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Stress-strain state of an elastic cylinder and a layer in joint torsion

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WE CONSIDER the mixed boundary value problem of theory of elasticity, namely the joint torsion of the cylinder and layer. We investigate the case of anisotropic materials. By means of the Fourier method and integral Hankel transforms the problem has been reduced to a Fredholm integral equation of the second kind. The proof of the convergence of the successive approximation method is given. Consequently, the problem has been reduced to the system of linear algebraic equations (infinite). All characteristics of the elastic state of the body have been determined. The numerical examples are presented in the diagram form.

Rozpatrzono mieszane zagadnienie brzegowe teroii sprężystości w przypadku równoczesnego skręcania walca i warstwy. Przedyskutowano również przypadek materiałów anizotropowych. Za pomocą metody Fouriera i przekształceń całkowych Hankela zagadnienie zostało sprowadzone do równania całkowego Fredholma drugiego rodzaju. Podano dowód zbieżności metody kolejnych przybliżeń. Zagadnienie zostało następnie sprowadzone do nieskończonego układu algebraicznych równań liniowych. Zostały wyznaczone wszystkie wielkości charakterystyczne stanu sprężystego. Przykłady liczbowe zostały przedstawione w postaci wykresów.

Рассматривается смещанная граничная задача теории упругости о совместном кручении цилиндра и слоя. Рассматривается случай анизотропных материалов. Методами Фурье и интегральных преобразований Ханкеля задача приведена к интегральному уравнению Фредгольма II-го рода. Дано доказательство сходимости метода последовательных приближений. Специальным приемом задача сведена к системе линейных уравнений (бесконечной). Определены все характеристики упругого состояния тела. Проведены численные подсчеты, которые представленны в виде графиков.

Introduction

THE PROBLEM of the joint torsion of an elastic cylinder and a semi-space and the influence of the shear moduli ratio on the stress-strain state of the elastic system has been discussed in the papers [1, 3]. In the present paper, the problem of the joint torsion of a cylinder and a layer is solved by the methods presented in [1, 4]. The effect of the layer thickness for various ratios of shear moduli is investigated.

1. Statement of the problem and basic relations

Let us consider the equilibrium of the system consisting of an elastic cylinder of radius R and length L and a layer of finite thickness H in joint torsion. The cylinder is fastened to the layer on a base and is subjected to the action of tractions rotating the upper base as a rigid whole. The cylindrical surface of the bar and the surface of the layer outside the contact region is free from stresses. The lower surface of the layer is rigidly clamped. The cylinder and the layer are made of different isotropic materials.

Introduce the cylindrical çoordinate system r, θ, z such that the surface $z = 0$ coincides with the layer surface while the axis $0z$ is directed along the symmetry axis of the cylinder.

Thus in order to determine the non-zero component of the displacement vector u_θ: and the stress tensor components $\tau_{\theta z}, \tau_{\theta r}$, we obtain the following boundary conditions:

$$\text{(1.1)} \qquad \text{for} \quad z = L, \quad u_\theta^{(1)} = \varepsilon r, \qquad r \leqslant R,$$

$$\text{(1.2)} \qquad \text{for} \quad z = 0, \quad u_\theta^{(1)} = u_\theta^{(2)}, \quad \tau_{\theta z}^{(1)} = \tau_{\theta z}^{(2)}, \quad r \leqslant R,$$

$$\text{(1.3)} \qquad \tau_{\theta z}^{(2)} = 0, \qquad r > R,$$

$$\text{(1.4)} \qquad \text{for} \quad r = R \quad \tau_{\theta r}^{(1)} = 0, \qquad 0 \leqslant z \leqslant L,$$

$$\text{(1.5)} \qquad \text{for} \quad z = -H. \quad u_\theta^{(2)} = 0.$$

Here and in what follows, all the quantities referring to the cylinder will be denoted by the index 1 and to the layer — by the index 2.

It is a known fact [2, 4] that in the case of pure torsion the displacement component $u_\theta^{(i)}$ $(i = 1, 2)$ satisfies the differential equation

$$\text{(1.6)} \qquad \frac{\partial^2 u_\theta^{(i)}}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta^{(i)}}{\partial r} - \frac{u_\theta^{(i)}}{r^2} + \frac{\partial^2 u_\theta^{(i)}}{\partial z^2} = 0, \quad i = 1, 2$$

and is related to $\tau_{\theta z}^{(i)}$ and $\tau_{\theta r}^{(i)}$ by the equations

$$\text{(1.7)} \qquad \tau_{\theta z}^{(i)} = G_i \frac{\partial u_\theta^{(i)}}{\partial z}, \quad \tau_{\theta r}^{(i)} = G_i\left(\frac{\partial u_\theta^{(i)}}{\partial r} - \frac{u_\theta^{(i)}}{r}\right), \quad i = 1, 2,$$

where G_i $(i = 1, 2)$ denote the shear moduli.

The solution for the layer and the cylinder will be determined separately. For the layer, we introduce the Hankel transform $\bar{u}_\theta^{(i)}$ of the function $u_\theta^{(i)}$

$$\text{(1.8)} \qquad \bar{u}_\theta^{(2)}(\xi, z) = \mathscr{H}_1[u_\theta^{(2)}(r, z), \quad r \to \xi];(^1)$$

consequently, we obtain for $\tau_\theta^{(2)}$, $u_{\theta z}$ and $\tau_{\theta r}$ the following formulae:

$$\text{(1.9)} \qquad \begin{aligned} u_\theta^{(2)} &= \mathscr{H}_1[A(\xi)e^{-\xi z} + B(\xi)e^{\xi z}, \qquad \xi \to r], \\ \tau_{\theta z}^{(2)} &= G_2\mathscr{H}_1[-A(\xi)e^{-\xi z} + B(\xi)e^{\xi z}, \qquad \xi \to r], \\ \tau_{\theta r}^{(2)} &= G_2\mathscr{H}_2[\xi(A(\xi)e^{-\xi z} + B(\xi)e^{\xi z}), \qquad \xi \to r]. \end{aligned}$$

Here, $A(\xi)$, $B(\xi)$ are arbitrary functions.

Taking into account the boundary condition (1.5), after some algebra, we obtain the formulae for the stresses and the displacements in the elastic laver

$$\text{(1.10)} \qquad \begin{aligned} u_\theta^{(2)} &= R\mathscr{H}_1\left[\eta^{-1}F(\eta)\frac{\operatorname{sh}\eta(\zeta+h)}{\operatorname{ch}(\eta h)}, \qquad \eta \to \varrho\right], \\ \tau_{\theta z}^{(2)} &= G_2\mathscr{H}_1\left[F(\eta)\frac{\operatorname{ch}\eta(\zeta+h)}{\operatorname{ch}(\eta h)}, \qquad \eta \to \varrho\right], \\ \tau_{\theta r}^{(2)} &= -G_2\mathscr{H}_2\left[\eta F(\eta)\frac{\operatorname{sh}(\zeta+h)\eta}{\operatorname{ch}(\eta h)}, \qquad \eta \to \varrho\right], \end{aligned}$$

(¹) Where $\mathscr{H}_\nu[F(\eta);\ \eta \to \varrho] = \int\limits_0^\infty \eta F(\eta) J_\nu(\eta\varrho)\,d\eta$.

where $\varrho = r/R$, $h = H/R$, $\eta = \zeta R$ are dimensionless parameters, $F(\eta)$ is an arbitrary function determined from the boundary conditions and the continuity conditions.

In order to solve the Eq. (1.6) for the case of the cylinder, Fourier's method will be applied. The solution is assumed in the form

$$u_\theta = A_0 rz + B_0 r + Z(z)\cdot R(r). \tag{1.11}$$

Substituting (1.11) in (1.6) and taking into account the finiteness of the solution for $r = 0$, and the Eq. (1.7), we obtain the particular solution:

$$\begin{aligned} u_\theta^{(1)} &= A_0 rz + B_0 r + J_1(\lambda r)[A\,\mathrm{sh}\,\lambda z + B\,\mathrm{ch}\,\lambda z],\\ \tau_{\theta z}^{(1)} &= G_1 A_0 r + G_1 \lambda J_1(\lambda r)[A\,\mathrm{ch}\,\lambda z + B\,\mathrm{sh}\,\lambda z],\\ \tau_{\theta r}^{(1)} &= -G_1 \lambda J_2(\lambda r)[A\,\mathrm{sh}\,\lambda z + B\,\mathrm{ch}\,\lambda z]. \end{aligned} \tag{1.12}$$

Here A and B denote arbitrary constants, λ is a parameter, $J_i(x)$ is the Bessel function of the first kind.

The boundary condition (1.4) is satisfied for $J_2(\lambda R) = 0$. Whence we obtain the latent values of the problem $\lambda_k = \mu_k/R$, where μ_k are the roots of the characteristic equation $J_2(\mu) = 0$. Then the general solution for the cylinder can be written down in the form

$$\begin{aligned} u_\theta^{(1)} &= R^2 A_0 \varrho\zeta + RB_0\varrho + \sum_{k=1}^{\infty} J_1(\mu_k\varrho)[A_k \mathrm{sh}\,\mu_k\zeta + B_k \mathrm{ch}\,\mu_k\zeta],\\ \tau_{\theta z}^{(1)} &= G_1 R A_0 \varrho + G_1/R \sum_{k=1}^{\infty} \mu_k J_1(\mu_k\varrho)[A_k \mathrm{ch}\,\mu_k\zeta + B_k \mathrm{sh}\,\mu_k\zeta],\\ \tau_{\theta r}^{(1)} &= -\frac{G_1}{R} \sum_{k=1}^{\infty} \mu_k J_2(\mu_k\varrho)[A_k \mathrm{sh}\,\mu_k\zeta + B_k \mathrm{ch}\,\mu_k\zeta], \end{aligned} \tag{1.13}$$

where $\zeta = z/R$.

2. Determination of stresses and displacements

The stresses and displacements in the layer and the cylinder will be determined if the constants A_k, B_k $(k = 0, 1, 2, \ldots)$ and the function $F(\eta)$ are known. In order to determine them, we make use of the boundary conditions and the continuity conditions (1.1)–(1.3). Satisfying the boundary condition (1.1) and the second continuity condition (1.2), we obtain

$$\varepsilon R\varrho = R^2 A_0 \varrho l + R\varrho B_0 + \sum_{k=1}^{\infty} J_1(\mu_k\varrho)\,[A_k \mathrm{sh}\,\mu_k l + B_k \mathrm{ch}\,\mu_k l], \tag{2.1}$$

$$\int_0^\infty \eta F(\eta) I_1(\eta\varrho)\,d\mu = A_0 B\varrho\delta + \frac{\delta}{R}\sum_{k=1}^{\infty} \mu_k J_1(\mu_k\varrho) A_k, \tag{2.2}$$

where $\delta = G_1/G_2$, $l = L/R$.

Multiplying both sides of the Eqs. (2.1) and (2.2) by ϱ^2 and $\varrho J_1(\mu_k\varrho)$, respectively, integrating them from 0 to 1, and taking into account the orthogonality of the Bessel functions, we obtain the relations for the determination of A_k and B_k by the function $F(\eta)$:

$$
(2.3)\qquad \begin{gathered}
A_0 Rl + B_0 = \varepsilon, \quad B_k = A_k \operatorname{th} \mu_k l, \quad A_0 = \frac{4}{R\delta}\int_0^\infty F(\eta) J_2(\eta)\, d\eta, \\
A_k = \frac{2R}{\delta \mu_k J_1^2(\mu_k)} \int_0^1 \varrho J_1(\varrho\mu_k)\, d\varrho \int_0^\infty \eta F(\eta) J_1(\eta\varrho)\, d\eta .
\end{gathered}
$$

Now, satisfying the first continuity condition (1.2), the boundary condition (1.3), and taking into account (2.3), we obtain the dual integral equations in the form

$$
(2.4)\qquad \begin{gathered}
\mathscr{H}_1[\eta^{-1}F(\eta); \eta \to \varrho] = \varrho(\varepsilon - A_0 Rl) - \frac{1}{R}\sum_{k=1}^{\infty} A_k J_1(\mu_k \varrho) \operatorname{th} \mu_k l \\
+ \int_0^\infty F(\eta)\,(1 - \operatorname{th}(\eta h) J_1(\eta\varrho)\, d\eta, \qquad \varrho < 1, \\
\mathscr{H}_1[F(\eta); \eta \to \varrho] = 0, \qquad \varrho > 1 .
\end{gathered}
$$

Applying the inverse transform to the dual integral equations [5], we obtain a Fredholm integral equation of the second kind

$$
\begin{gathered}
F(\eta) = \frac{4}{\pi}(\varepsilon - A_1 Rl)\frac{1}{\eta}\left(\frac{\sin\eta}{\eta} - \cos\eta\right) - \frac{2}{\pi R}\sum_{k=1}^{\infty} A_k \operatorname{th} \mu_k l \times \\
\times \int_0^1 \sin\mu_k y \sin\eta y\, dy + \frac{2}{\pi\eta}\int_0^1\int_0^\infty F\left(\frac{u}{\eta}\right)(1 - \operatorname{th} u)\sin\frac{uy}{h}\sin\eta y\, du\, dy .
\end{gathered}
$$

To complete the solution to the problem it is necessary to determine the constants A_k and the function $F(\eta)$, which are mutually related by the relations (2.3) and (2.5).

Equation (2.5) can be solved by the method of successive approximations. The solution for an elastic semi-space ($h = \infty$), obtained in [1] is taken as the zero approximation, i.e.:

$$
(2.6)\qquad F_0(\eta) = \frac{4}{\pi}(\varepsilon - A_0 Rl)\frac{1}{\eta}\left(\frac{\sin\eta}{\eta} - \cos\eta\right) - \frac{2}{\pi R}\sum_{k=1}^{\infty} A_k \operatorname{th} \mu_k l \int_0^1 \sin\mu_k y \sin\eta y\, dy .
$$

Then the subsequent approximations can be determined from the formula:

$$
(2.7)\qquad F_k(\eta) = \frac{2}{\pi\eta}\int_0^1\int_0^\infty F_{k-1}\left(\frac{u}{\eta}\right)(1 - \operatorname{th} u)\sin\frac{uy}{R}\sin\eta y\, du\, dy
$$

and the solution assumes the following form:

$$
(2.8)\qquad F(\eta) = \sum_{k=0}^{\infty} F_k(\eta) .
$$

To justify the application of the method of successive approximations, and to find the interval of convergence, we make use of the contracted mapping principle. The Eq. (2.5) can be written in the form $F = U(F)$, where U is an operator in the space M of bounded functions. We shall prove that the following conditions hold:

$$\begin{aligned} &1. \quad U(F) \in M \quad \text{if} \quad F \in M, \\ &2. \quad |U(F_1) - U(F_2)| \leqslant m \sup |F_1 - F_2|, \quad F_1, F_2 \in M. \end{aligned} \tag{2.9}$$

The convergence of the process is ensured for $0 \leqslant m < 1$. The zero approximation is bounded — i.e., $F_0(\eta) \in M$, since the solution for the semi-space, derived in [1], is bounded.

Let $F(\eta) \in M$. Performing the relevant estimations, we obtain

$$U(F) \leqslant C + \frac{1}{\pi h^2} \int_0^\infty u(1 - \operatorname{th} u)\, du = C + \frac{1}{2\pi h^2} G_1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} < C_3, \qquad U(F) \in M.$$

Then

$$|U(F_1) - U(F_2)| \leqslant \frac{1}{2\pi h^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sup |F_1 - F_2|.$$

If

$$\frac{1}{2\pi h^2} \sum_{n=1}^{\infty} \frac{(1-)^{n+1}}{n^2} < 1,$$

the operator $U(F)$ is a contracted operator which by Banach's theorem possesses the fixed point. Performing the relevant calculations, we find that the Eq. (2.5) can be solved by the method of successive approximations, provided that $h > 0.362$.

The problem is solved in the following way. As it was shown above, the successive approximations are found by the formula (2.7) and the solution by the formula (2.8). Since, except the first approximation, the straightforward integration is not feasible, each approximation is expanded into series in terms of h^{-1}, up to the terms with h^{-7}. Substituting the value thus found of $F(\eta)$ into (2.3) and introducing new constants C_0 and C_k for A_0 and A_k,

$$C_0 = \frac{Rl}{\varepsilon} A_0, \quad C_k = A_k \frac{\operatorname{th} \mu_k l}{2\varepsilon R}; \tag{2.10}$$

after some arrangements, we obtain the infinite system of linear algebraic equations

$$\alpha_k C_k + \sum_{n=0}^{\infty} \alpha_{kn} C_n = f_k, \quad k_1 = 0, 1, 2 \ldots \tag{2.11}$$

The coefficients α_k, α_{kn} and f_k have the following values:

$$\alpha_{00} = f_0 = \frac{2}{3} + \frac{2N_3}{9\pi h^3} - \frac{2N_5}{15\pi h^5} + \frac{4N_3^2}{54\pi^2 h^6} + \frac{3N_7}{35\pi R^7}, \tag{2.12}$$

$$\alpha_{kn} = \alpha_{nk} = \beta_{kn} + \frac{N_3\beta'_k\beta'_n}{\pi h^3} - \frac{N_5}{2\pi h^5}(\beta'_k\beta''_n + \beta'_n\beta''_k) + \frac{15N_7}{8\pi h^7}\left(\frac{\beta'_k\beta'''_n}{10} + \frac{\beta''_k\beta''_n}{3} + \frac{\beta'''_k\beta'_n}{10}\right) + \frac{N_3^2\beta'_k\beta'_n}{3\pi^2h^6}, \tag{2.12 [cont.]}$$

$$\alpha_{k0} = f_k = \beta'_k + \frac{N_3\beta'_k}{3\pi h^3} - \frac{N_5}{2\pi h^5}\left(\frac{\beta'_k}{5} + \frac{\beta''_k}{3}\right) + \frac{N_3^2\beta'_k}{9\pi^2h^6} + \frac{N_7}{8\pi h^7}\left(\frac{3\beta'_k}{14} + \beta''_k + \frac{\beta'''_k}{2}\right),$$

$$\alpha_{0n} = 2\alpha_{k0}, \quad \alpha_0 = \frac{\pi\delta}{16l}, \quad \alpha_k = \frac{\pi\delta}{4}\mu_k J_1^2(\mu_k).$$

Here, the following symbols have been introduced:

$$\beta'_k = \frac{1}{\mu_k}\left(\frac{\sin\mu_k}{\mu_k} - \cos\mu_k\right),$$

$$\beta''_k = \frac{\cos\mu_k}{\mu_k}\left(\frac{6}{\mu_k^2} - 1\right) + \frac{3\sin\mu_k}{\mu_k^2}\left(1 - \frac{2}{\mu_k^2}\right), \tag{2.13}$$

$$\beta'''_k = \frac{\cos\mu_k}{\mu_k}\left(\frac{5\sin\mu_k}{\mu_k} - 1\right) - \frac{20}{\mu_k^2}\left[\frac{\cos\mu_k}{\mu_k}\left(\frac{6}{\mu_k^2} - 1\right) + \frac{3\sin\mu_k}{\mu_k^2}\left(1 - \frac{2}{\mu_k^2}\right)\right],$$

$$\beta_{kn} = \frac{\beta_k\sin\mu_n\cos\mu_k - \mu_n\sin\mu_k\cos\mu_n}{\mu_n^2 - \mu_k^2},$$

$$N_i = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^i}, \quad i = 3, 5, 7.$$

After the determination of C_k from the system (2.11), and taking into account the relations between A, B, C, A_0, B_0 and C_0 according to (2.3) and (2.10), we obtain the following equations for the stresses and displacements in the cylinder:

$$u_\theta^{(1)} = R\varepsilon\left\{\varrho\left[1 + \frac{C_0(\zeta - l)}{l}\right] - 2\sum_{k=1}^{\infty}\frac{C_kJ_1(\mu_k\varrho)}{\operatorname{sh}\mu_k l}\operatorname{sh}\mu_k(l-\zeta)\right\},$$

$$\tau_{\theta z}^{(1)} = G_1\frac{\varepsilon}{l}C_0\varrho + 2\varepsilon G_1\sum_{k=1}^{\infty}\frac{\mu_kC_k}{\operatorname{sh}\mu_k l}J_1(\mu_k\varrho)\operatorname{ch}\mu_k(l-\zeta), \tag{2.14}$$

$$\tau_{\theta r}^{(1)} = 2\varepsilon G_1\sum_{k=1}^{\infty}\mu_kJ_2(\mu_k\varrho)\frac{C_k}{\operatorname{sh}\mu_k l}\operatorname{sh}\mu_k(l-\zeta).$$

The stresses and displacements in the contact region can be determined from (2.14) for $\zeta = 0$.

Note that according to (1.10), for $\zeta = 0$, another form of the solution for the stresses and displacements in the contact region can be found. Since the expressions for $F(\eta)$, as well as for $\tau_{\theta z}^{(2)}$, $u_\theta^{(2)}$, $\tau_{\theta r}^{(2)}$ are cumbersome, the solution is not given here.

In order to obtain the relation between the rotation angle of the flat end of the cylinder and the torque, we make use of the equilibrium condition

$$M = 2\pi R^3 \int_0^1 \varrho^2 \tau_{\theta z}^{(1)} d\varrho .$$

After computation we obtain

$$M = \frac{\pi}{2} \frac{R^3 G_1 C_0}{l} \varepsilon .$$

3. Solution for particular case of anisotropic materials

The method discussed in the preceding sections can be applied to solve the problems when the cylinder and the layer materials are transversally anisotropic or cylindrically orthotropic. If the geometrical axis of the cylinder coincides with the anisotropy axis of the cylinder and layer, we find, exactly as in the case of isotropy [4, 5], that the only non-vanishing components of the stress tensor $\tau_{\theta z}^{(i)}$, $\tau_{\theta r}^{(i)}$ and the component of the displacement vector $u_\theta^{(i)}$ $(i = 1, 2)$ are related by the formulae

$$\tau_{\theta z}^{(i)} = A_{44}^{(i)} \frac{\partial u_\theta^{(i)}}{\partial z}, \quad \tau_{\theta r}^{(i)} = A_{66}^{(i)} \left(\frac{\partial u_\theta^{(i)}}{\partial r} - \frac{u_\theta^{(i)}}{r} \right), \tag{3.1}$$

where $A_{44}^{(i)}$, $A_{66}^{(i)}$ are the moduli of elasticity.

Making use of the equilibrium equations, we find that the displacements $u_\theta^{(i)}$, each in its region, satisfy the equation

$$\frac{A_{44}^{(i)}}{A_{66}^{(i)}} \frac{\partial^2 u_\theta^{(i)}}{\partial z^2} + \frac{\partial^2 u_\theta^{(i)}}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta^{(i)}}{\partial r} - \frac{u_\theta^{(i)}}{r^2} = 0, \quad i = 1, 2. \tag{3.2}$$

Substituting

$$z = \sqrt{\frac{A_{44}^{(i)}}{A_{66}^{(i)}}} z_i$$

we transform the Eq. (3.2) to the form

$$\frac{\partial^2 u_\theta^{(i)}}{\partial z_i^2} + \frac{\partial^2 u_\theta^{(i)}}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta^{(i)}}{\partial r} - \frac{u_\theta^{(i)}}{r^2} = 0. \tag{3.3}$$

The boundary conditions of the problem remain unchanged, i.e., (1.1)–(1.5) — only the quantities Z, L and H have to be replaced by z_1 or z_2, L_1 and H_2, respectively, where

$$L_1 = \sqrt{\frac{A_{66}^{(1)}}{A_{44}^{(1)}}} L, \quad H_2 = \sqrt{\frac{A_{66}^{(2)}}{A_{44}^{(2)}}} H.$$

Thus it is evident that in the particular cases of anisotropy the torsion problem can be reduced to the isotropic case. We obtain the following formulae for the stresses and displacement in the cylinder and the layer:

$$u_\theta^{(1)} = R\varepsilon \left\{ \varrho \left[1 + \frac{C_0'(\zeta_1 - l_1)}{l_1} \right] - 2 \sum_{k=1}^{\infty} \frac{C_k' J_1(\mu_k \varrho)}{\operatorname{sh} \mu_k l_1} \operatorname{sh} \mu_k (l_1 - \zeta_1) \right\}, \tag{3.4}$$

(3.4) [cont.]
$$\tau_{\theta z}^{(1)} = \frac{\varepsilon C_0'}{l_1}\sqrt{A_{44}^{(1)}A_{66}^{(1)}}\,\varrho + 2\varepsilon\sqrt{A_{44}^{(1)}A_{66}^{(1)}}\sum_{k=1}^{\infty} C_k'\mu_k J_1(\mu_k\varrho)\frac{\operatorname{ch}\mu_k(l_1-\zeta_1)}{\operatorname{sh}\mu_k l_1},$$

$$\tau_{\theta r}^{(1)} = 2\varepsilon A_{66}^{(1)}\sum_{k=1}^{\infty} C_k'\mu_k J_2(\mu_k\varrho)\frac{\operatorname{sh}(l_1-\zeta_1)}{\operatorname{sh}\mu_k l_1},$$

$$u_\theta^{(2)} = R\mathscr{H}_1\left[\eta^{-1}F(\eta)\frac{\operatorname{sh}\eta(\zeta_2+h_2)}{\operatorname{ch}\eta h_2},\quad \eta\to\varrho\right],$$

(3.5)
$$\tau_{\theta z}^{(2)} = \sqrt{A_{44}^{(2)}A_{66}^{(2)}}\,\mathscr{H}_1\left[F(\eta)\frac{\operatorname{ch}\eta(\zeta_2+h_2)}{\operatorname{ch}\eta h_2},\quad \eta\to\varrho\right],$$

$$\tau_{\theta r} = -A_{66}\mathscr{H}_2\left[F(\eta)\frac{\operatorname{sh}\eta(\zeta_2+h_2)}{\operatorname{ch}\eta h_2},\quad \eta\to\varrho]\right],$$

where

$$\zeta_i = z_i/R,\quad l_1 = L_1/R,\quad h_2 = H_2/R.$$

The constants C_k' can be determined from the system of equations (2.11), provided that we replace l and $\delta_1 l_1$ and $\delta_1 = A_{44}^{(1)}/A_{44}^{(2)}$ in the expressions for the coefficients (2.12).

4. Numerical results and the analysis of the solution

It has been shown that the solution to the problem can be reduced to the determination of the constants C_k' from the infinite system of linear algebraic Eq. (2.11). Since

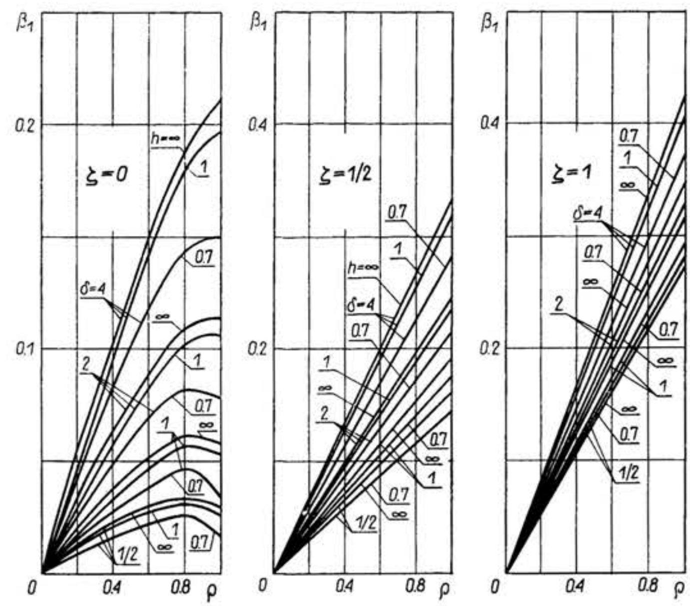

FIG. 1.

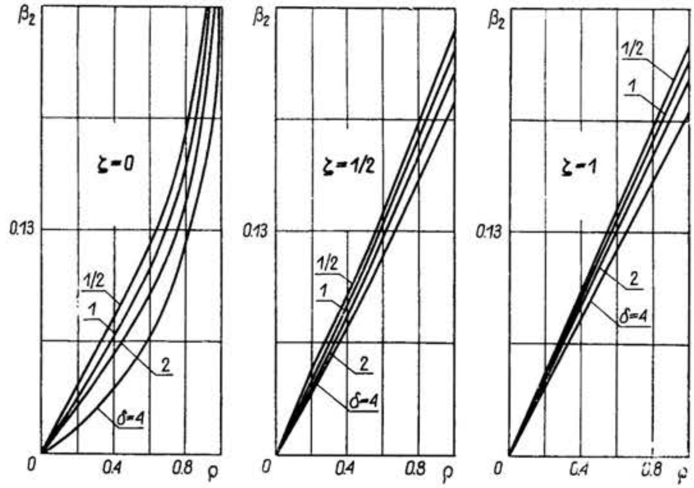

FIG. 2.

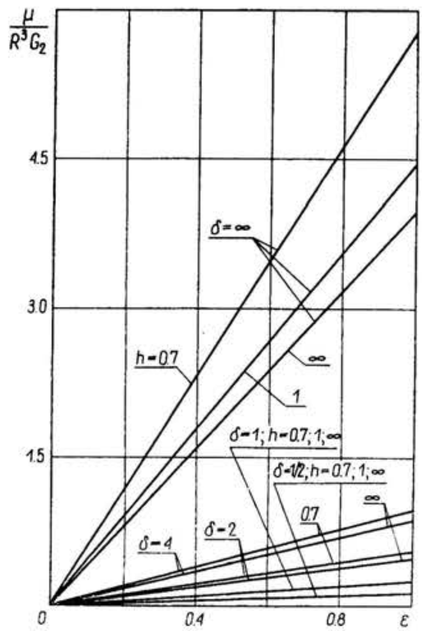

FIG. 3.

the final expressions are cumbersome, we do not dwell on the investigation of the system. However, we should point out that the coefficients of the given system depend on the dimensionless parameters: the height of the cylinder l, the width of the layer h, and the ratio of shear moduli δ. Independently of the parameters, the system has a symmetric matrix the diagonal elements of which are much greater than others. The non-diagonal terms are alternating and tend to zero for n and k tending to infinity. This enables us to determine the constants C_k^l, in the convergence region, by the iteration method.

16 constants for $l = 4$, $\delta = 1/2, 1, 2, 4$ and $h = 0.7$, 1 have been determined. The results of the computations are shown in Figs. 1–3. In Fig. 1 the quantity $\beta_1 = u_\theta/R\varepsilon$ is presented for the cross-sections of the cylinder $\zeta = 0$, 1/2, 1 for $\delta = 4, 2, 1, 1/2$ and for $h = 0.7$, 1 and ∞.

The distribution of the quantity $\beta_2 = \tau_{0z}/\varepsilon G_1$ for $h = 0.7$ is shown in Fig. 2 for the same values of δ and ζ.

Figure 3 illustrates the relation between the torque M and the rotation angle ε for $\delta = 0, 1/2, 1, 2, 4$ and $h = 0.7, 1, \infty$.

On the basis of the numerical calculations and the analysis of the solution, we arrive at the following conclusions:

1. The width of the layer, such that the process of the successive approximations is convergent, does affect the character of the distribution of stresses and displacements, i.e., it is exactly the same as in the case of a semi-space [1].

2. The width of the layer affects the value of the stresses and displacements. This influence depends on the magnitude δ. It increases with the increase of δ and decreases with the decrease of it. For $\delta < 1/2$ it is negligible, i.e., the solution $h \to \infty$ is sufficiently accurate, in practice, for the determination of the stresses and displacements in the contact region as well as in the cylinder.

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