depend with Lagrange, upon expressions for the coordinates and their differential coefficients in terms of the time and the elements; with Poisson, on expressions for the elements in terms of the time, and of the coordinates and their differential coefficients. The distinction is far more important than would at first sight appear, and the theory of Poisson gives rise to developments which seem to have nothing corresponding to them in the theory of Lagrange. The reason is as follows: when the system of differential equations is completely integrated, it is of course the same thing whether we have the integral equations in the form made use of by Lagrange, or in that by Poisson, the two systems are precisely equivalent the one to the other; but when the equations are not completely integrated, suppose, for instance, we have an expression for one of the coordinates in terms of the time and the elements, it is impossible to judge whether this is or is not one of the integral equations; the differential equations are not satisfied by means of this equation alone, but only by this equation with the assistance of the other integral equations. On the other hand, when we have an expression for one of the constants of integration in terms of the time and of the coordinates and their differential coefficients, it is possible, by mere substitution in the differential equations, and without the knowledge of any other integral equations, to see that the differential equations are satisfied, and that the assumed expression is, in fact, one of the system of integral equations. An expression of the form just referred to, viz., $c = \phi(t, x, y, \ldots x', y' \ldots)$, where the right-hand side does not contain any of the arbitrary constants, may, with great propriety, be termed an "integral," as distinguished from an integral equation, in which the constants and variables may enter in any conceivable manner; it is convenient also to speak of such equation simply as the integral c. [These locutions were introduced by Jacobi.]

12. Returning now to the consideration of Poisson's memoir, the equations of motion are considered under the same form as by Lagrange, viz., putting $T - V = R$, under the form

$$\frac{d}{dt}\frac{dR}{d\phi'} - \frac{dR}{d\phi} = \frac{d\Omega}{d\phi};$$

$$\vdots$$

but Poisson writes

$$\frac{dR}{d\phi} = s, \ldots$$

thus, in effect, introducing a new set of variables, $s, \ldots$ equal in number to the coordinates $\phi, \ldots$, but he does not complete the transformation of the differential equations by the introduction therein of the new variables $s, \ldots$ in the place of the differential coefficients $\phi', \ldots$; this very important transformation was only effected a considerable time afterwards by Sir W. R. Hamilton. Poisson then assumes that the undisturbed equations are integrated in the form above adverted to, viz., that the several elements $a, b, \ldots$ are given as functions of the time t, and of the coordinates ϕ, &c. and their differential coefficients ϕ', &c. or, what is the form ultimately assumed, as functions of the time t, of the coordinates $\phi, \ldots$, and of the new variables s, &c.; and he then forms the functions

$$(a, b) = \frac{\partial (a, b)}{\partial (s, \phi)} + \ldots$$

where

$$\frac{\partial (a, b)}{\partial (s, \phi)} = \frac{da}{ds}\frac{db}{d\phi} - \frac{db}{ds}\frac{da}{d\phi},$$

(the notation is the abbreviated one before referred to), and he proves by differentiation that the differential coefficient of (a, b) with respect to the time vanishes: that is, that (a, b) which, by its definition is given as a function of t and of the coordinates $\phi, \ldots$, and of the new variables $s, \ldots$, is really a constant. Upon which Poisson remarks—"On conçoit que la constante...sera en général une fonction de a et b et des constantes arbitraires contenues dans les autres intégrales des équations du mouvement; quelquefois il pourra arriver que sa valeur ne renferme ni la constante a ni la constante b; dans d'autres cas elle ne contiendra aucune constante arbitraire, et se réduira à une constante déterminée; mais afin &c."

13. The importance of the remark seems to have been overlooked until the attention of geometers was called to it by Jacobi; it has since been developed by Bertrand and Bour.

It is clear from the definition that $(a, b) = -(b, a)$. It may be as well to remark that the denominator of the functional symbol is (s, ϕ) and not (ϕ, s), which would reverse the sign. [It may be noticed that throughout the Report, I speak of the Lagrange's Coefficients (a, b), and Poisson's Coefficients (a, b), distinguishing them in this manner, and not by any difference of notation.]

14. The equations for the variations of the elements are without difficulty shown to be

$$\frac{da}{dt} = (a, b)\frac{d\Omega}{db} + \ldots,$$

which have the advantage over those of Lagrange of giving directly $\frac{da}{dt}$, &c. in terms of $\frac{d\Omega}{da}$, &c., instead of these expressions having to be determined from the value of $\frac{d\Omega}{da}$, &c. in terms of $\frac{da}{dt}$, &c.

15. Poisson applies his formulæ to the case of a body acted upon by a central force varying as any function of the distance, and also to the case of a solid body revolving round a fixed point. There is, as Poisson remarks, a complete similarity between the formulæ for these apparently very different problems, but this arises from the analogy which exists between the arbitrary constants chosen in the memoir for the two problems. The formulæ obtained form a very simple and elegant system, and one which, although not actually of the canonical form (the meaning of the term will be presently explained), might by a slight change be reduced to that form.

16. I may notice here a problem suggested by Poisson in a report to the Institute in the year 1830, on a manuscript work by Ostrogradsky on Celestial Mechanics, viz., in the case of a body acted upon by a central force, the effect of a disturbing function, *which is a function only of the distance from the centre,* is merely to alter the amount of the central force; and the expressions for the variations of the elements should therefore, in the case in question, admit of exact integration; the report is to be found in *Crelle*, t. VII. [1831], pp. 97—101.

17. The two memoirs of Lagrange and Poisson, which have been considered, establish the general theory of the variation of the arbitrary constants, and there is not, I think, very much added to them by Lagrange's memoir of 1810, the second edition of the *Mécanique Analytique*, 1811, or Poisson's memoir of 1816. The memoir by Maurice, in 1844, belongs to this part of the subject, and as its title imports, it is in fact a development of the theories of Lagrange and Poisson.

18. There is, however, one important point which requires to be adverted to. Lagrange, in the memoir of 1810, and the second edition of the *Mécanique Analytique*, remarks, that for a particular system of arbitrary constants, viz., if $\alpha, \ldots$ denote the initial values of the coordinates $\xi, \ldots$ and $\lambda, \ldots$ denote the initial values of $\dfrac{dT}{d\xi'}, \ldots$ then the equations for the variations of the elements take the very simple form

$$\frac{d\alpha}{dt} = -\frac{d\Omega}{d\lambda}, \ldots, \quad \frac{d\lambda}{dt} = \frac{d\Omega}{d\alpha}, \ldots;$$

this is, in fact, the original idea and simplest example of a system of canonical elements; viz. of a system composed of pairs of elements, α, λ, the variations of which are given in the form just mentioned.

19. The "Avertissement" to the second edition of the *Mécanique Analytique*, contains the remark, that it is not necessary that the disturbing function Ω should actually exist; $\dfrac{d\Omega}{dx}, \dfrac{d\Omega}{dy}, \dfrac{d\Omega}{dz}$ may be considered as mere conventional symbols standing for forces X, Y, Z, not the differential coefficients of one and the same function, and then $\dfrac{d\Omega}{da}$ will be a conventional symbol standing for $\dfrac{d\Omega}{dx}\dfrac{dx}{da} + \dfrac{d\Omega}{dy}\dfrac{dy}{da} + \dfrac{d\Omega}{dz}\dfrac{dz}{da}$, and similarly for $\dfrac{d\Omega}{db}$, &c.; and this being so, all the formulæ will subsist as in the case of an actually existing disturbing function.

20. Cauchy, in a note in the *Bulletin de la Société Philomatique* for 1819 (reproduced in the "Mémoire sur l'Intégration des Equations aux Dérivées Partielles du Premier Ordre," *Exer. d'Anal. et de Physique Math.*, t. II. pp. 238—272 (1841)), showed that the integration of a partial differential equation of the first order could be reduced to that of a single system of ordinary differential equations. A particular case of this general theorem was afterwards obtained by Jacobi in the course of his investigations (founded on those of Sir W. R. Hamilton) on the equations of dynamics, and he was thence led to a slightly different form of the general theorem previously established by Cauchy, viz., Cauchy's method gives the *general*, Jacobi's the *complete* integral, of the partial differential equation. The investigations of the geometers who have written on the theory of dynamics are based upon those of Sir W. R. Hamilton and Jacobi, and it is therefore unnecessary, in the present report, to advert more particularly to Cauchy's very important discovery.

21. I come now to Sir W. R. Hamilton's memoirs of 1834 and 1835, which are the commencement of a second period in the history of the subject. The title of the first memoir shows the object which the author proposed to himself, viz., the discovery of a function by means of which the integral equations can be all of them actually represented. The method given for the determination of this function, or rather of each of the several functions which answer the purpose, presupposes the knowledge of the integral equations; it is therefore not a *method of integration*, but a theory of the representation of the integral equations assumed to be known. I venture to dissent from what appears to have been Jacobi's opinion, that the author missed the true application of his discovery; it seems to me, that Jacobi's investigations were rather a theory collateral to, and historically arising out of the Hamiltonian theory, than the course of development which was of necessity to be given to such theory. But the new form obtained in Sir W. R. Hamilton's memoirs for the equations of motion, is a result of not less importance than that which was the professed object of the memoirs.

22. Hamilton's principal function V.—The formulæ are given for the case of any number of free particles, but, for simplicity, I take the case of a single particle. The equations of motion are taken to be

$$m\frac{d^2x}{dt^2}=\frac{dU}{dx},$$

$$m\frac{d^2y}{dt^2}=\frac{dU}{dy},$$

$$m\frac{d^2y}{dt^2}=\frac{dU}{dz};$$

so that the *vis viva* function is

$$T=\tfrac{1}{2}m\,(x'^2+y'^2+z'^2),$$

and the force function, taken with Lagrange's sign, would be $-U$. It is assumed that the condition of *vis viva* holds, that is, that U is a function of x, y, z only. The initial values of the coordinates are denoted by a, b, c, and those of the velocities by a', b', c'. The equation of *vis viva* is $T = U + H$, and this gives rise to an equation $T_0 = U_0 + H$ of the same form for the initial values of the coordinates. The author then writes

$$V = \int_0^t 2T dt,$$

an equation, the form of which implies that T is expressed as a function of the time and of the constants of integration a, b, c, a', b', c'. The method of the calculus of variations leads to the equation

$$\delta V = m(x'\delta x + y'\delta y + z'\delta z) - m(a'\delta a + b'\delta b + c'\delta c) + t\delta H,$$

to understand which, it should be remarked that the coordinates x, y, z, and the velocities x', y', z', being functions of t and of a, b, c, a', b', c', then V is, in the first instance, given as a function of these quantities. But x, y, z being functions of a, b, c, a', b', c', t, we may conversely consider a', b', c' as functions of x, y, z, a, b, c, t, and thus V becomes a function of x, y, z, a, b, c, t. In like manner H is a function of x, y, z, a, b, c, t, and, eliminating t, we have V a function of x, y, z, a, b, c, H, which is the form in which in the last equation V is considered to be expressed. The equation then gives

$$\frac{dV}{dx} = mx', \quad \frac{dV}{dy} = my', \quad \frac{dV}{dz} = mz',$$

$$\frac{dV}{da} = -ma', \quad \frac{dV}{db} = -mb', \quad \frac{dV}{dc} = -mc'$$

$$\frac{dV}{dH} = t;$$

and, considering V as a known function of x, y, z, a, b, c, H, the elimination of H gives a set of equations which are in fact the integral equations of the problem, viz., the first three equations and the last equation give equations containing x, y, z, x', y', z', t and a, b, c, that is, the intermediate integrals; the second three equations and the last equation, give equations containing x, y, z, t, a, b, c, a', b', c', that is, the final integrals.

The function V satisfies the two partial differential equations

$$\frac{1}{2m}\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} = U + H,$$

$$\frac{1}{2m}\left\{\left(\frac{dV}{da}\right)^2 + \left(\frac{dV}{db}\right)^2 + \left(\frac{dV}{dc}\right)^2\right\} = U_0 + H;$$

which, if they could be integrated, would give V as a function of x, y, z, a, b, c, H, and thus determine the motion of the system.

23. Hamilton's principal function S.—This is connected with the function V by the equation

$$V = tH + S;$$

or, what is the same thing, the new principal function S is defined by the equation

$$S = \int_0^t (T + U)\, dt\,;$$

but S is considered (not like V as a function of x, y, z, a, b, c, H, but) as a function of x, y, z, a, b, c, t. The expression for the variation of S is

$$\delta S = - H\delta t + m\,(x'\delta x + y'\delta y + z'\delta z) - m\,(a'\delta a + b'\delta b + c'\delta c)$$

which is equivalent to the system

$$\frac{dS}{dx} = mx', \quad \frac{dS}{dy} = my', \quad \frac{dS}{dz} = mz',$$

$$\frac{dS}{da} = -ma', \quad \frac{dS}{db} = -mb', \quad \frac{dS}{dc} = -mc',$$

$$\frac{dS}{dt} = -H\,;$$

the first three and the second three of which give, respectively, the intermediate and the final integrals; the last equation leads only to the expression of the supernumerary constant H in terms of the initial coordinates a, b, c, and it may be omitted from the system.

The function S satisfies the partial differential equations

$$\frac{dS}{dt} + \frac{1}{2m}\left\{\left(\frac{dS}{dx}\right)^2 + \left(\frac{dS}{dy}\right)^2 + \left(\frac{dS}{dz}\right)^2\right\} = U,$$

$$\frac{dS}{dt} + \frac{1}{2m}\left\{\left(\frac{dS}{da}\right)^2 + \left(\frac{dS}{db}\right)^2 + \left(\frac{dS}{dc}\right)^2\right\} = U_0\,;$$

which, if they could be integrated, would give S as a function of x, y, z, a, b, c, t, and thus determine the motion of the system.

24. Hamilton's form of the equations of motion.—This is in fact the form obtained by carrying out the idea of introducing into the differential equations, in the place of the differential coefficients of the coordinates, the derived functions (with respect to these differential coefficients) of the *vis viva* function T. Taking η to denote any one of the series of coordinates, then the original system may be denoted by

$$\frac{d}{dt}\,\frac{dT}{d\eta'} - \frac{dT}{d\eta} = \frac{dU}{d\eta},$$
$$\vdots$$

(U is the force function taken with a contrary sign to that of Lagrange), and writing in like manner ϖ to denote any one of the new variables connected with the coordinates η by the equations

$$\frac{dT}{d\eta'} = \varpi,$$
$$\vdots$$

then T, in its original form, is a function of $\eta, \ldots \eta', \ldots$, homogeneous of the second order as regards the differential coefficients $\eta', \ldots$; and, consequently, these being linear functions (without constant terms) of the new variables ϖ, the *vis viva* function T can be expressed as a function of $\eta, \ldots \varpi, \ldots$, homogeneous of the second order as regards the variables $\varpi, \ldots$. And when T has been thus expressed, the equations of motion take the form

$$\frac{d\eta}{dt} = \frac{dH}{d\varpi}, \quad \frac{d\varpi}{dt} = -\frac{dT}{d\eta} + \frac{dU}{d\eta},$$
$$\vdots \qquad\qquad \vdots$$

which is the required transformation. The force function U is independent of the differential coefficients $\eta', \ldots$ and, consequently, of the variables $\varpi, \ldots$, hence, writing $H = T - U$, the equations take the form

$$\frac{d\eta}{dt} = \frac{dH}{d\varpi}, \quad \frac{d\varpi}{dt} = -\frac{dH}{d\eta}, \text{ [1]}$$
$$\vdots \qquad\qquad \vdots$$

which correspond to the condensed form obtained by writing $T - V = R$ in Lagrange's equations. It is hardly necessary to remark that H is to be considered as a given function of $\eta, \ldots \varpi, \ldots$, viz., it is what $T - U$ becomes when the differential coefficients $\eta', \ldots$ are replaced by their values in terms of the new variables $\varpi, \ldots$.

25. I have, for greater simplicity, explained the theory of the functions V and S in reference to a very special form of the equations of motion; but the theory is, in fact, applicable to any form whatever of these equations; and, as regards the function V, is in the first memoir examined in detail with reference to Lagrange's general form of the equations of motion. The function S is considered at the end of the memoir, in reference only to the special form. The new form of the equations of motion is first established in the second memoir, and the theory of the functions V and S is there considered in reference to this form. The author considers also another function Q, which, when the matter is looked at from a somewhat more general point of view, is not really distinct from the function S.

26. The first memoir contains applications of the method to the problem of two bodies, and the problem of three or more bodies, and researches in reference to the approximate integration of the equations of motion by the separation of the function V into two parts, one of them depending on the principal forces, the other on the disturbing forces. The method, or one of the methods, given for this purpose, involves the consideration of the variation of the arbitrary constants, but it is not easy to single out any precise results, or explain their relation to the results of Lagrange and Poisson. The like remark applies to the investigations contained in Nos. 7 to 12 of the second memoir, but it is important to consider the theory described in the heading

[1] I find it stated in a note to M. Houel's "Thèse sur l'intégration des équations différentielles de la Mécanique," Paris, 1855, that this form of the equations of motion had been previously employed in an *unpublished* memoir by Cauchy, written in 1831. [Cauchy "Extrait du Mémoire présenté à l'Academie de Turin le 11 Oct. 1831" published in lithograph under the date Turin, 1832, with an Addition dated 6 Mar. 1833.]

of No. 13, as "giving formulæ for the variation of elements more analogous to those already known." The function H is considered as consisting of two parts, one of them being treated as a disturbing function; the equations of motion assume therefore the form

$$\frac{d\eta}{dt}=\frac{dH}{d\varpi}+\frac{d\Upsilon}{d\varpi},\quad \frac{d\varpi}{dt}=-\frac{dH}{d\eta}-\frac{d\Upsilon}{d\eta},$$

(I have written H, Υ instead of the author's H_1, H_2). The terms involving Υ are in the first instance neglected, and it is assumed that the integrals of the resulting equations are presented in the form adopted by Poisson, viz., the constants of integration a, b, &c. are considered as given in terms of t, and of the two sets of variables $\eta, \ldots$ and $\varpi, \ldots$; the integrals are then extended to the complete equations by the method of the variation of the elements. The resulting expressions are the same in form as those of Poisson, viz.:

$$\frac{da}{dt}=(a,\, b)\frac{d\Upsilon}{db}+\ldots,$$

where

$$(a,\, b)=\frac{\partial\,(a,\, b)}{\partial\,(\eta,\, \varpi)}+\ldots,$$

if, for shortness,

$$\frac{\partial\,(a,\, b)}{\partial\,(\eta,\, \varpi)}=\frac{da}{d\eta}\frac{db}{d\varpi}-\frac{db}{d\eta}\frac{da}{d\varpi},$$

and conversely the values of $\frac{d\Upsilon}{da}$, &c. in terms of $\frac{da}{dt}$, &c. might have been exhibited in a form such as that of Lagrange. The expressions (a, b), considered as functional symbols, have the same meanings as in the theories of Poisson and Lagrange; and, as in these theories, the differential coefficient of (a, b) with respect to the time, vanishes, or (a, b) is a function of the elements only.

27. It is to be observed that the disturbing function Υ is not necessarily in the same problem identical with the disturbing function Ω of Lagrange and Poisson (indeed, in any problem, the separation of the forces into principal forces and disturbing forces is an arbitrary one). Sir W. R. Hamilton, in the second memoir, gives a very beautiful application of his theory to the problem of three or more bodies, which has the peculiar advantage of making the motion of all the bodies depend upon one and the same disturbing function [1]. This disturbing function contains (as in the last-mentioned general formulæ) both sets of variables, and the consequence is that, as the author remarks, the varying elements employed by him are essentially different from those made use of in the theories of Lagrange and Poisson; the velocities cannot, in his theory, be obtained by differentiating the coordinates as if the elements were

[1] Lagrange has given formulæ for the determination of the motion of three or more bodies referred to their common centre of gravity by means of one and the same disturbing function. In Sir W. R. Hamilton's theory there is one central body to which all the others are referred. The method of Sir W. R. Hamilton is made use of in M. Houel's "Thèse d'Astronomie: Application de la Méthode de M. Hamilton au Calcul des Perturbations de Jupiter."—Paris, 1855.

constant. The investigation applies to the case where the attracting force is any function whatever of the distance, and the six elements ultimately adopted form a canonical system.

28. The precise relation of Sir W. R. Hamilton's form of the equations of motion to that of Lagrange, is best seen by considering Lagrange's equations, not as a system of differential equations of the second order between the coordinates and the time t, but as a system of twice as many differential equations of the first order between the coordinates, their differential coefficients treated as a new system of variables, and the time. It will be convenient to write $-U$, instead of Lagrange's force-function V, and (to conform to the usage of later writers who have treated the subject in the most general manner) to represent the coordinates by $q, \dots$, their differential coefficients by $q', \dots$, and the new variables which enter into the Hamiltonian form by $p, \dots$; then the Lagrangian system will be

$$\frac{dq}{dt} = q', \quad \frac{d}{dt}\frac{dT}{dq'} - \frac{dT}{dq} = \frac{dU}{dq}; \\ \vdots \qquad\qquad \vdots$$

or putting $T + U = Z$ (this is the same as Lagrange's substitution, $T - V = R$), the system becomes

$$\frac{dq}{dt} = q', \quad \frac{d}{dt}\frac{dZ}{dq'} = \frac{dZ}{dq}, \\ \vdots \qquad\qquad \vdots$$

while the Hamiltonian system is

$$\frac{dq}{dt} = \frac{dT}{dp}, \quad \frac{dp}{dt} = -\frac{dT}{dq} + \frac{dU}{dq}; \\ \vdots \qquad\qquad \vdots$$

or putting as before $T - U = H$, the system is

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq}; \\ \vdots \qquad\qquad \vdots$$

where, in the Lagrangian systems, T and U, and consequently Z, are given functions of a certain form of t, $q, \dots q' \dots$, and in like manner, in the Hamiltonian system, T and U, and consequently H, are given functions of a certain form of t, $q, \dots p \dots$. The generalisation has since been made (it is not easy to say precisely when first made) of considering Z as standing for any function whatever of t, $q, \dots q', \dots$, and in like manner of considering H as standing for any function whatever of t, $q, \dots p, \dots$. It is to be noticed that in Sir W. R. Hamilton's memoir, the demonstration which is given of the transformation from Lagrange's equations to the new form depends essentially on the special form of the function T as a homogeneous function of the second order in regard to the differential coefficients of the coordinates; indeed the transformation itself, as regards the actual value of the new function T ($=T$ expressed in terms of the new variables), which enters into the transformed equations, depends essentially upon the special form just referred to of the function T, although, as will be seen in the sequel, there is a like transformation applying to the most general form of the function T.

29. In the greater part of what has preceded, and especially in the above-mentioned substitutions $T+U=Z$ and $T-U=H$, it is of course assumed that the force function U exists; when there is no force function these substitutions cannot be made, but the forms corresponding to the untransformed forms in T and U are as follows, viz. the Lagrangian form is

$$\frac{dq}{dt}=q', \quad \frac{d}{dt}\frac{dT}{dq'}-\frac{dT}{dq}=Q,$$
$$\vdots \qquad \vdots$$

and the Hamiltonian form is

$$\frac{dq}{dt}=\frac{dT}{dp}, \quad \frac{dp}{dt}=-\frac{dT}{dq}+Q;$$
$$\vdots \qquad \vdots$$

that is, the only difference is, that the functions Q, instead of being the differential coefficients with respect to the variables $q\ldots$ of one and the same force function U, are so many separate and distinct functions of the variables $q, \ldots$, or more generally of the variables $q, \ldots p, \ldots$ of both sets.

30. Jacobi's letter of 1836.—This is a short note containing a mere statement of two results. The first is as follows, viz. the equations for the motion of a point *in plano* being taken to be

$$\frac{d^2x}{dt^2}=\frac{dU}{dx}, \quad \frac{d^2y}{dt^2}=\frac{dU}{dy},$$

where U is a function x, y without t; one integral is the equation of *vis viva* $\frac{1}{2}(x'^2+y'^2)=U+h$. Assume that another integral is $a=F(x, y, x', y')$, then x', y' will in general be functions of x, y, a, h, and considering them as thus expressed, it is stated that not only $x'dx+y'dy$ will be an exact differential, but its differential coefficients with respect to a, h will be so likewise, and the remaining integrals are

$$b=\int\left(\frac{dx}{da}dx+\frac{dy'}{da}dy\right),$$

$$t+T=\int\left(\frac{dx'}{dh}dx+\frac{dy'}{dh}dy\right),$$

a theorem, the relation of which to the general subject will presently appear.

The second result does not relate to the general subject, but I give it in a note for its own sake[1].

31. Poisson's memoir of 1837.—This contains investigations suggested by Sir W. R. Hamilton's memoir, and relating to the aid to be derived from a system of given integral equations (equal in number to the coordinates) in the determination of the principal function V. The equations $\frac{dV}{dx} = mx'$, &c. give $dV = m(x'dx + y'dy + z'dz)$, or in the case of a system of points, $dV = \Sigma m(x'dx + y'dy + z'dz)$. If the points, instead of being free, are connected together by any equations of condition, then, by means of these equations, the coordinates x, y, z of the different points and their differential coefficients, x', y', z', can be expressed as functions of a certain number of independent variables ϕ, ψ, θ, &c., and of their differential coefficients ϕ', ψ', θ', &c.; dV then takes the form $dV = Xd\phi + Yd\psi + Zd\theta + \ldots$, where X, Y, Z are functions of ϕ, ψ,... ϕ', ψ',.... Imagine now a system of integrals (one of them the equation of *vis viva*) equal in number to the independent variables ϕ, ψ, θ...; then, by the aid of these equations, ϕ', ψ', θ'..., and, consequently, X, Y, Z,... can be expressed as functions (of the constants of integration and) of the variables ϕ, ψ, θ,.... Hence, attending only to the variables, $dV = Xd\phi + Yd\psi + Zd\theta + \ldots$ is a differential expression involving only the variables ϕ, ψ, θ...; but, as Poisson remarks, this expression is not in general a complete differential. In the cases in which it is so, V can of course be obtained directly by integrating the differential expression, viz. the function so obtained is in value, but not in form, Sir W. R. Hamilton's principal function V; for, with him, V is a function of the coordinates, and of a particular set of the constants of integration, viz. the constant of *vis viva* h, and the initial values of the coordinates. Poisson adds the very important remark, that V being determined by his process as above, then h being the constant of *vis viva*, and the constants of the other given integral equations being e, f, &c., the remaining integrals of the problem are[2]

$$\frac{dV}{dh} = t + \tau, \quad \frac{dV}{de} = l, \quad \frac{dV}{df} = m, \ldots$$

[1] Jacobi imagines a point *without mass* revolving round the sun and disturbed by a planet moving in a circular orbit, which is taken for the plane of x, y; the coordinates of the point are x, y, z, those of the planet $a' \cos n't$, $a' \sin n't$, m' is the mass of the planet, M the mass of the sun; then we have accurately

$$\tfrac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} - n'\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) =$$

$$\frac{M}{(x^2+y^2+z^2)^{\frac{1}{2}}} + m'\left\{\frac{1}{(x^2+y^2+z^2-2a'(x\cos n't + y\sin n't) + a'^2)^{\frac{1}{2}}} - \frac{x\cos n't + y\sin n't}{a'^2}\right\} + \text{const.}$$

which Jacobi suggests might be found useful in the lunar theory. The point being without mass, means only that it is considered as not disturbing the circular motion of the planet; the problem is properly a case of the problem of two centres, viz. one centre is fixed, and the other one revolves round it in a circle with a uniform velocity.

[2] Poisson writes $\frac{dV}{dh} = -t + \epsilon$; there seems to be a mistake as to the sign of h running through the memoir. Correcting this, and putting $-\tau$ for ϵ, we have the formula $\frac{dV}{dh} = t + \tau$ given in the text.

where τ, l, m, ... are new arbitrary constants. But, as before remarked, the expression for dV is not always a complete differential. Poisson accordingly inquires into and determines (but not in a precise form) the conditions which must be satisfied, in order that the expression in question may be a complete differential. He gives, as an example, the case of the motion of a body in space under the action of a central force; and, secondly, the case considered in Jacobi's letter of 1836, which he refers to, viz., here $dV = x'dx + y'dy$, and when the two integral equations are one of them, the equations of *vis viva* $\frac{1}{2}(x'^2 + y'^2) = U + h$, and the other of them any integral equation $a = F(x, y, x', y')$ whatever (subject only to the restriction that a is not a function of x, y, $x'^2 + y'^2$, the necessity of which is obvious) the condition is satisfied *per se*, and, consequently, $x'dx + y'dy$ is a complete differential, and its integral gives (in value, although as before remarked not in form) the principal function V; and such value of V gives the two integral equations obtained in Jacobi's letter.

32. Jacobi's note of the 29th of November, 1836, "On the Calculus of Variations, and the Theory of Differential Equations."—The greater part of this note relates to the differential equations which occur in the calculus of variations, including, indeed, the differential equations of dynamics, but which belong to a different field of investigation. The latter part of the note relates more immediately to the differential equations of dynamics. The author remarks, that, in any dynamical problem of the motion of a single particle for which the principle of *vis viva* holds good, if, besides the integral of *vis viva*, there is given any other integral, the problem is reducible to the integration of an ordinary differential equation of two variables, and that it is always possible to integrate this equation, or at least *discover by a precise and general rule the factor which renders it integrable.* This would seem to refer to Jacobi's researches on the theory of the ultimate multiplier, but the author goes on to refer to a preceding communication to the Academy of Paris (the before-mentioned letter of 1836), which does not belong (or, at least, does not obviously belong) to this theory. He speaks also of a class of dynamical problems, viz. that of the motion of a system of bodies which mutually attract each other, and which may besides be acted upon by forces in parallel lines, or directed to fixed centres, or even to centres the motion of which is given; and, he remarks, in the solution of such a problem, the system of differential equations being in the first instance of the order $2n$ (that is, being a system admitting of $2n$ arbitrary constants), then if one integral is known, it is possible by a proper choice of the quantities selected for variables to reduce the system to the order $2n-2$. If another integral is known, the equation may in like manner be reduced to a system of the order $2n-4$, and so on until there are no more equations to be integrated; and thus the operations to be effected depend only upon quadratures. All this seems to refer to researches of Jacobi, which, so far as I am aware, have not hitherto been published. The results correspond with those recently obtained by Bour, *post*, Nos. 66 and 67.

33. Jacobi's memoir of 1837.—Jacobi refers to the memoirs of Sir W. R. Hamilton, and he reproduces, in a slightly different form, the investigation of the fundamental property of the principal function S. The case considered is that of a system of n

particles, the coordinates of which are connected together by any number of equations; but it will be sufficient here to attend to the case of a single free particle. The equations of motion are assumed to be

$$m\frac{d^2x}{dt^2}=\frac{dU}{dx},\quad m\frac{d^2y}{dt^2}=\frac{dU}{dy},\quad m\frac{d^2z}{dt^2}=\frac{dU}{dz};$$

but U is considered as being a function of x, y, z and of the time t, that is, it is assumed that the condition of *vis viva* is not of necessity satisfied. The definition of the function S is

$$S+\int_0^t\left[U=\tfrac{1}{2}m\,(x'^2+y'^2+z'^2)\right]dt,$$

which, when the equation of *vis viva* is satisfied, that is, when

$$T=\tfrac{1}{2}m\,(x'^2+y'^2+z'^2)=U+h,$$

agrees with Sir W. R. Hamilton's definition $S=2\int_0^t Udt+ht$. The function S is considered as being, by means of the integral equations assumed as known, expressed as a function of t, of the coordinates x, y, z, and of their initial values a, b, c. And then it is shown that S satisfies the equations

$$\frac{dS}{dx}=\quad mx',\quad \frac{dS}{dy}=\quad my',\quad \frac{dS}{dz}=\quad mz',$$

$$\frac{dS}{da}=-\,ma',\quad \frac{dS}{db}=-\,mb',\quad \frac{dS}{dc}=-\,mc';$$

so that the intermediate and final integrals are expressed by means of the principal function S.

34. But Jacobi proceeds, "the definition assumes the integration of the differential equations of the problem. The results, therefore, are only interesting in so far as they have reduced the system of integral equations into a remarkable form. We may, however, define the function S in a quite different and *very much more general* manner." And then, attending only to the case of a system of free particles, he gives a definition, which, in the case of a single particle, is as follows:

Jacobi's principal function S.—The equations of motion being as before

$$\frac{d^2x}{dt^2}=m\frac{dU}{dx},\quad \frac{d^2y}{dt^2}=m\frac{dU}{dy},\quad \frac{d^2z}{dt^2}=m\frac{dU}{dz},$$

(where U is in general a function of x, y, z and t), then S is defined to be a *complete* solution of the partial differential equation

$$\frac{dS}{dt}+\frac{1}{2m}\left\{\left(\frac{dS}{dx}\right)^2+\left(\frac{dS}{dy}\right)^2+\left(\frac{dS}{dz}\right)^2\right\}=U.$$

A complete solution, it will be recollected, means a solution containing as many arbitrary constants as there are independent variables in the partial differential equation;

in the present case, therefore, four arbitrary constants. But one of these constants may be taken to be a constant attached to the function S by mere addition, and which disappears from the differential coefficients, and it is only necessary to attend to the other three arbitrary constants. S is consequently a function of t, x, y, z, and of the arbitrary constants α, β, γ, satisfying the partial differential equation. And this being so, it is shown that the integrals of the problem are

$$\frac{dS}{dx} = mx', \quad \frac{dS}{dy} = my', \quad \frac{dS}{dz} = mz',$$

$$\frac{dS}{d\alpha} = \lambda, \quad \frac{dS}{d\beta} = \mu, \quad \frac{dS}{d\gamma} = \nu;$$

where λ, μ, ν are any other arbitrary constants, viz., the first three equations give the intermediate integrals, and the last three equations give the final integrals of the problem.

Jacobi proceeds to give an analogous definition of the principal function V as follows:

35. Jacobi's principal function V.—First, when the condition of *vis viva* is satisfied. Here V is a complete solution of the partial differential equation

$$\frac{1}{2m}\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} = U + h,$$

where h is the constant of *vis viva*. The partial differential equation contains only three independent variables; and since as before one of the constants of the complete solution may be taken to be a constant attached to V by mere addition, and which disappears from the differential coefficients, we may consider V as a function of t, x, y, z, and of the two constants of integration α and β. But V will of course also contain the constant h, which enters into the partial differential equation. The integrals of the problem are then shown to be

$$\frac{dV}{dx} = mx', \quad \frac{dV}{dy} = my', \quad \frac{dV}{dz} = mz',$$

$$\frac{dV}{dh} = t + \tau, \quad \frac{dV}{d\alpha} = \lambda, \quad \frac{dV}{d\beta} = \mu,$$

where τ, λ, μ are new arbitrary constants.

36. Jacobi's principal function V.—Secondly, when the equation of *vis viva* is not satisfied. Here U contains the time t, and we have no such equation as $T = U + h$, but along with the coordinates x, y, z there is introduced a new variable H, and V is defined to be a complete integral of the partial differential equation

$$\frac{1}{2m}\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} = U + H;$$

where, in the expression for U, it is assumed that t is replaced by $\frac{dV}{dH}$. There are, consequently, four independent variables, and a complete solution must contain, exclusively of the constant attached to V by mere addition, and which disappears from the differential coefficients, three arbitrary constants α, β, γ. The integral equations are shown to be

$$\frac{dV}{dx} = mx', \quad \frac{dV}{dy} = my', \quad \frac{dV}{dz} = mz',$$

$$\frac{dV}{d\alpha} = \lambda, \quad \frac{dV}{d\beta} = \mu, \quad \frac{dV}{d\gamma} = \nu,$$

$$\frac{dV}{dH} = t;$$

where λ, μ, ν are arbitrary constants, viz., eliminating H from the first three equations by the assistance of the last equation, we have the intermediate integrals; and eliminating H from the second three equations by the assistance of the last equation, we have the final integrals. The substitution of the above values $\frac{dV}{dx}$, &c. in the partial differential equation gives $T = U + H$, that is, $H (= T - U)$ is that function which, when the condition of *vis viva* is satisfied, becomes equal to h, the constant of *vis viva*.

Jacobi's extension of the theory to the case where the condition of *vis viva* is not satisfied, appears to have attracted very little attention; it is indeed true, as will be noticed in the sequel, that this general case can be reduced to the particular one in which the condition of *vis viva* is satisfied, but there is not it would seem any advantage in making this reduction; the formulæ for the general case are at least quite as elegant as those for the particular case.

37. Jacobi, after considering some particular dynamical applications, proceeds to apply the theory developed in the first part of the memoir to the general subject of partial differential equations; the differential equations of a dynamical problem lead to a partial differential equation, a complete solution of which gives the integral equations. Conversely, the integral equations give the complete solution of the partial differential equation, and applying similar considerations to any partial differential equation of the first order whatever, it is shown (what, but for Cauchy's memoir of 1819, which Jacobi was not acquainted with [1]), would have been a new theorem) that the solution of the partial differential equation depends on the integration of a single system of differential equations. The remainder of the memoir is devoted to the discussion of this theory and of the integration of the Pfaffian system of ordinary differential

[1] Jacobi refers to Lagrange's "Leçons sur la Théorie des Fonctions," and to a memoir by Pfaff in the Berlin Transactions for 1814, as containing, so far as he was aware, everything essential which was known in reference to the integration of partial differential equations of the first order; he refers also to his own memoir "Ueber die Pfaffsche Methode u. s. w." *Crelle*, t. II. pp. 347—358 (1827), as presenting the method in a more symmetrical and compendious form, but without adding to it anything essentially new.

equations, a system which is also treated of in Jacobi's memoir of 1844, "Theoria Novi Multiplicatoris &c." I take the opportunity of referring here to a short note by Brioschi, "Intorno ad una Proprietà delle Equazioni alle Derivate Parziali del Primo Ordine," *Tortolini*, t. VI. pp. 426—429 (1855), where the theory of the integration of a partial differential equation of the first order is presented under a singularly elegant form.

38. Jacobi's note of 1837, "On the Integration of the Differential Equations of Dynamics."—Jacobi remarks that it is possible to derive from Lagrange's form of the equations of motion an important profit for the integration of these equations, and he refers to his communication of the 29th of November 1839 to the Academy of Berlin, and to his former note to the Academy of Paris. He proceeds to say, that whenever the condition of *vis viva* holds good, he had found that it was possible in the integration of the equations of motion to follow a course such that each of the given integrals successively lowers by two unities the order of the system; and that the like theorem holds good when the condition of *vis viva* is not satisfied, that is, when the force function involves the time (this seems to be a restatement, in a more general form, of the theorems referred to in the note of the 29th of November 1836 to the Academy of Berlin); and he mentions that he had been, by his researches on the theory of numbers, led away from composing an extended memoir on the subject. The note then passes on to other subjects, and it concludes with two theorems, which are given without demonstration as extracts from the intended work he had before spoken of. These theorems are in effect as follows:

I. Let

$$m\frac{d^2x}{dt^2}=\frac{dU}{dx},\quad m\frac{d^2y}{dt^2}=\frac{dU}{dy},\quad m\frac{d^2z}{dt^2}=\frac{dU}{dz},\ \text{\&c.}$$

be the $3n$ differential equations of the motion of a free system, and

$$\tfrac{1}{2}\Sigma m\,(x'^2+y'^2+z'^2)\,dt=U+h,$$

the equation of *vis viva*.

Let V be a complete solution of the partial differential equation

$$\tfrac{1}{2}\Sigma\frac{1}{m}\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}=U+h,$$

that is, a solution containing, besides the constants attached to V by mere addition, $3n-1$ constants $\alpha\,(\alpha_1,\ \alpha_2,\dots\ \alpha_{3n-1})$, then first the integral equations are

$$\frac{dV}{d\alpha}=\beta,\ \dots\ \ \frac{dV}{dh}=t+\tau;$$

where $\beta\,(\beta_1,\ \beta_2,\dots\ \beta_{3n-1})$ and τ are new arbitrary constants: this is in fact the theorem already quoted from Jacobi's memoir of 1837, and it is in the present place referred to as an easy generalisation of Sir W. R. Hamilton's formulæ. But Jacobi proceeds (and this is given as entirely new) that the disturbed equations being

$$m\frac{d^2x}{dt^2}=\frac{dU}{dx}+\frac{d\Omega}{dx},\quad m\frac{d^2y}{dt^2}=\frac{dU}{dy}+\frac{d\Omega}{dy},\quad m\frac{d^2z}{dt^2}=\frac{dU}{dz}+\frac{d\Omega}{dz},$$

then the equations for the variations of the above system of arbitrary constants are

$$\frac{d\alpha}{dt}=\frac{d\Omega}{d\beta},\ldots\quad \frac{dh}{dt}=\frac{d\Omega}{d\tau},$$

$$\frac{d\beta}{dt}=-\frac{d\Omega}{d\alpha},\ldots\quad \frac{d\tau}{dt}=-\frac{d\Omega}{dh},$$

so that the constants form (I think the term is here first introduced) a canonical system.

Jacobi observes that, in the theory of elliptic motion, certain elements which he mentions form a system of canonical elements, and he remarks that, since one complete solution of a partial differential equation gives all the others, the theorem leads to the solution of another interesting problem, viz. "Given one system of canonical elements, to find all the other systems." This is effected by means of the second theorem, which is as follows:

II. Given the systems of differential equations between the variables $a\,(a_1,\ a_2\ldots a_m)$ and $b\,(b_1,\ b_2\ldots b_m)$

$$\frac{da}{dt}=-\frac{dH}{db},\ldots\quad \frac{db}{dt}=\frac{dH}{da},\ldots$$

where H is *any function* of the variables $a,\ldots$ and $b,\ldots$; let $\alpha\,(\alpha_1,\ \alpha_2,\ldots\alpha_m)$ and $\beta\,(\beta_1,\ \beta_2,\ldots\beta_m)$ be two new systems of variables connected with the preceding ones by the equations

$$\frac{d\psi}{d\alpha}=\beta,\ldots\quad \frac{d\psi}{da}=-b,\ldots$$

where ψ is a function of $\alpha,\ldots b,\ldots$ without t or the other variables, then expressing H as a function of t and the new variables $\alpha,\ldots$ and $\beta,\ldots$, these last variables are connected together by equations of the like form with the original system, viz.:

$$\frac{d\alpha}{dt}=-\frac{dH}{d\beta},\ldots\quad \frac{d\beta}{dt}=-\frac{dH}{d\alpha},\ldots$$

Jacobi concludes with the remark, that other theorems no less general may be deduced by putting $\psi+\lambda\psi_1+\mu\psi_2+\ldots$ instead of ψ, and eliminating the multipliers $\lambda,\ \mu,\ldots$ by means of the equations $\psi_1=0,\ \psi_2=0,\ldots$, and that the demonstrations of the theorems are obtained without difficulty.

39. Jacobi's note of the 21st of November, 1838.—Jacobi refers to a memoir by Encke in the Berlin *Ephemeris* for 1837, "über die speciellen Störungen," where expressions are given for the partial differential coefficients of the values in the theory

of elliptic motion of the coordinates x, y, z and the velocities x', y', z' with respect to the elements; and he remarks, that if Encke's elements are replaced by a system of elements $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ which he mentions, connected with those of Encke by equations of a simple form, then considering first x, y, z, x', y', z' as given functions of t and the elements, and afterwards the elements $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ as given functions of t and x, y, z, x', y', z', there exists the remarkable theorem that the thirty-six partial differential coefficients $\frac{d\alpha}{dx}, \frac{d\beta}{dx}$, &c., and the thirty-six partial differential co-coefficients $\frac{dx}{d\alpha}, \frac{dx}{d\beta}$, &c., are equal to each other, or differ only in their sign, viz.

$$\frac{dx}{d\alpha} = -\frac{d\alpha'}{dx'}, \quad \frac{dx}{d\alpha'} = \frac{d\alpha}{dx'}, \quad \frac{dx'}{d\alpha} = \frac{d\alpha'}{dx}, \quad \frac{\partial x'}{\partial \alpha'} = -\frac{d\alpha}{dx};$$

thirty-six equations in all, viz. the pair α, α' of corresponding elements may be replaced by the pair β, β' or γ, γ': and then in each of the twelve equations y, y' or z, z' may be written instead of x, x'. The like applies to a system of constants which are the initial values of any system whatever of coordinates $p, \ldots$, and the initial values of the differential coefficients $q' = \frac{dT}{dp}$, &c. of the force function T with respect to $p, \ldots$; *and for every system of elements which possess the property first mentioned, the formulæ for the variations assume the simplest possible form, inasmuch as the variations of each element is equal to a single partial differential coefficient of the disturbing function with the coefficient* $+1$ *or* -1, *as is known to be the case with the last-mentioned system of elements;* in other words, if $a, \ldots$ and $b, \ldots$ be a system of elements corresponding to each other in pairs, such that

$$\frac{dp}{da} = -\frac{db}{dq}, \quad \frac{dp}{db} = \frac{da}{dq}, \quad \frac{dq}{da} = \frac{db}{dp}, \quad \frac{dq}{db} = -\frac{da}{dp}$$
$$\vdots$$

(where a, b may be replaced by any other corresponding pair of elements, and p, q by any other corresponding pair of variables), then the elements $a, \ldots$ and $b, \ldots$ form a canonical system.

40. Jacobi's note of 1840 in the *Comptes Rendus*, calls attention to the theorem contained in the passage quoted above from Poisson's memoir of 1808, a theorem which Jacobi characterizes as "la plus profonde découverte de M. Poisson," and as the theorem "le plus important de la Mécanique et de cette partie du calcul intégral qui s'attache à l'intégration d'un système d'équations différentielles ordinaires"; and he proceeds, "le théorème dont il est question énoncé convenablement est le suivant—un nombre quelconque de points matériels étant tirés par des forces et soumis à des conditions telles que le principe des forces vives ait lieu, si l'on connaît outre que l'intégrale fournie par ce principe deux autres intégrales, on en peut déduire une troisième d'une manière directe et sans même employer des quadratures. En poursuivant le même procédé on pourra trouver une quatrième, une cinquième intégrale, et en général on parviendra à cette manière à déduire des deux intégrales données toutes

les intégrales, ou ce qui revient au même l'intégration complète du problême. *Dans des cas particuliers on retombera sur une combinaison des intégrales déjà trouvées avant qu'on soit parvenu à toutes les intégrales du problême, mais alors les deux intégrales données jouissent des propriétés particulières dont on peut tirer un autre profit pour l'intégration des équations dynamiques proposées.* C'est ce qu'on verra dans un ouvrage auquel je travaille depuis plusieurs années et dont peut-être je pourrai bientôt faire commencer l'impression."

41. Liouville's addition to Jacobi's letter of 1840.—This contains the demonstration of a theorem similar to that given in Jacobi's letter of 1836, and Poisson's memoir of 1837, but somewhat more general; the system considered is a system of four differential equations of the first order:

$$\frac{dx}{dt} = \lambda \frac{dU}{dx'}, \quad \frac{dx'}{dt} = -\lambda \frac{dU}{dx}, \quad \frac{dy}{dt} = \lambda \frac{dU}{dy'}, \quad \frac{dy'}{dt} = -\lambda \frac{dU}{dy},$$

where U is a function of x, y, x', y', and λ is a function of x, y, x', y' and t. One integral is $U = a$, and if there be another integral $V = b$ where V is a function of x, y, x', y' only, then x', y' being by means of these two integrals expressed as a function of x, y, a, b, it is shown that $x'dx + y'dy$ is an exact differential, and putting $\int (x'dx + y'dy) = \theta$, then that $\frac{d\theta}{db} = \beta$ is a new integral of the given equations; and in the case where λ is a function of t only, the remaining integral is $\frac{d\theta}{da} = \int \lambda dt + \alpha$.

42. Binet's memoir of 1841 contains an exposition of the theory of the variation of the arbitrary constants as applied to the general system of equations

$$\frac{d}{dt}\frac{dF}{dx'} = \frac{dF}{dx'} + \lambda \frac{dL}{dx} + \mu \frac{dM}{dx} + \dots,$$
$$\vdots$$

where F is any function of t, and of the coordinates x, y, z... of the different points of the system, and of their differential coefficients x', y', z', &c., and $L = 0$, $M = 0$, &c. are any equations of equation between the coordinates x, y, z, ... of the different points of the system; these equations may contain t, but they must not contain the differential coefficients x', y', z',... The form is a more general one than that considered by Lagrange and Poisson. The memoir contains an elegant investigation of the variations of the elements of the orbit of a body acted upon by a central force, the expressions for the variations being obtained in a canonical form; and there is also a discussion of the problem suggested in Poisson's report of 1830 on the manuscript work of Ostrogradsky.

43. Jacobi's note of 1842, in the *Comptes Rendus*, announces the general principle (being a particular case of the theorem of the ultimate multiplier) stated and demonstrated in the memoir next referred to, and gives also the rule for the formation of the multiplier in the case to which the general principle applies.

44. Jacobi's memoir of 1842, "De Motu Puncti singularis": the author remarks, that the greater the difficulties in the general integration of the equations of dynamics, the greater the care which should be bestowed on the examination of the dynamical problems in which the integration can be reduced to quadratures; and the object of the memoir is stated to be the examination of the simplest case of all, viz. the problems relating to the motion of a single point. The first section, entitled, "De Extensione quadam Principii Virium vivarum," contains a remark which, though obvious enough, is of considerable importance: the forces X, Y, Z which act upon a particle, may be such that $Xdx + Ydy + Zdz$ is not an exact differential, so that if the particle were free, there would be no force function, and the equations of motion would not be expressible in the standard form. But if the point move on a surface or a curve, then in the former case $Xdx + Ydy + Zdz$ will be reducible to the form $Pdp + Qdq$, which will be an exact differential if a single condition (instead of the three conditions which are required in the case of a free particle) be satisfied, and in the latter case it will be reducible to the form Pdp, which is, *per se*, an exact differential. In the case of a surface, the requisite transformation is given by the Hamiltonian form of the equations of motion, which Jacobi demonstrates for the case in hand; and then in the third section, with a view to its application to the particular case, he enumerates the general proposition "quæ pro novo principio mechanico haberi potest," which is as follows:

"Consider the motion of a system of material points subjected to any conditions, and let the forces acting on the several points in the direction of the axes be functions of the coordinates alone: if the determination of the orbits of the several points is reduced to the integration of a single differential equation of the first order between two variables, for this equation there may be found, by a general rule, a multiplier which will render it integrable by quadratures only."

And for the particular case the theorem is thus stated:

"Given three differential equations of the first order between the four quantities q_1, q_2, p_1, p_2,

$$dq_1 : dq_2 : dp_1 : dp_2 = \frac{dT}{dp_1} : \frac{dT}{dp_2} : -\frac{dT}{dq_1} + Q_1 : -\frac{dT}{dq_2} + Q_2,$$

in which Q_1, Q_2 are functions of q_1, q_2 only; suppose that there are known two integrals α, β, and that by the aid of these p_1, p_2, $\frac{dT}{dp_1}$, $\frac{dT}{dp_2}$ are expressed by means of the quantities q_1, q_2 and the arbitrary constants α, β; there then remains to be integrated an equation of the first order, $\frac{dT}{dp_1} dq_2 - \frac{dT}{dp_2} dq_1 = 0$ between the quantities q_1, q_2, by which is determined the orbit of the point on the given surface: I say that the left-hand side of the equation multiplied by the factor

$$\frac{dp_1}{d\alpha}\frac{dp_2}{d\beta} - \frac{dp_2}{d\alpha}\frac{dp_1}{d\beta},$$

will be a complete differential, or will be integrable by quadratures alone," and the demonstration of the theorem is given. The remainder of the memoir, sections 4 to 7, is occupied by a very interesting discussion of various important special problems.

45. There is an important memoir by Jacobi, which, as it relates to a special problem, I will merely refer to, viz. the memoir "Sur l'Elimination des Nœuds dans le problême des trois Corps," *Crelle*, t. XXVII. pp. 115—131 (1843). The solution is made to depend upon six differential equations, all of them of the first order except one, which is of the second order, and upon a quadrature.

46. Jacobi's memoir of 1844, "Theoria Novi Multiplicatoris &c."—This is an elaborate memoir establishing the definition and developing the properties of the "multiplier" of a system of ordinary differential equations, or of a linear partial differential equation of the first order, with applications to various systems of differential equations, and in particular to the differential equations of dynamics. The definition of the multiplier is as follows, viz. the multiplier of the system of differential equations

$$dx : dy : dz : dw \ldots = X : Y : Z : W \ldots$$

or of the linear partial differential equation of the first order

$$X \frac{df}{dx} + Y \frac{df}{dy} Z \frac{df}{dz} + W \frac{df}{dw} + \ldots = 0$$

is a function M, such that

$$\frac{dMX}{dx} + \frac{dMY}{dy} + \frac{dMZ}{dz} + \frac{dMW}{dw} + \ldots = 0.$$

One of the properties of the multiplier is that contained in the theorem of the ultimate multiplier, viz. that when all the integrals (except one) of the system of partial differential equations are known, and the system is thereby reduced to a single differential equation between two variables, then the multiplier (in the ordinary sense of the word) of this last equation is $M\nabla$, where M is the multiplier of the system, and ∇ is a given derivative of the known integrals, so that the multiplier of the system being known, the integration of the last differential equation is reduced to a mere quadrature. To explain the theorem more particularly, suppose that the system of given integrals, that is, all the integrals (except one) of the system are represented by $p = \alpha$, $q = \beta$, ..., and let u, v be any two functions whatever of the variables, so that $p, q, \ldots u, v$ are in number equal to the system $x, y, z, w, \ldots$ then if

$$X \frac{du}{dx} + Y \frac{du}{dy} + Z \frac{du}{dz} + W \frac{du}{dw} + \ldots = U,$$

$$X \frac{dv}{dx} + Y \frac{dv}{dy} + Z \frac{dv}{dz} + W \frac{dv}{dw} + \ldots = V,$$

the last differential equation takes the form

$$Udv - Vdu = 0,$$

where it is assumed that U and V are, by the assistance of the given integrals, expressed as functions of u, v and the constants of integration. The multiplier of the last-mentioned equation is $M\nabla$, where M is the multiplier of the system, and ∇ may be expressed in either of the two forms

$$\nabla = \frac{\partial(x, y, z, w, \ldots)}{\partial(\alpha, \beta, \ldots u, v)}$$

and

$$\nabla = \left\{\frac{\partial(p, q, \ldots u, v)}{\partial(x, y, z, w, \ldots)}\right\}^{-1};$$

where the symbols on the right-hand sides represent functional determinants; in the first form it is assumed that $x, y, z, w, \ldots$ are expressed as functions of $\alpha, \beta, \ldots u, v$, and in the second form that $p, q, \ldots u, v$ are expressed as functions of $x, y, z, w, \ldots$, but that ultimately $p, q, \ldots$ are replaced by their values in terms of the constants and u, v; the first of the two forms, from its not involving this transformation backwards, appears the more convenient.

47. I have thought it worth while to quote the theorem in its general form, but we may take for u, v any two of the original variables, and if, to fix the ideas, it is assumed that there are in all the four variables x, y, z, w, then the theorem will be stated more simply as follows:—given the system of differential equations

$$dx : dy : dz : dw = X : Y : Z : W,$$

and suppose that two of the integrals are $p = \alpha$, $q = \beta$, the last equation to be integrated will be

$$Wdz - ZdW = 0,$$

where, by the assistance of the given integrals, W, Z are expressed as functions of z, w. And the multiplier of this equation is $M\nabla$, where M is the multiplier of the system, and ∇, attending only to the first of the two forms, is given by the equation

$$\nabla = \frac{\partial(x, y)}{\partial(\alpha, \beta)},$$

which supposes that x, y are expressed as functions of α, β, z, w.

48. Jacobi applies the theorem of the ultimate multiplier to the differential equations of dynamics, considered first in the unreduced Lagrangian form, where the coordinates are connected by any given system of equations of condition; secondly, in the reduced or ordinary Lagrangian form; and, thirdly, in the Hamiltonian form. The multiplier can be found for the first two forms, and the expressions obtained are simple and elegant; but, as regards the third form, there is a further simplification: the multiplier M of the system is equal to unity, and the multiplier of the last equation is therefore equal to ∇. The two cases are to be distinguished in which t does not, or does enter into the equations of motion; in the latter case the theorem furnished by the principle of the ultimate multiplier is the same as for the general

case of a system the multiplier of which is known, viz., the theorem is, given all the integrals except one, the remaining integral can be found by quadratures only. But in the former case, which is the ordinary one, including all the problems in which the condition of *vis viva* is satisfied, there is a further consequence deduced. In fact, the time t may be separated from the other variables, and the system of differential equations reduced to a system not involving the time, and containing a number of equations less by unity than the original system, and the theorem of the ultimate multiplier applies to this new system. But when the integrals of the new system have been obtained, the system may be completed by the addition of a single differential equation involving the time, and which is integrable by quadratures; the theorem consequently is, given all the integrals except two, these given integrals being independent of the time, the remaining integrals can be found by quadratures only. This is, in fact, the "Principium generale mechanicum" of the memoir of 1842.

The last of the published writings of Jacobi on the subject of dynamics is the "Auszug zweier Schreiben des Professors Jacobi an Herrn Director Hansen," *Crelle*, t. XLII. pp. 12—31 (1851): these relate chiefly to Hansen's theory of ideal coordinates.

49. The very interesting investigations contained in several memoirs by Liouville (*Liouville*, t. XI. XII. and XIV., and the additions to the "Connaissance des Temps" for 1849 and 1850) in relation to the cases in which the equations of motion of a particle or system of particles admit of integration, are based upon Jacobi's theory of the S function, that is, of the function which is the complete solution of a certain partial differential equation of the first order; the equation is given, in the first instance, in rectangular coordinates, and the author transforms it by means of elliptic coordinates or otherwise, and he then inquires in what cases, that is, for what forms of the force function, the equation is one which admits of solution. A more particular account of these memoirs does not come within the plan of the present report.

50. Desboves' memoir of 1848 contains a demonstration of the two theorems given in Jacobi's note of 1837, in the *Comptes Rendus;* and, as the title imports, there is an application of the theory to the problem of the planetary perturbations; the author refers to the above-mentioned memoirs of Liouville as containing a solution of the partial differential equation on which the problem depends, and also to a memoir of his own relating to the problem of two centres, where the solution is also given; and from this he deduces the solution just referred to, and which is employed in the present memoir. Jacobi's theorem gives at once the formulæ for the variation of the arbitrary constants contained in the solution. The material thing is to determine the signification of these constants, which can of course be done by a comparison of the formulæ with the known formulæ of elliptic motion; the author is thus led to a system of canonical elements similar to, but not identical with, those obtained by Jacobi.

51. Serret's two notes of 1848 in the *Comptes Rendus*.—These relate to the theory of Jacobi's S function, that is, of the function considered as the complete solution of a given partial differential equation of the first order. In the first of the two notes,

which relates to a single particle, the author gives a demonstration founded on a particular choice of variables, viz., those which determine orthotomic surfaces to the curve described by the moving point. The process seems somewhat artificial.

52. Sturm's note of 1848, in the *Comptes Rendus*, relates to the theory of Jacobi's S function, that is, of this function considered as the complete solution of a given partial differential equation of the first order. The force function is considered as involving the time t, which, however, is no more than had been previously done by Jacobi.

53. Ostrogradsky's note of 1848.—This contains an important step in the theory of the forms of the equations of motion, viz. it is shown how, in the case where the force function contains the time, the equations of motion may be transformed from the form of Lagrange to that of Sir W. R. Hamilton. If, as before, the force function (taken with the contrary sign to that of Lagrange) is represented by U, then putting, as before, $T+U=Z$ (the author writes V instead of Z), in the case under consideration Z will contain not only terms of the second order and terms of the order zero in the differential coefficients of the coordinates $q, \ldots$, but also terms of the first order, that is, Z will be of the form $Z=Z_2+Z_1+Z_0$, and putting $H=Z_2-Z_0$, this new function H being expressed as a function of the coordinates $q, \ldots$ and of the new variables $p, \ldots$, then the equations of motion take the Hamiltonian form, viz.—

$$\frac{dq}{dt}=\frac{dH}{dp}, \quad \frac{dp}{dt}=-\frac{dH}{dq};$$
$$\vdots$$

in the theory of the transformation, as originally given by Sir W. R. Hamilton, $Z_2=T$, $Z_1=0$, $Z_0=U$, and, consequently, $H=Z_2-Z_0=T-U$ as before.

[Ostrogradsky's memoir of 1850.—This among other important researches contains, and that *in the most general form*, the transformation of the equations of motion from the Lagrangian to the Hamiltonian form, and indeed the transformation of the general isoperimetric system (that is the system arising from any problem in the calculus of variations) to the Hamiltonian form.]

54. Brassinne's memoir of 1851.—The author reproduces for the Lagrangian equations of motion $\frac{d}{dt}\frac{dZ}{d\xi'}-\frac{dZ}{d\xi}=0$, &c. the demonstration of the theorem

$$\frac{d}{dt}\left(\delta\,\frac{dZ}{d\xi'}\,\Delta\xi-\Delta\,\frac{dZ}{d\xi'}\,\delta\xi+\ldots\right)=0\,;$$

and he shows that a similar theorem exists with regard to the system

$$-\frac{d^2}{dt^2}\,\frac{dZ}{d\xi''}+\frac{d}{dt}\,\frac{dz}{d\xi'}-\frac{dZ}{d\xi}=0\,;$$
$$\vdots$$

and with respect to the corresponding system of the mth order. The system in question, which is, in fact, the general form of the system of equations arising from a problem in the calculus of variations, had previously been treated of by Jacobi, but the theorem is probably new. In conclusion, the author shows in a very elegant manner the interdependence of the theorem relating to Lagrange's coefficients (a, b), and of the corresponding theorem for the coefficients of Poisson.

55. Bertrand's memoir of 1851, "On the Integrals common to several Mechanical Problems," is one of great importance, but it is not very easy to explain its relation to other investigations. The author remarks that, given the integral of a mechanical problem, it is in general a question admitting of determinate solution to find the expression for the forces; in other words, to determine the problem which has given rise to the integral; at least, this is the case when it is assumed that the forces are functions of the coordinates, without the time or the velocities; and he points out how the solution of the question is to be obtained. But, in certain cases, the method fails, that is, it leads to expressions which are not sufficient for the determination of the forces; these are the only cases in which the given integral can belong to several different problems; and the method shows the conditions necessary in order that these cases may present themselves. It is to be remarked that the given integral must be understood to be one of an absolutely definite form, such for instance as the equations of the conservation of the motion of the centre of gravity or of areas, but not such as the equation of *vis viva*, which is a property common indeed to a variety of mechanical problems, but which involves the forces, and is therefore not the *same* equation for different problems. The author studies in particular the case where the system consists of a single particle; he shows, that when the motion is in a plane, the integrals capable of belonging to two or more different problems are two in number, each of them involving as a particular case the equation of areas. When the point moves on a surface, he arrives at the remarkable theorem—"In order that the equations of motion of a point moving on a surface may have an integral independent of the time, and common to two or more problems, it is necessary that the surface should be a surface of revolution, or one which is developable upon a surface of revolution." When the condition is satisfied, he gives the form of the integral, and the general expression of the forces in the problems for which such integral exists. He examines, lastly, the general case of a point moving freely in space. The number of integrals common to several problems is here infinite. After giving a general form which comprehends them all, the author shows how to obtain as many particular forms as may be desired: it is, in fact, only necessary to resolve any problem relative to motion in a plane, and to effect a certain simple transformation on the integrals; one thus obtains a new equation which is the integral of an infinite number of different problems relating to the motion in space.

As an instance of the analytical forms on which these remarkable results depend, I quote the following, which is one of the most simple:—"If an integral of the equations of motion of a point in a plane belongs to two different problems, it is of the form

$$\alpha = F(\phi', x, y, t),$$

where ϕ' is the derivative with respect to t, of a function of x, y, which equated to zero gives the equation of a system of right lines."

56. Bertrand's memoir of 1851, "On the Integration of the Differential Equations of Dynamics."—The author refers to Jacobi's note of 1840, in relation to Poisson's theorem; and after remarking that there are very few problems of which two integrals are not known, and which therefore might not be solved by the method if it never failed; he observes that unfortunately there are (as was known) cases of exception, and that, as his memoir shows, these cases are far more numerous than those to which the method applies; thus for example the equation of *vis viva*, combined with any other integral whatever, leads to an illusory result. The theorem of Poisson may lead to an illusory result in two ways; either the resulting integral may be an identity $0 = 0$, or it may be an integral contained in the integrals already known, and which consequently does not help the solution. It appears by the memoir that the two cases are substantially the same, and that it is sufficient to study the case in which the two integrals lead to the identity $0 = 0$. Suppose that one integral is given, the author shows that there always exist integrals which, combined with the given integral, lead to an illusory result, and he shows how the integrals which, combined with the given integral, leads to such illusory result, are to be obtained.

For instance, in the case of a body moving round a fixed centre [and attracted to it by a force which is a function of the distance], there are here two known integrals; first, the equation of *vis viva* (but this, as already remarked, combined with any other integral whatever, leads to an illusory result); secondly, the equation of areas. The question arises, what are the integrals which, combined with the equation of areas, lead to an illusory result? The integrals in question are, in fact, the other two integrals of the problem; so that the inquiry into the integrals which give an illusory result, leads here to the completion of the solution.

The like happens in two other cases which are considered, viz. first, the problem of motion *in plano* when the force function is a homogeneous function of the coordinates of the degree -2; secondly, the problem of two fixed centres. Indeed the case is the same for all problems whatever, where the coordinates of the points of the system can be expressed by means of two independent variables.

The next problem considered is that where two bodies attract each other, and are attracted to a fixed centre. Suppose, first, the motion is *in plano*, then as in the former case *all* the integrals will be found by seeking for the integrals which, combined with the equation of areas, give an illusory result. When the motion is in space, the principle of areas furnishes three integrals (the equation of *vis viva* is contained in these three equations); the integrals which, combined with the integrals in question, give illusory results, are eight in number, and, to complete the solution, there must be added to these one other integral, which alone does not put the method in default. The problem of three bodies is then shown to be reducible to the last-mentioned problem; and the same consequences therefore hold good with respect to the problem of three bodies, viz., there are eight integrals which, combined with the integrals furnished by the principle of areas, give illusory results. To complete the solution it would be necessary to add to these a ninth integral, which alone would not put the method into default.

57. The author remarks that it appears by the preceding enumeration that the method of integration, based on the theorem of Poisson, is far from having all the

importance attributed to it by Jacobi. The cases of exception are numerous; they constitute, in certain cases, the complete solutions of the problems, and embrace in other cases eleven integrals out of twelve. But it would be a misapprehension of his meaning to suppose that, according to him, the cases in which Poisson's theorem is usefully applicable ought to be considered as exceptions. The expression would not be correct even for the problems which are completely resolved in seeking for the integrals which put the method into default; there exists for these problems, it is true, a system of integrals which give illusory results; but these integrals, combined in a suitable manner, might furnish others to which the theorem could be usefully applied.

The author remarks, that, in seeking the cases of exception to Poisson's theorem, there is obtained a new method of integration, which may lead to useful results; and, after referring to Jacobi's memoir on the elimination of the nodes in the problem of three bodies, he remarks that, by his own new method, the problem is reduced to the integration of six equations, all of them of the first order; so that he effectuates one more integration than had been done by Jacobi [this is incorrect]; and he refers to a future memoir, (not, I believe, yet published) [? the Memoir of 1857] for the further development of his solution.

58. To give an idea of the analytical investigations, the equations of motion are considered under the Hamiltonian form

$$\frac{dq}{dt}=\frac{dH}{dp}, \quad \frac{dp}{dt}=-\frac{dH}{dq},$$

where H is any function whatever of $q, \dots p, \dots$ without t, and then a given integral being

$$\alpha=\phi\,(q, \dots p, \dots),$$

the question is shown to resolve itself into the determination of an integral $\beta=\psi\,(q, \dots p, \dots)$ such that identically $(\alpha, \beta)=0$ or else $(\alpha, \beta)=1$, where (α, β) represents, as before, Poisson's symbol, viz.

$$(\alpha, \beta)=\frac{\partial\,(\alpha, \beta)}{\partial\,(q, p)}+\dots,$$

if for shortness

$$\frac{\partial\,(\alpha, \beta)}{\partial\,(q, p)}=\frac{\partial\alpha}{dq}\frac{d\beta}{dp}-\frac{d\alpha}{dp}\frac{d\beta}{dq}.$$

The partial differential equations $(\alpha, \beta)=0$ or $(\alpha, \beta)=1$, satisfied by certain integrals β, are in certain cases, as Bertrand remarks, a precious method of integration leading to the classification of the integrals of a problem, so as to facilitate their ulterior determination: it is in fact by means of them that the several results before referred to are obtained in the memoir.

59. Bertrand's note of 1852 in the *Comptes Rendus.*—This contains the demonstration of a theorem analogous to Poisson's theorem $(\alpha, \beta)=\text{const.}$, but the function on the left-hand side is a function involving four of the arbitrary constants and binary combinations of pairs of corresponding variables, instead of two arbitrary constants and the series of pairs of corresponding variables.

60. Bertrand's notes, VI. and VII., to the third edition of the *Mécanique Analytique*, 1853, contain a concise and elegant exposition of various theorems which have been considered in the present report. The latter of the two notes relates to the above-mentioned theorem of Poisson, and places the theorem in a very clear light, in fact, establishing its connexion with the theory of canonical integrals. Bertrand in fact shows, that, given any integral α of the differential equations (in the last-mentioned form, the whole number of equations being $2k$), then the solution may be completed by joining to the integral α a system of integrals $\beta_1, \beta_2 \ldots \beta_{2k-1}$, which, combined with the integral α, give to Poisson's equation an identical form, viz. which are such that

$$(\alpha, \beta_1) = 1, \; (\alpha, \beta_2) = 0, \ldots (\alpha, \beta_{2k-1}) = 0.$$

This, he remarks, shows, that, given any integral α, the solution of the problem *may* be completed by integrals $\beta_1, \beta_2 \ldots \beta_{2k-1}$, which, combined with α, give all of them an identical form to the theorem of Poisson. But it is not to be supposed that all the integrals of the problem are in the same case. In fact, the most general integral is $\eta = \varpi(\alpha, \beta_1, \beta_2 \ldots \beta_{2k-1})$, and it is at once seen that $(\alpha, \eta) = (\alpha, \beta_1)\dfrac{d\eta}{d\beta_1} = \dfrac{d\eta}{d\beta_1}$, consequently the expression (α, η) will not be identically constant unless $\dfrac{d\eta}{d\beta_1}$ is so: but the integrals, in number infinite, which result from the combination of α, with $\beta_2, \beta_3 \ldots \beta_{2k-1}$ combined with the integral α, give identical results. Only the integrals which contain β_1 lead to results which are not identical. The integrals α and β_1, connected together in the above special manner, are termed by the author *conjugate integrals.*

61. Brioschi's two notes of 1853.—The memoir "Sulla Variazione &c." contains reflections and developments in relation to Bertrand's method of integration and to canonical systems of integrals, but I do not perceive that any new results are obtained.

The note, "Intorno ad un Teorema di Meccanica," contains a demonstration of the theorem in Bertrand's note of 1852 in the *Comptes Rendus,* and an extension of the theorem to the case of a combination of any even number of the arbitrary constants; the value of the symbol is shown by the theory of determinants to be a function of the Poissonian coefficients (α, β), and as these are constants, the value of the symbol considered is also constant.

62. Liouville's note of the 29th of June 1853 [1], contains the enunciation of a theorem which completes the investigations contained in Poisson's memoir of 1837. The equations considered are the Hamiltonian equations in their most general form, viz., H is any function whatever of t and the other variables: it is assumed that

[1] The date is that of the communication of the note to the Bureau of Longitudes, but the note is only published in Liouville's *Journal* in the May Number for 1855, which is subsequent to the date of the second part of Professor Donkin's memoir in the *Philosophical Transactions*, which contains the theorem in question. I have not had the opportunity of seeing a thesis by M. Adrien Lafon, Paris, 1854, where Liouville's theorem is quoted and demonstrated.

half of the integrals are known, and that the given integrals are such that for any two of them α, β, Poisson's coefficient (α, β) is equal to zero; this being so, the expression $pdq + \ldots - Hdt$, where, by means of the known integrals, the variables $p, \ldots$ are expressed in terms of $q, \ldots t$, is a complete differential in respect to $q, \ldots t$, viz. it will be the differential of Sir W. R. Hamilton's principal function V, which is thus determined by means of the known integrals, and the remaining integrals are then given at once by the general theory.

63. Professor Donkin's memoir of 1854 and 1855, Part I. (sections 1, 2, 3, articles 1 to 48).—The author refers to the researches of Lagrange, Poisson, Sir W. R. Hamilton, and Jacobi, and he remarks that his own investigations do not pretend to make any important step in advance. The investigations contained in section 1, articles 1 to 14, establish by an inverse process (that is, one setting out from the integral equations) the chief conclusions of the theories of Sir W. R. Hamilton and Jacobi, and in particular those relating to the canonical system of elements as given by Jacobi's theory. The theorem (3), article 1, which is a very general property of functional determinants, is referred to as probably new. The most important results of this portion of the memoir are recapitulated in section 4, in the form of seven theorems there given without demonstration; some of these will be presently again referred to. Articles 17 and 18 contain, I believe, the only demonstration which has been given of the equivalence of the generalised Lagrangian and Hamiltonian systems. The transformation is as follows: the generalised Lagrangian system is

$$\frac{d}{dt}\frac{dZ}{dq'} = \frac{dZ}{dq},$$
$$\vdots$$

where Z is any function of t and of $q, \ldots q', \ldots$; writing herein $\frac{dZ}{dq'} = p, \ldots$, and also $H = -Z + q'p + \ldots$, where, on the right-hand side, $q', \ldots$ are expressed in terms of $t, q, \ldots p, \ldots$, so that H is a function of $t, q, \ldots p, \ldots$; then the theorems in the preceding articles show that

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq},$$
$$\vdots \qquad \vdots$$

which is the generalised Hamiltonian system.

In section 2, articles 21 and 22, there is an elegant demonstration, by means of the Hamiltonian equations, of the theorem in relation to Poisson's coefficients (a, b), viz., that these coefficients are functions of the elements only. And there are contained various developments as to the consequences of this theorem; and as to systems of canonical, or, as the author calls them, *normal* elements. The latter part of the section and section 3, relate principally to the special problems of the motion of a body under the action of a central force, and of the motion of rotation of a solid body.

64. Part II. (sections 4, 5, 6 and 7, articles 49—93, appendices).—Section 4 contains the seven theorems before referred to. Although not given as new theorems, yet, to a considerable extent, and in form and point of view, they are new theorems.

Theorem 1 is a theorem standing apart from the others, and which is used in the demonstration of the transformation from the Lagrangian to the Hamiltonian system. It is as follows: viz., if X be a function of the n variables $x, \ldots$, and if $y, \ldots$ be n other variables connected with these by the n equations

$$\frac{dX}{dx} - y, \ldots$$

then will the values of $x, \ldots$, expressed by means of these equations in terms of $y, \ldots$, be of the form

$$x = \frac{dY}{dy}, \ldots$$

and if p be any other quantity explicitly contained in X, then also

$$\frac{dX}{dp} + \frac{dY}{dp} = 0,$$

the differentiation with respect to p being in each case performed only so far as p appears explicitly in the function.

The value of Y is given by the equation

$$Y = -X + xy + \ldots$$

where, on the right-hand side, $x, \ldots$ are expressed in terms of $y, \ldots$

Theorems 2, 3 and 4, and a supplemental theorem in article 50, relate to the deduction of the generalised Hamiltonian system of differential equations from the integral equations assumed to be known. In fact (writing V, $q, \ldots$ $p, \ldots$ $b, \ldots$ $a, \ldots$, instead of the author's $X, x_1, \ldots x_n, y_1, \ldots y_n, a_1, \ldots a_n, b_1, \ldots b_n$), it is assumed that V is a given function of t, of the n variables $q, \ldots$, and of the n constants $b, \ldots$, and that the n variables $p, \ldots$, and the n constants $a, \ldots$, are determined by the conditions

$$\frac{dV}{dq} = p, \ldots \quad (1)$$

$$\frac{dV}{db} = a, \ldots \quad (2)$$

so that in fact by virtue of these $2n$ equations the $2n$ variables X, $q, \ldots$ $p, \ldots$ may be considered as functions of t, and the $2n$ constants $b, \ldots$ $a, \ldots$ (hypothesis 1), or conversely, the $2n$ constants $b, \ldots$ $a, \ldots$, may be considered as functions of t and of the $2n$ variables $q, \ldots$ $p, \ldots$ (hypothesis 2).

Theorem 2 is as follows: viz., if from the $2n$ equations (1, 2) and their total differential coefficients with respect to t, the $2n$ constants be eliminated, there will result the following $2n$ simultaneous differential equations of the first order, viz.:

$$\frac{dq}{dt} = \frac{dH}{dp}, \ldots$$

$$\frac{dp}{dt} = -\frac{dH}{dq}, \ldots$$

H is here a function of $q, \ldots\ p, \ldots$ (which will in general also contain t explicitly), given by the equation

$$H = -\frac{dV}{dt},$$

where, on the right-hand side, the differential coefficient $\frac{dV}{dt}$ is taken with respect to t, in so far as t appears explicitly in the original expression for V in terms of $q, \ldots\ b, \ldots$ and t, and after the differentiation, $b, \ldots$, are to be expressed in terms of the variables and t, by means of the equations (1).

Theorem 3 is, that there exists the following relations, viz.:

$$\frac{dq}{db} = -\frac{da}{bp}, \quad \frac{dq}{da} = \frac{db}{dp}, \ldots$$

$$\frac{dp}{db} = \frac{da}{dq}, \quad \frac{dp}{da} = -\frac{db}{dq}, \ldots$$

where (p, q) are *any* corresponding pair out of the systems $p, \ldots$ and $q, \ldots$, and (b, a) are any corresponding pair out of the systems $b, \ldots$ and $a, \ldots$, so that the total number of equations is $4n^2$: in each of the equations the left-hand side refers to hypothesis 1, and the right-hand side to hypothesis 2.

To these theorems should be added the supplemental theorem contained in article 50, viz., that there subsists also the system of equations

$$\frac{db}{dt} = \frac{dH}{da}, \ldots$$

$$\frac{da}{dt} = -\frac{dH}{db}, \ldots$$

where the left-hand sides refer to hypothesis 2, while the right-hand sides refer to hypothesis 1; as before $H = -\frac{dV}{dt}$, but here H is differentially expressed, being what the H of theorem 3 becomes when the variables are expressed according to hypothesis 1.

In theorem 4 the author's symbol (p, q) has a signification such as Poisson's (a, b), and if we write as before

$$(a, b) = \frac{\partial (a, b)}{\partial (p, q)} + \dots,$$

where

$$\frac{\partial (a, b)}{\partial (p \;\; q)} = \frac{da}{dp}\frac{db}{dq} - \frac{db}{dp}\frac{da}{dq}$$

(this refers of course to hypothesis 2), the theorem is, that the following equations subsist identically, viz., b, a being corresponding constants out of the two series $b, \dots$ and $a, \dots$, then

$$(b, a) = -(a, b) = 1,$$

but that for any other pairs b, a, or for any pairs whatever b, b or a, a, the corresponding symbol is $= 0$: in fact, that the constants $b, \dots$ and $a, \dots$ form a canonical system of elements.

Theorem 5 is a theorem including theorem 4, and relating to any two functions u, v either of the two $2n$ constants or else of the $2n$ variables, and which may besides contain t explicitly; it establishes, in fact, a relation between Poisson's coefficient (u, v) and the corresponding coefficient of Lagrange.

Theorem 6 is as follows: viz., if $q, \dots\ p, \dots$ are any $2n$ variables concerning which no supposition is made, except that they are connected by the n equations

$$b = \phi\,(q, \dots\ p, \dots),$$

which equations are only subject to the condition of being sufficient for the determination of $p, \dots$ in terms of $q, \dots$ and $a, \dots$, and they may contain explicitly any other quantities, for example, a variable t. Then, in order that the $\frac{1}{2}n\,(n-1)$ equations

$$\frac{dp_i}{dq_j} = \frac{dp_j}{dq_i}$$

may subsist identically, it is only necessary that each of the $\frac{1}{2}n\,(n-1)$ equations $(b_i, b_j) = 0$ may be satisfied identically.

Theorem 7 is, in fact, the theorem previously established in its general form in Liouville's note of the 29th of June, 1853, viz., if, of the system of $2n$ differential equations

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq},$$

there be given n integrals involving the n arbitrary constants $b, \dots$, so that each of these constants can be expressed as a function of the variables $q, \dots, p, \dots$ (with or without t); then, *if the* $\frac{1}{2}n\,(n-1)$ *conditions* $(b_i, b_j) = 0$ *subsist identically, the remaining* n *integrals can be found as follows:*—By means of the n integrals, let the n variables

$p, \ldots$ be expressed in terms of $x, \ldots\ b, \ldots$ and t, and let H stand for what H, as originally given, becomes when $q, \ldots$ are thus expressed. Then the values of $p, \ldots$ and $-H$ are the differential coefficients of one and the same function of $p, \ldots$ and t; call this function V, then, since its differential coefficients are all given (by the equations $\frac{dV}{dq} = p, \ldots\ \frac{dV}{dt} = -H$), V may be found by integration; and it is therefore to be considered as a given function of $p, \ldots$ and t and of the constants $b, \ldots$. The remaining n integrals are given by the n equations

$$\frac{dV}{db} = a, \ldots$$

where the n quantities $a, \ldots$ are new arbitrary constants.

65. Section 5 of the memoir relates to the theory of the variation of the elements considered in relation to the following very general problem: viz., $Q, \ldots\ P, \ldots$ being any functions whatever of the $2n$ variables $q, \ldots\ p, \ldots$ and t; it is required to express the integrals of the system $2n$ differential equations

$$\frac{dq}{dt} = P, \quad \frac{dp}{dt} = Q,$$
$$\vdots \qquad \vdots$$

in the same form as the integrals (supposed given) of the standard system

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq}$$
$$\vdots \qquad \vdots$$

by substituting functions of t for the constant elements of the latter system. And section 6 contains some very general researches on the general problem of the transformation of variables, a problem of which, as the author remarks, the method of the variation of elements is a particular, and not the only useful case. In particular, the author considers what he terms a normal transformation of variables, and he obtains the theorem 8, which includes as a particular case the second of the two theorems in Jacobi's note of 1837, in the *Comptes Rendus*. This theorem is as follows: viz., if the original variables $q, \ldots\ p, \ldots$ are given by the $2n$ equations

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq};$$
$$\vdots \qquad \vdots$$

and if the new variables $\eta, \ldots\ \varpi, \ldots$ are connected with the original variables by the equations

$$\frac{dK}{dp} = q, \quad \frac{dK}{d\eta} = \varpi;$$
$$\vdots \qquad \vdots$$

where K is any function of $\eta, \ldots p, \ldots$ which may also contain t explicitly, then will the transformed equations be

$$\frac{d\eta}{dt}=\frac{d\Phi}{d\varpi}, \quad \frac{d\varpi}{dt}=-\frac{d\Phi}{d\eta};$$

in which Φ is defined by the equation

$$\Phi = H - \frac{dK}{dt},$$

and is to be expressed in terms of the new variables, the substitution of the new variables in $\frac{dK}{dt}$ being made after the differentiation. In particular, if K does not contain t explicitly, then $\frac{dK}{dt}=0$ and $\Phi = H$, so that, in this case, the transformation is effected merely by expressing H in terms of the new variables. There is also an important theorem relating to the *transformation of coordinates.* To explain this, it is necessary to go back to the generalised Lagrangian form

$$\frac{d}{dt}\frac{dZ}{dq'}=\frac{dZ}{dq};$$
$$\vdots$$

where the variables $q, \ldots$ correspond to the coordinates of a dynamical problem; if the new variables $\eta, \ldots$ are any given functions whatever of the original variables $q, \ldots$ and of t, this is what may be termed a transformation of coordinates. But the proposed system can be expressed, as shown in the former part of the memoir, in the generalised Hamiltonian form with the variables $q, \ldots$ and the derived variables $p, \ldots$ (the values of which are given by $\frac{dZ}{dq}=p, \ldots$): the problem is to transform the last-mentioned system by introducing, instead of the original coordinates $q, \ldots$, the new coordinates $\eta, \ldots$, and instead of the derived variables $p, \ldots$ the new derived variables $\varpi, \ldots$ defined by the analogous equations $\frac{dZ}{d\eta}=\varpi, \ldots$, in which Z is supposed to be expressed as a function of $\eta, \ldots$ and t. The method of transformation is given by the theorem 9, which states that the transformation is a normal transformation, and that the modulus of transformation (that is, the function corresponding to K in theorem 8) is

$$K = qp + \ldots$$

where $q, \ldots$ are to be expressed in terms of $\eta, \ldots$. The latter part of the same section contains researches relating to the case where the proposed equations are symbolically, but not actually, in the Hamiltonian form, viz., where the function H is considered as containing functions of $q, \ldots p, \ldots$ which are exempt from differentiation in forming the differential equations (the author calls this a pseudo-canonical system), and where, in like manner, the transformation of variables is a pseudo-normal transformation; the theorems 10 and 11 relate to this question, which is treated still more generally in Appendix C. The general methods are illustrated by applications to the problem of three bodies and the problem of rotation; the former problem is specially discussed in section 7; but the results obtained (and which, as the author remarks, affords an example of the so-called "elimination of the nodes") do not come within the plan of the present report.

66. Bour's memoir of 1855, "On the Integration of the Differential Equations of Analytical Mechanics."—It has been already seen that the knowledge of half the entire system of the integrals of the differential equations (these known integrals satisfying certain conditions) leads by quadratures only to the knowledge of the remaining integrals; the researches contained in this most interesting and valuable memoir show that this theorem is, in fact, only the last of a series of theorems, here first established, relating to the successive reduction which results from the knowledge of each new integral. Speaking in general terms, it may be stated that the author operates on the linear partial differential equation of the first order, which is satisfied by the integrals of the differential equations; and that he effectuates upon this equation a reduction of two unities in the number of variables for every suitable new integral which is obtained [1]. The author shows also that an equal or greater reduction may sometimes be obtained by means of integrals which appear at first foreign to his method. Before going further, it may be convenient to remark that the author restricts himself to the case in which H is independent of the time, and where, consequently, the condition of *vis viva* is satisfied; it was, however, remarked by Liouville that the analysis, slightly modified, applies to the most general case where H is any function of t and the variables, and it is possible that when the entire memoir is published (it is given in Liouville's *Journal* as an extract), the theory will be exhibited under this more general form.

67. To give an idea of the analytical results, the equations are considered under the form

$$\frac{dp_i}{dt}=\frac{dH}{dq_i}, \quad \frac{dq_i}{dt}=-\frac{dH}{dq_i} \; (i=1 \text{ to } i=n)$$

(where, as already remarked, H is independent of t). The integrals admit, therefore, of representation in the canonical form $\alpha, \beta, \alpha_1, \alpha_2, \ldots \alpha_{2n-2}$ where $\alpha\,(=H)$ is the equation of *vis viva*; $\beta\,(=G-t)$ is the integral conjugate to this, and the only integral involving the time; and the remaining integrals α_1 and α_2, α_3 and $\alpha_4 \ldots \alpha_{2n-3}$ and α_{2n-2} are conjugate pairs: we have

$$(\alpha_1, \alpha)\;(=(\alpha_1, H))=0, \quad (\alpha_1, \beta)\;(=(\alpha_1, G))=0, \quad (\alpha_1, \alpha_2)=1, \quad (\alpha_1, \alpha_3)=0, \ldots (\alpha_1, \alpha_{2n-2})=0.$$

The integrals $\alpha_1, \alpha_2, \ldots \alpha_{2n-2}$ verify the linear partial differential equation

$$\sum_{i=1}^{i=n}\left(\frac{dH}{dq_i}\frac{d\zeta}{dp_i}-\frac{dH}{dp_i}\frac{d\zeta}{dq_i}\right)=0 \text{ or, } (H, \zeta)=0 \qquad (1)$$

[1] I have borrowed this and the next sentence from Liouville's report. It would, I think, be more accurate to say, for every suitable new integral after the first one; in the case considered in the memoir, the condition of *vis viva* is satisfied, and there is always one integral, the equation of *vis viva*, which is known; but this alone, and in the general case the first known integral, will not cause a reduction of two unities.

which is also satisfied by $\zeta = H$, and of which the general solution is $\zeta = \phi(H, \alpha_1, \alpha_2 \ldots \alpha_{2n-2})$, while, on the contrary, the first member of the equation (1), becomes unity for $\zeta = G$, in other words $(H, G) = 1$. The equation (1) replaces the original differential equations; it is to the equation (1) that the theorems of Poisson and Bertrand may be supposed to be applied, and it is this equation (1) which is studied in the memoir, where it it shown how the order may be diminished when one or more integrals are known.

In the first place, the integral $\alpha = H$ which is known, may be made use of to eliminate one of the variables, suppose p_n; the result is found to be

$$\sum_{i=1}^{l=n-1} \left(\frac{dp_n}{dq_i}\frac{d\zeta}{dp_i} - \frac{dp_n}{dp_i}\frac{d\zeta}{dq_i}\right) + \frac{d\zeta}{dq_n} = 0, \qquad (2)$$

which has the same integrals as the equation (1), except the integral of *vis viva* $\zeta = H$; it is this equation (2) which would have to be integrated if only the integral α were known.

Suppose now there is known a new integral α_1; this gives rise to the partial differential equation

$$\sum_{i=1}^{l=n} \left(\frac{d\alpha_1}{dq_i}\frac{d\zeta}{dp} - \frac{d\alpha_1}{dp_i}\frac{d\zeta}{dq_i}\right) = 0 \text{ or } (\alpha_1, \zeta) = 0, \qquad (4)$$

which is satisfied by $\zeta = H, G, \alpha_1, \alpha_3, \alpha_4 \ldots \alpha_{2n-2}$, but not by α_2, which gives $(\alpha_1, \alpha_2) = 0$. The equation (4) is satisfied by $\zeta = H$, and it may be therefore transformed in the same manner as the equation (1) was, viz. p_n may be expressed in terms of the other variables and of α. The author remarks that it will happen, what causes the success of the method, that this operation (the object of which is to get rid of the solution $\zeta = H$) conducts to two different equations, according as $\zeta = G$ or $\zeta =$ any other integral of the equation (4); so that in the second form of the transformed equation the unknown integral $\zeta = G$ is also eliminated. This second form is found to be

$$\sum_{i=1}^{l=n-1} \left(\frac{d\alpha_1}{dq_i}\frac{d\zeta}{dp_i} - \frac{d\alpha_1}{dp_i}\frac{d\zeta}{dq_i}\right) = 0 \text{ or } (\alpha_1, \zeta) = 0, \qquad (5)$$

which is precisely similar to the equation (1) (only the number of variables is diminished by two unities), and is possessed of the same properties. Its integrals are $\alpha_1, \alpha_3, \alpha_4 \ldots \alpha_{2n-2}$, which are all of them integrals of the problem, and give $(\alpha_1, \alpha_i) = 0$. And the theorems of Poisson and Bertrand apply equally to this equation; the only difference is, that the number of terms in the expressions (α, β) is less by two unities. A new integral (α_3) leads in like manner to an equation (8) similar to (5), but with the number of variables further diminished by two unities, and so on, until the half series of integrals $\alpha, \alpha_1, \alpha_3 \ldots \alpha_{2n-3}$ are known; the conjugate integrals $\beta, \alpha_2, \alpha_4 \ldots \alpha_{2n-2}$ are then obtained by quadratures only, in the method explained in the memoir, and which is in fact identical with that given by the theorem of Liouville and Donkin. The memoir contains other results, which have been already alluded to in a general manner; some of these are made use of by the author in his "Mémoire sur le problême des trois corps," *Journal École Polyt.*, t. XXI. pp. 35—58 (1856).

68. Liouville's note of July, 1855, on the occasion of Bour's memoir, mentions that the author of the memoir had recognized that, according to the remark made to him, his formulæ subsist with even increased elegance when H is considered as a function of t and the other variables. But (it is remarked) the general case can be always reduced to the particular one considered in the memoir, provided that the number of equations is augmented by two unities by the introduction of the new variables τ and u, the former of them, τ, equal to t + constant, so that

$$\frac{dt}{d\tau}=1$$

the latter of them, u, defined by the equation

$$\frac{du}{d\tau}=-\frac{dH}{dt}.$$

Suppose in fact that

$$V=H+u,$$

then, since τ and u do not enter into H, which is a function only of t and the variables $q, \ldots p, \ldots$, we have

$$\frac{dV}{du}=1=\frac{dt}{d\tau};$$

and, moreover, the differential coefficients with respect to $t, q, \ldots p, \ldots$ of the functions H and V are equal. The system may be written

$$\frac{dt}{d\tau}=\frac{dV}{du},\quad \frac{du}{d\tau}=-\frac{dV}{dt},$$

$$\frac{dp}{d\tau}=\frac{dV}{dq},\quad \frac{dq}{d\tau}=-\frac{dV}{dp},$$

$$\vdots \qquad\qquad \vdots$$

which is a system containing two more variables, but in which V is independent of the variable τ, which stands in the place of t. The transformation is an elegant and valuable one, but it is not in anywise to be inferred that there is any advantage in considering the particular case (which is thus shown to be capable of including the general one), rather than the general one itself: such inference does not seem to be intended, and would, I think, be a wrong one.

69. Brioschi's note of 1855 contains an elegant demonstration (founded on the theory of skew determinants) of a property, which appears to be a new one, of the canonical integrals of a dynamical problem, viz. if q, p stand for a corresponding pair of the variables $q, \ldots p, \ldots$ then

$$\Sigma\frac{\partial(\alpha,\ \beta)}{\partial(q,\ p)}=1$$

where the summation refers to all the different pairs of conjugate integrals α, β of the canonical system, the pair q, p in the denominator being the same in each term;

but if the variables in the denominator are a non-corresponding pair out of the two series $q, \ldots$ and $p, \ldots$, or else a pair out of one series only (that is, both q's or both p's), then the expression on the left-hand side is equal to zero. This is in fact a sort of reciprocal theorem to the theorem which defines the canonical system of integrals. There are two or three memoirs of Brioschi in Crelle's *Journal* connected with this note and the note of 1853; but as they relate professedly to skew determinants and not to the equations of dynamics, it is not necessary here to refer to them more particularly.

70. Bertrand's memoir of 1857 forms a sequel to the memoir of 1851, on the integrals common to several problems of mechanics. The author calls to mind that he has shown in the first memoir, that, given an integral of a mechanical problem, and assuming only that the forces are functions of the coordinates, it is possible to determine the problem and find the forces which act upon each point; and (he proceeds) it is important to remark, that the solution leads often to contradictory results,—that, in fact, an equation assumed at hazard is not in general an integral of any problem whatever of the class under consideration: and he thereupon proposes to himself in the present memoir to develope some of the consequences of this remark, and to seek among the most simple forms, the equations which can present themselves as integrals, and the problems to which such integrals belong. The various special results obtained in the memoir are interesting and valuable.

71. In what precedes I have traced as well as I have been able the series of investigations of geometers in relation to the subject of analytical dynamics. The various theorems obtained have been in general stated with sufficient fulness to render them intelligible to mathematicians; the attempt to state them in a uniform notation and systematic order would be out of the province of the present report. The leading steps are,—first, the establishment of the Lagrangian form of the equations of motion; secondly, Lagrange's theory of the variation of the arbitrary constants, a theory perfectly complete in itself; and it would not have been easy to see *à priori* that it would be less fruitful in results than the theory of Poisson; thirdly, Poisson's theory of the variation of the arbitrary constants, and the method of integration thereby afforded; fourthly, Sir W. R. Hamilton's representation of the integral equations by means of a single characteristic function determinable *à posteriori* by means of the integral equations assumed to be known, or by the condition of its simultaneous satisfaction of two partial differential equations; fifthly, Sir W. R. Hamilton's form of the equations of motion; sixthly, Jacobi's reduction of the integration of the differential equations to the problem of finding a complete integral of a single partial differential equation, and the general theory of the connexion of the integration of a system of ordinary differential equations, and of a partial differential equation of the first order, a theory, however, of which Jacobi can only be considered as the second founder; seventhly, the notion (arising from the researches of Lagrange and Poisson) and ulterior development of the theory of a system of canonical integrals.

I remark in conclusion, that the differential equations of dynamics (including in the expression, as I have done throughout the report, the generalized Lagrangian and

Hamiltonian forms) are only one of the classes of differential equations which have occupied the attention of geometers. The greater part of what has been done with respect to the general theory of a system of differential equations is due to Jacobi, and he has also considered in particular, besides the differential equations of dynamics, the Pfaffian system of differential equations (including therein the system of differential equations which arise from any partial differential equation of the first order), and the so-called isoperimetric system of differential equations, that is, the system arising from any problem in the calculus of variations. In a systematic treatise it would be proper to commence with the general theory of a system of differential equations, and as a branch of this general theory, to consider the generalized Hamiltonian system, and in relation thereto to develope the various theorems which have a dynamical application. It would be shown that the generalized Lagrangian form could be transformed into the Hamiltonian form, but the first-mentioned form would, I think, properly be treated as a particular case of the isoperimetric system of differential equations.

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