Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 2, pp. 105–118, Warszawa 1973

Initial strains in multiphase media

CZ. EIMER (WARSZAWA)

IN THIS PAPER, the problem is solved of determination of the strains of a multiphase medium when the initial stresses or strains (distortions) of the individual phases are known. Distortions which are homogeneous within the grains are considered, and they may follow from various sources (such as thermal expansion). The phase strain — i.e. the mean real strain for a given phase following from the interaction of distortions and elastic strains is determined, and also the macro-strain — that is, the mean strain for the entire medium together with the macro-effect of distortions (e.g. the thermal expansion coefficient of the material). The internal geometry of the medium is of random character and may be described by the method of correlations. The solution in the elastic range is represented in an integral form by means of the Green tensor and the polarization tensor method.

W pracy rozwiązano problem wyznaczenia odkształceń ośrodka wielofazowego, gdy znane są naprężenia lub odkształcenia początkowe czyli dystorsje poszczególnych faz. Rozważa się dystorsje jednorodne w obrębie ziaren jakiegokolwiek rodzaju, np. wynikające z rozszerzalności cieplnej, i określa się odkształcenie fazowe, tzn. średnie odkształcenie rzeczywiste dla danej fazy, wynikające ze współdziałania dystorsji i odkształceń sprężystych, oraz makro-odkształcenie, tzn. odkształcenie średnie dla całego ośrodka, a tym samym makro-efekt dystorsji, np. współczynnik rozszerzalności cieplnej materiału. Geometria wewnętrzna ośrodka jest przypadkowa i opisuje się ją metodami korelacyjnymi. Rozwiązanie w zakresie sprężystym przedstawia się w postaci całkowej za pomocą tensora Greena i korzysta się z metody tensora polaryzacji.

В работе решена задача об определении деформаций в многофазной среде в случае, когда известны начальные напряжения или деформации, то есть дисторсии отдельных фаз. Рассматриваются дисторсии произвольного типа однородные внутри отдельных зерен, например возникающие в результате теплового расширения. Определяются фазовые деформации, то есть средние действительные деформации для данной фазы, следующие из взаимодействия дисторсий и упругих деформаций, а также макро-деформации, то есть средние деформации для всей среды. Тем самым изучается макро-эффект дисторсий, например коэффициента теплового расширения. Внутренняя геометрия среды является случайной и описывается корреляционными методами. Решение в упругой области представлено в интегральном виде с помощью тензора Грина, причем использован метод тензора поляризации.

1

INITIAL strains and stresses are usually understood as strains and stresses which appear independently of the external loads. They are mainly defined according to their cause by calling them thermal, shrinkage, casting, etc. strains or stresses, though the terminology usually refers to the stresses. If they result from certain loads which have been removed, we term them residual stresses. From the point of view of the scale of the phenomenon, certain authors distinguish between the stresses of first, second and third orders (or even a greater number of orders). They correspond to the stress field nonhomogeneity in the macroscopic scale or in the scales of grains, particles or lattice constants. The first nonhomogeneity is of a deterministic character and often follows from the material non-

homogeneity; the second is stochastic and follows from the grain structure (multiphase media), the third one results from the atom-scale phenomena.

In the present paper, we shall deal with initial strains of the second kind, within the framework of the linear theory of elastic multiphase media. The medium is assumed to consist of a finite number of phases (regions filled by one type of material), each phase consisting of a countable number of grains (regions filled by a homogeneous material, including its spatial orientation). The internal geometry is of random type (distribution and density, mutual position and form of grains), and is described by probabilistic methods; its changes occurring during the physical process are negligible — a more detailed description of the theory is given in [1]; thus the assumption of infinitesimal strains holds true. The stress and strain fields are assumed to result from initial strains (distortions) which are homogeneous — i.e., strains which would be homogeneous within a single given grain after removing the constraints (the continuity condition on passing the grain boundary). In various grains belonging to the same phase, the initial strains may differ only by their orientation — i.e., the corresponding strain tensors are fixed (constant) with accuracy up to an orthogonal transformation (in the sense of the full group of orthogonal transformations, including the mirror reflection). For different phases they are generally different, possibly also equal to zero. The source of such initial strains may be, for example, the temperature, and we shall confine ourselves to a stationary and homogeneous temperature field. Owing to the generally different elastic and distortional properties of the phases, they produce initial strains and stresses.

The aim of the present paper is to determine the macroscopic effect of distortions — that is, the macrodeformations of the medium and of the individual phases. In such a manner, we can calculate such magnitudes as the macroscopic coefficient of thermal expansion, the drying shrinkage coefficient, the electro- and magnetostriction effect, etc. The problem has been inadequately treated in the literature so far, and — in addition — only in the context of the thermal expansion coefficient, where certain approximate formulae were obtained. They were derived, in particular, by W. M. LEVIN [2] for an isotropic two-phase medium in which the relations are considerably simplified. B. W. ROSEN and Z. HASHIN [3] generalized the problem to a two-phase anisotropic medium and to multiphase media. In the both cases, the variational Hashin-Hill approach is utilized, making it possible to bound the macro-value sought for in a certain interval, provided the concentrations of individual phases are known. The bounding formulae contain the macro-moduli of elasticity (the bulk modulus in particular), and for a two-phase medium the inequalities are transformed into equalities. Thus the problem is reduced — under known elastic and thermal properties of the phases — to the determination of the macrotensor of elasticity which, by means of such a variational approach, may be bound in a definite interval. A discussion of thermal stresses in the context of rheological phenomena in fiber-reinforced media was presented by A. R. T. Da SILVA and G. A. CHADWICK [4]. All those papers try to "circumvene" in a certain sense the effect of internal geometry of the medium, giving up the determination of an exact solution. In the present paper, in contrast to those mentioned above, we shall concetrate our attention upon the problem and derive an accurate solution (i.e. with a prescribed accuracy), while the physical background of the phenomenon will play a secondary role.

2

In the present section, we shall consider the relations between the macro-quantities and the phase quantities, and define the quantities sought for, beginning from the description of a distortion. The problem may be formulated in two ways, reflecting the dualism of the methods of forces and deformations.

Let us separate a grain from the medium and consider it under the conditions of the second boundary value problem, the stress vector at the boundary being equal to zero. Subjecting the grain to a homogegeous initial strain (distortion), we obtain the respective strains and stresses:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^d, \quad \boldsymbol{\sigma} = 0. \tag{2.1}$$

Let us consider, on the other hand, the same problem under the conditions of the first boundary value problem with vanishing displacements at the boundary; then we have:

$$\boldsymbol{\epsilon} = 0, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^d. \tag{2.2}$$

Here $\boldsymbol{\epsilon}^d$ and $\boldsymbol{\sigma}^d$ denote the initial strain and stress tensors, respectively — i.e. the distortions and selfstresses. The tensor quantities are written here and henceforth in absolute notation (unless otherwise explicitly indicated), the indices at the symbols denote the numbers of phases, which enables us to avoid double index notation.

In the first case [Eqs. (2.1)], in order to eliminate the results of distortions, a homogeneous stress field has to be applied

$$\boldsymbol{\sigma} = -C\boldsymbol{\epsilon}^d = \boldsymbol{\sigma}^d, \tag{2.3}$$

where C is the elasticity tensor of the medium; in the second case [Eqs. (2.2)], a homogeneous strain field must be applied

$$\boldsymbol{\epsilon} = -S\boldsymbol{\sigma}^d = \boldsymbol{\epsilon}^d, \tag{2.4}$$

where S is the deformability tensor. The senses of the vectors $\boldsymbol{\sigma}^d$ and $\boldsymbol{\epsilon}^d$ are clearly opposite; strictly speaking, the scalar products $\boldsymbol{\sigma}^d \cdot \boldsymbol{\epsilon}^d$ are negative owing to the positive definiteness of the matrices C, S. Tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ will be considered as 6-dimensional vectors, and the fourth rank tensors C and S — as symmetric matrices containing 6×6 terms, while $CS = I$, I being the unit matrix.

By $\boldsymbol{\epsilon}_i^d$ we denote the *distortion tensor* for the phase i having a fixed reference orientation in a given (arbitrary) phase, usually such that the principal axes of the tensor have the same senses and directions as the coordinate axes of the Cartesian coordinate frame (uniform for the entire medium). By $\bar{\boldsymbol{\epsilon}}_i^d$ is denoted the *phase distortion tensor*; this is the strain averaged over all orientations of grains belonging to the given phase and separated from the phase without changing their positions and orientations, and remaining in the conditions indicated by the Eqs. (2.1). The symbol $\bar{\boldsymbol{\epsilon}}_i$ represents the *phase strain tensor* — that is, the strain occurring in the real medium (with internal constraints), averaged over the given phase. $\bar{\boldsymbol{\epsilon}}$ denotes the macrostrain — i.e., the strain of the real medium averaged over all the phases; if it is exclusively produced by distortions, it will be called the initial strain. In a similar manner are defined the quantities $\boldsymbol{\sigma}_i^d$, $\bar{\boldsymbol{\sigma}}_i^d$, $\bar{\boldsymbol{\sigma}}_i$, $\bar{\boldsymbol{\sigma}}$. A dash over the symbol denotes everywhere the result of the averaging operation, and if the symbol does not possess the phase

index, the averaging includes all the phases — e.g., $\bar{\boldsymbol{\epsilon}} = \langle \boldsymbol{\epsilon} \rangle$, while in the opposite case, averaging is limited to the phase indicated — e.g., $\bar{\boldsymbol{\epsilon}}_i = \langle \boldsymbol{\epsilon} \rangle_i$. It should be remembered that the symbol $\langle\ \rangle$ denotes averaging at a fixed point over all elements of the population, by contrast with the symbol $\langle\ \rangle_i$, which denotes averaging over those elements whose points are located within the phase i. If the random fields considered are stochastically homogeneous and ergodic, then the process of averaging over the population may be replaced by averaging over the volume, either total or concerning a single phase i.

The relations between the macro-magnitudes and the phase magnitudes are as follows:

$$\bar{\boldsymbol{\epsilon}} = \sum_i \nu_i \bar{\boldsymbol{\epsilon}}_i, \tag{2.5}$$

$$\bar{\boldsymbol{\sigma}} = \sum_i \nu_i \bar{\boldsymbol{\sigma}}_i, \tag{2.6}$$

where ν_i denotes the concentration of the phase i — that is, the ratio of its volume to the entire volume (mixture rule). Remaining within the linear domain, we may postulate the following relations (which will result later from accurate considerations):

$$\bar{\boldsymbol{\epsilon}}_i = \sum_k \mathbf{a}_{ik} \boldsymbol{\sigma}_k^d = \sum_k \boldsymbol{\alpha}_{ik} \boldsymbol{\epsilon}_k^d = \sum_k \mathbf{a}'_{ik} \bar{\boldsymbol{\sigma}}_k^d = \sum_k \boldsymbol{\alpha}'_{ik} \bar{\boldsymbol{\epsilon}}_k^d, \tag{2.7}$$

$$\bar{\boldsymbol{\sigma}}_i = \sum_k \mathbf{b}_{ik} \boldsymbol{\epsilon}_k^d = \sum_k \boldsymbol{\beta}_{ik} \boldsymbol{\sigma}_k^d = \sum_k \mathbf{b}'_{ik} \bar{\boldsymbol{\epsilon}}_k^d = \sum \boldsymbol{\beta}'_{ik} \bar{\boldsymbol{\sigma}}_k^d. \tag{2.8}$$

Here $\mathbf{a}_{ik}$, ... are certain constant tensor coefficients of the fourth rank (in absolute notation); the indices denote the phase numbers. Our considerations are aimed at determination of the coefficients since they combine the known initial values in the reference system with the phase quantities. It should be observed that in a body of finite dimensions they depend, in general, on the position of the point and on the type of (macro-) boundary conditions; thus such a relation also holds true in the case of phase strains and stresses.

Knowing the phase magnitudes, we may easily determine the global quantities by inserting the Eqs. (2.7), (2.8) into (2.5), (2.6); for instance,

$$\bar{\boldsymbol{\epsilon}} = \sum_{i,k} \nu_i \boldsymbol{\alpha}_{ik} \boldsymbol{\epsilon}_k^d = \sum_k \mathbf{A}_k \boldsymbol{\epsilon}_k^d, \tag{2.9}$$

where

$$\mathbf{A}_k = \sum_i \nu_i \boldsymbol{\alpha}_{ik}$$

and in a similar manner for the remaining relations. In the case of distortions depending on a single parameter, as in the typical case of thermal deformations, we obtain

$$\boldsymbol{\epsilon}_k^d = \boldsymbol{\vartheta}_k t, \tag{2.10}$$

$\boldsymbol{\vartheta}_k$ being the tensor coefficient of thermal expansion for the material of the phase k, and t denoting the temperature. In such a manner, the expansion coefficient of the medium is expressed by the formula

$$\boldsymbol{\vartheta} = \sum_k \mathbf{A}_k \boldsymbol{\vartheta}_k. \tag{2.11}$$

It is seen that it depends on the elastic constants of the medium contained in $\mathbf{A}_k$.

All relations are considerably simplified in the case in which the medium is isotropic. It should be borne in mind, however, that in the theory of multiphase media various notions of isotropy occur (cf. [1]); in particular, micro-isotropy — i.e., the physical isotropy (the elastic and distortional properties of the grains) — should be distinguished from the geometric isotropy or macro-isotropy following from the form and distribution of the grains. The directional characteristics of the both types may be correlated and the problem becomes, in general, rather complicated. In the case of full isotropy (both physical and geometric), the phase deformations become voluminal, the phase stresses — uniform, while the coefficients in the Eqs. (2.7), (2.8), (2.15), etc. — are transformed to scalar magnitudes.

3

Let us now pass to determination of the global and phase macro-strains defined in the preceding section. The first step will be the derivation of a deterministic solution in a form suitable for averaging.

The multiphase medium is characterized by elastic and distortional nonhomogeneity. The derivations are considerably simplified if the medium is replaced by a homogeneous medium loaded by certain tensor fields of stresses $\boldsymbol{\tau}$ or strains $\boldsymbol{\eta}$, defined by means of the following formulae:

$$\boldsymbol{\sigma} = C(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^d) = C_0\boldsymbol{\epsilon}+\boldsymbol{\tau}, \tag{3.1}$$

$$\boldsymbol{\epsilon} = S(\boldsymbol{\sigma}-\boldsymbol{\sigma}^d) = S_0\boldsymbol{\sigma}-\boldsymbol{\eta}, \tag{3.2}$$

where

$$\boldsymbol{\tau} = c\boldsymbol{\epsilon}+\boldsymbol{\sigma}^d, \quad c = C-C_0; \tag{3.3}$$

$$\boldsymbol{\eta} = s\boldsymbol{\sigma}-\boldsymbol{\epsilon}^d, \quad s = S_0-S. \tag{3.4}$$

Here C_0, S_0 denote the respective tensors of elasticity and deformability for the reference medium which may be assumed to be an arbitrary (though subject to certain limitations to be mentioned later) elastic and homogeneous medium; for instance, it might be the medium with the properties of one of the isotropic phases, or with the mean elasticity tensor. The terms $c\boldsymbol{\epsilon}$ and $s\boldsymbol{\sigma}$ are called the tensors of stress and strains of *polarization*, respectively. Together with the fields of $\boldsymbol{\sigma}^d$, $\boldsymbol{\epsilon}^d$ they constitute the load of the reference medium. In this manner, an analogy is established with the electric polarization vector induced by the electric field. Introducing the Eqs. (3.3), (3.4) into (3.1), (3.2), we may easily verify the formulae. Inserting (3.1) into (3.2) and conversely, we arrive at the formulae:

$$\boldsymbol{\tau} = C_0\boldsymbol{\eta}, \quad \boldsymbol{\eta} = S_0\boldsymbol{\tau}. \tag{3.5}$$

The signs in the Eqs. (3.1)–(3.4) were so selected that the relations (3.5) become symmetric, and that — under a proper choice of the reference medium — the matrices c and s are simultaneously positive definite. Let us observe that once a certain isotropic phase is selected as the reference medium, then for that phase we have $c = s = 0$.

The Eqs. (3.1), (3.2) furnish the stress-strain relations (corresponding to Hooke's law), the prescribed values being in Eqs. (3.1) the strains, and in the Eqs. (3.2) — the stresses.

In particular, we obtain from the Eqs. (3.1) $\boldsymbol{\sigma} = 0$ when $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^d$, and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^d$ when $\boldsymbol{\epsilon} = 0$, while from the Eqs. (3.2) we have $\boldsymbol{\epsilon} = 0$ when $\boldsymbol{\sigma} = \boldsymbol{\sigma}^d$, and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^d$ when $\boldsymbol{\sigma} = 0$; these relations may be interpreted as conditions of equality of the corresponding fields.

Consider now the equations of equilibrium

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{P} = 0,$$

where $\mathbf{P}$ is the intensity vector of the deterministic body force per unit volume. Strictly speaking, $\mathbf{P}$ also contains the fluctuations of the specific weight which, in general, varies from phase to phase; that influence is, at least in static problems, negligible. Thus P will contain the possible body forces in the averaged sense — i.e., calculated as for a homogeneous medium and other external loadings. Substituting the Eq. (3.1), we obtain:

$$\operatorname{div}(C_0 \boldsymbol{\epsilon}) + \operatorname{div} \boldsymbol{\tau} + \mathbf{P} = 0.$$

The solution is decomposed into two parts $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0 + \boldsymbol{\epsilon}'$, the first of which satisfies the equation

$$\operatorname{div}(C_0 \boldsymbol{\epsilon}_0) + \mathbf{P} = 0, \tag{3.6}$$

and the second — the equation

$$\operatorname{div}(C_0 \boldsymbol{\epsilon}') + \mathbf{K} = 0, \quad \mathbf{K} = \operatorname{div} \boldsymbol{\tau}. \tag{3.7}$$

Expressing the strains in terms of displacements and assuming for the reference medium C_0 an isotropic medium, we obtain, in particular, the Lamé equations in the classical form, the loading term in the second equation having the form $\mathbf{K} = \operatorname{div} \boldsymbol{\tau}$. The solution is found to consist of the solution $\boldsymbol{\epsilon}_0$ for the (homogeneous) reference medium loaded by given deterministic forces, and of the solution $\boldsymbol{\epsilon}'$ for the reference medium loaded by the tensor field $\boldsymbol{\tau}$. In the subsequent part of the paper the solution $\boldsymbol{\epsilon}_0$ will be considered as known, the considerations being confined to the Eq. (3.7).

The solution of the Eqs. (3.7) for an isotropic medium has the form (written in coordinates):

$$u'_p(\mathbf{x}) = \int G_{p\varsigma}(\mathbf{x}, \boldsymbol{\xi}) K_q \, dV,$$

$$G_{pq} = \frac{\lambda_0 + \mu_0}{8\pi\mu_0 (\lambda_0 + 2\mu_0)} \left[\frac{(x_p - \xi_p)(x_q - \xi_q)}{r^3} + \frac{\lambda_0 + 3\mu_0}{\lambda_0 + \mu_0} \frac{\delta_{pq}}{r,} \right], \tag{3.8}$$

$$r = \sqrt{\sum (x_p - \xi_p)^2},$$

G denoting the Green tensor of the Lamé problem for the reference medium which was assumed to be isotropic (Kelvin's solution), $\mathbf{x}$ is the given point, $\boldsymbol{\xi}$ is the point of application of the force $\mathbf{K} dV$; λ_0, μ_0 are the Lamé constants of the reference medium. Passing from $\mathbf{K}$ to $\boldsymbol{\tau}$, we assume the existence of div $\boldsymbol{\tau}$, and hence the proper smoothness of $\boldsymbol{\tau}(\mathbf{x})$; we shall return to this problem later. This leads to approximation of the multiphase medium by a medium with smoothly changing nonhomogeneity at the grain boundaries and releases us from discussion of smoothness of the surfaces bounding the grains; the multiphase medium can now be achieved by means of a suitable limit procedure. In a more general approach, the notion of a surface divergence may be applied, the function $\boldsymbol{\tau}(\mathbf{x})$ being assumed to be a regionally smooth function and discontinuous at the grain bound-

aries. The derivations require a certain generalization, though the changes will not be substantial and the notations will remain the same.

For the time being (provided that no assumptions concerning the behaviour of $\boldsymbol{\tau}(\mathbf{x})$ at infinity have been made), let us separate from the infinite medium a finite region D outside of which $\boldsymbol{\tau} \equiv 0$ (i.e. there are no inclusions and distortions). The point $\mathbf{x}$ is surrounded by a small sphere K ($K \subset D$) of area S_K, and the integral (3.8) is decomposed in the following manner:

$$u'_p = \int_K G_{pq} K_q dV + \int_{D-K} G_{pq} K_q dV.$$

In order to pass to the strains, the derivatives of that expression have to be calculated. The first term may be differentiated under the sign of integration, which can readily be demonstrated similarly as in the proof of existence of the first derivative of the voluminal potential, since G_{pq} tends to zero as $1/r$. The integral obtained, $\int G_{pq,s} K_q dV$, tends to zero together with the radius of the sphere. In the second integral, $\mathbf{K}$ is expressed in terms of $\boldsymbol{\tau}$ and differentiation is performed under the sign of integration; the use of the Gauss transformation is justified by the fact that the integration domain does not contain singular points. Disregarding the first integral, we obtain:

$$u'_{p,s} \approx \int_{D-K} G_{pq,s} K_q dV = \int_{D-K} G_{pq,s} \tau_{qr,r} dV = \int_{D-K} (G_{pq,s} \tau_{qr})_{,r} dV$$

$$- \int_{D-K} G_{pq,sr} \tau_{qr} dV = \oint_S G_{pq,s} \tau_{qr} n_r dS - \oint_{S_K} G_{pq,s} \tau_{qr} n_r dS_K + \int_{D-K} G_{pq,sr} \tau_{qr} dV,$$

where S is the surface bounding D. The signs on the right-hand side result from replacing the differentiation of $G(\mathbf{x}-\boldsymbol{\xi})$ with respect to ξ_r into differentiation with respect to x_r, and from the assumption that the outward unit normal $\mathbf{n}$ to S_K is positive. The first right-hand integral vanishes together with τ_{qr}, provided the surface S bounding D lies outside the region containing the inclusions, which may always be assumed. The third integral is calculated in the sense of its principal value when the radius of K approaches zero. The conditions of existence of the integral connected with the properties of the function $\boldsymbol{\tau}(\mathbf{x})$ will be given later. In the second integral, under the assumptions presented, the function $\boldsymbol{\tau}(\boldsymbol{\xi}) \to \boldsymbol{\tau}(\mathbf{x})$ and may be taken outside of the integration sign. Taking this into account, we finally obtain:

$$\varepsilon'_{ps} = u'_{(p,s)} = a_{psqr} \tau_{qr} + \bar{\int}_D \Lambda_{psqr}(\mathbf{x}, \boldsymbol{\xi}) \tau_{qr}(\boldsymbol{\xi}) dV, \tag{3.9}$$

where

$$\Lambda_{psqr} = G_{q(p,s)r},$$

$$a_{psqr} = - \lim_{r_K \to 0} \oint_{S_K} G_{q(p,s)} n_r dS_K.$$

Putting the indices into parentheses denotes symmetrization; the dash over the integral means that the principal value of the integral should be evaluated. It is easily verified that the tensor Λ exhibits symmetry properties similar to those of the elasticity tensor C and, moreover, that its sign is independent of whether G is differentiated (twice) with

respect to either $\mathbf{x}$ or $\boldsymbol{\xi}$. Once the differentiations prescribed are performed in the Eq. (3.6), we obtain for the isotropic reference medium the formula (detailed calculations are presented in [1])

$$\Lambda_{psqr} = \frac{\lambda_0+\mu_0}{8\pi\mu_0(\lambda_0+2\mu_0)r^3}\left\{\frac{\mu_0}{\lambda_0+\mu_0}(\delta_{sq}\delta_{pr}+\delta_{pq}\delta_{sr})-\delta_{qr}\delta_{ps}\right.$$
$$+\frac{3}{r^2}\left[-\frac{\mu_0}{\lambda_0+\mu_0}\delta_{sq}(x_p-\xi_p)(x_r-\xi_r)+\delta_{qr}(x_p-\xi_p)(x_s-\xi_s)+\delta_{ps}(x_q-\xi_q)(x_r-\xi_r)\right.$$
$$\left.+\delta_{pr}(x_s-\xi_s)(x_q-\xi_q)+\delta_{sr}(x_p-\xi_p)(x_q-\xi_q)-\frac{\mu_0}{\lambda_0+\mu_0}\delta_{pq}(x_s-\xi_s)(x_r-\xi_r)\right]$$
$$\left.-\frac{15}{r^4}(x_p-\xi_p)(x_s-\xi_s)(x_q-\xi_q)(x_r-\xi_r)\right\}. \tag{3.10}$$

The tensor $\mathbf{a}$ is calculated by taking into account [after performing the differentiations in the Eqs. (3.8)] that

$$\oint_{S_1} G_{pq,s}n_r dS_1 = \frac{\lambda_0+\mu_0}{8\pi\mu_0(\lambda_0+2\mu_0)}\oint_{S_1}\left[(\delta_{pq}n_s+\delta_{qs}n_p+\delta_{sp}n_q-3n_pn_qn_s)n_r\right.$$
$$\left.-\frac{\lambda_0+2\mu_0}{\lambda_0+\mu_0}\delta_{pq}(\delta_{nn}n_s+\delta_{ns}n_n+\delta_{sn}n_n-3n_nn_nn_s)r_r\right]dS_1,$$

where $n_q = (x_q-\xi_q)/r$ are the corresponding direction cosines, and the integration is performed over the surface $r = 1$.
Taking into account that

$$\int_{S_1} n_sn_r dS_1 = \frac{4}{3}\pi\delta_{sr}, \qquad \int_{S_1} n_pn_qn_sn_r dS_1 = \frac{4}{15}\pi(\delta_{pq}\delta_{sr}+\delta_{ps}\delta_{qr}+\delta_{pr}\delta_{qs}),$$

we obtain, after certain calculations, the result

$$a_{psqr} = \frac{1}{6\mu_0}(\delta_{pq}\delta_{sr}+\delta_{pr}\delta_{sq})-\frac{\lambda_0+\mu_0}{15\mu_0(\lambda_0+2\mu_0)}(\delta_{pq}\delta_{sr}+\delta_{qs}\delta_{pr}+\delta_{sp}\delta_{qr}). \tag{3.11}$$

There remains the problem of conditions to be imposed on the function $\boldsymbol{\tau}(\mathbf{x})$. When using the Gauss transformation we assumed the function $\boldsymbol{\tau}(\mathbf{x})$ to be continuous together with its derivatives, and hence $\boldsymbol{\tau}(\mathbf{x}) \in \mathscr{C}^1$ (using the divergence in its classical sense). This ensures the existence of (3.9) (for a finite region), since there exists the integral $\int_D G_{pq,s}K_q dV$ (used in the derivation) in which the expression $G_{pq,s}$ increases as $1/r^2$ for $\boldsymbol{\xi} \to \mathbf{x}$. Observe that in deriving the generalized divergence, it suffices to require the function $\boldsymbol{\tau}(\mathbf{x})$ to satisfy the Lipschitz condition for the exponent $0 < \alpha \leqslant 1$ (cf. [1]). In passing to the unbounded region we have to assume, moreover, that $\boldsymbol{\tau}(\mathbf{x})$ is bounded and that the integral (3.9) exists in the improper sense. Passing to polar coordinates and taking into account the Eqs. (3.10), we can express the integral in the form

$$\int \frac{1}{r^3}f_{psqr}(\mathbf{n})\tau_{qr}(\mathbf{n},r)r^2 d\omega\, dr = \int_0^\infty \frac{1}{r}\left\{\int f_{psqr}(\mathbf{n})\tau_{qr}(\mathbf{n},r)d\omega\right\}dr,$$

where $\mathbf{r} = \mathbf{x}-\boldsymbol{\xi}$, ω is the solid angle, $\mathbf{n}$ — the direction vector (unit with direction consines n_q), $f(\mathbf{n})$ — a certain function of the direction. It is seen that $\boldsymbol{\tau}(\mathbf{n}, r)$ must be such as to satisfy the condition

$$|F_{ps}(r)| = \left| \int f_{psqr}(\mathbf{n})\tau_{qr}(\mathbf{n}, r)d\omega \right| \leqslant \frac{B}{r^{\alpha}}, \quad \alpha > 0, \tag{3.12}$$

where B is a positive number independent of r. The meaning of this condition is particularly clear on the background of probabilistic considerations. Let us observe that the condition (3.12) should hold true (for fixed α and B) for each $\mathbf{x}$, independently of the choice of the origin of the coordinate system. This translative invariance characterizes the stochastically homogeneous medium.

The basic relation (3.9) is conveniently written in the operator form

$$\boldsymbol{\epsilon}' = \boldsymbol{\Lambda}\boldsymbol{\tau}, \tag{3.13}$$

where $\boldsymbol{\Lambda}$ is the linear tensor operator [acting on $\boldsymbol{\tau}(\boldsymbol{\xi})$],

$$\boldsymbol{\Lambda} = \left(\mathbf{a} \int dV\delta + \overline{\int} dV\Lambda\right). \tag{3.14}$$

Tensors $\boldsymbol{\Lambda}$, $\mathbf{a}$ are given by the formulae (3.10), (3.11), and $\delta = \delta(\mathbf{x}, \boldsymbol{\xi})$ is the Dirac function. Let us observe that, owing to the symmetry properties of tensors $\mathbf{a}$, $\boldsymbol{\Lambda}$, $\boldsymbol{\Lambda}$ may be treated as a matrix, as was explained in Sec. 1.

In a similar manner, the dual formulae may be derived in the case of loading the medium by the strain field $\boldsymbol{\eta}$; this problem will not be dealt with in the present paper.

4

Let us consider the medium in which the only sources of deformations and stresses are the initial strains (distortions) of certain phases. Then $\boldsymbol{\epsilon}_0 = \boldsymbol{\sigma}_0 = 0$ and the problem is reduced to solution of the Eq. (3.7). Using the Eqs. (3.3), (3.4) and (3.13), we obtain

$$\boldsymbol{\epsilon} = \boldsymbol{\Lambda}\boldsymbol{\sigma}^d + (\boldsymbol{\Lambda}c)\boldsymbol{\epsilon}. \tag{4.1}$$

The field $c(\mathbf{x})$ is known from our assumptions (in the deterministic approach), and hence it is contained in the operator parentheses. The above relation represents an integral equation for the unknown function $\boldsymbol{\epsilon}(\mathbf{x})$.

The solution becomes almost trivial in the case of elastically homogeneous (but distortionally nonhomogeneous) media. Then $c = 0$ and we obtain

$$\boldsymbol{\epsilon} = \boldsymbol{\Lambda}\boldsymbol{\sigma}^d. \tag{4.2}$$

The appearance of the second term of the Eq. (4.1) is then due to the elastic nonhomogeneity of the medium. Simultaneously we see that, for example, the macroscopic coefficient of thermal expansion will depend on the elastic properties of the phases.

We shall try to obtain the solution of the Eq. (4.1) in such a form as will yield, after averaging, the explicit correlation functions. To that end, they are expanded into the Neumann type series

$$\boldsymbol{\epsilon} = [\mathbf{I} + (\boldsymbol{\Lambda}c) + (\boldsymbol{\Lambda}c)^2 + (\boldsymbol{\Lambda}c)^3 + \ldots]\boldsymbol{\Lambda}\boldsymbol{\sigma}^d, \tag{4.3}$$

where I is the identity operator. This expansion is obtained in an iterational process, by applying the operator Λc consecutively to both sides of the Eq. (4.1) and adding the corresponding results. Thus we obtain

$$\epsilon = [\mathrm{I}+(\Lambda c)+(\Lambda c)^2+ \dots +(\Lambda c)^{(n-1)}]\Lambda\sigma^d+(\Lambda c)^{(n)}\epsilon. \tag{4.4}$$

The last term on the right-hand side denotes the remainder of the series, and the entire expression represents the iterated integral equation for ϵ.

The consecutive terms of the expansion have the form:

$$\Lambda\sigma^d = \Lambda(\mathbf{x}_0, \mathbf{x}_1)\sigma^d(\mathbf{x}_1) = \Lambda_{01}\sigma_1^d,$$
$$\Lambda c\Lambda\sigma^d = \Lambda(\mathbf{x}_0, \mathbf{x}_1)c(\mathbf{x}_1)\Lambda(\mathbf{x}_1, \mathbf{x}_2)\sigma^d(\mathbf{x}_2) = \Lambda_{01}c_1\Lambda_{12}\sigma_2^d = \Lambda_{012}c_1\sigma_2^d,$$
$$(\Lambda c)^{(2)}\Lambda\sigma^d = \Lambda_{01}c_1\Lambda_{12}c_2\Lambda_{23}\sigma_3^d = \Lambda_{0123}c_1c_2\sigma_3^d,$$

etc. The notation is clear and does not require any explanation; the indices refer to the points which are the arguments of the function. Let us observe that, when the tensors are written in coordinates, no contractions (in tensor indices) appear in the quantities denoted by a common symbol; we have, for instance,

$$(\Lambda_{01})_{ghpq}(c_1)_{pqrs}(\Lambda_{12})_{rsuw}(\sigma_2^d)_{uw} = (\Lambda_{012})_{ghpqrsuw}(c_1)_{pqrs}(\sigma_2^d)_{uw},$$

etc., which simplifies the notation. The expansion (4.4) written in the same formalism has the form:

$$\epsilon = \Lambda_{01}\sigma_1^d+\Lambda_{012}c_1\sigma_2^d+ \dots +\Lambda_{01\dots n}c_1 \dots c_{n-1}\sigma_n^d+\Lambda_{01n\dots n}c_1\dots c_n\epsilon_n, \tag{4.5}$$

the last right-hand term being the remainder of the series.

Now, the passage from deterministic to random fields does not present any difficulties. Taking into account the random character of c, ϵ, σ^d, preserving the deterministic operators Λ and averaging the Eq. (4.5) on both sides (which can be done owing to finite sums), we obtain

$$\langle\epsilon\rangle = \Lambda_{01}\langle\sigma_1^d\rangle+\Lambda_{012}\langle c_1\sigma_2^d\rangle+ \dots (+\Lambda_{01\dots n}\langle c_1 \dots c_n\epsilon_n\rangle). \tag{4.6}$$

The quantities in averaging brackets are, in general, correlated and cannot be separated. They represent the correlation functions (non-centred), and we shall return to the problem in the sequel.

In an unbounded, stochastically homogeneous medium, loaded by fields of distortion also stochastically homogeneous, both the operators Λ and all the correlation functions are independent of position (translatory invariance), and the expression (4.6) may be reduced to a simpler form. We have namely:

$$\Lambda_{012\dots} = \Lambda(\mathbf{x}_1-\mathbf{x}_0, \mathbf{x}_2-\mathbf{x}_1, \dots) = \Lambda(\mathbf{r}_{01}, \mathbf{r}_{12}, \dots),$$
$$\langle c_1c_2c_3..\rangle = f(\mathbf{r}_{12}, \mathbf{r}_{23}, \dots),$$

where $\mathbf{r}_{ik} = \mathbf{x}_k-\mathbf{x}_i$. Thus we obtain, in accordance with (3.14),

$$\Lambda_{01}\langle\sigma_1^d\rangle = \mathbf{a}\langle\sigma_0^d\rangle+\overline{\int}\Lambda_{01}\langle\sigma_1^d\rangle\,dV_1 = \mathbf{a}\bar{\sigma}^d,$$

and generally,

$$\Lambda_{01\dots n}\langle c_1c_2 \dots \sigma_n^d\rangle = \Lambda_{01}\Lambda_{12\dots n}\langle c_1\dots\sigma_n^d\rangle = (\mathbf{a}\int dV_1\,\delta_{01}+\int dV_1\,\Lambda_{01})\Lambda_{12\dots n}\langle c_1\dots\sigma_n^d\rangle$$
$$= \mathbf{a}\Lambda_{02\dots n}\langle c_0c_2\dots\sigma_n^d\rangle = \text{const},$$

use being made of the fact that $\int \boldsymbol{\Lambda}(\mathbf{r})kdV = 0$, provided k is a tensor constant [which is easily verified by expressing the kernel (3.10) in polar coordinates], of the formulae applied in the derivation of (3.11), and of the fact that $\boldsymbol{\Lambda}_{12\ldots n}\langle c_{1\ldots}\boldsymbol{\sigma}_n^d\rangle = \text{const}$. Changing the indices appropriately, we finally obtain

$$\bar{\boldsymbol{\epsilon}} = \mathbf{a}[\bar{\boldsymbol{\sigma}}^d + \boldsymbol{\Lambda}_{12}\langle c_1\boldsymbol{\sigma}_2^d\rangle + \boldsymbol{\Lambda}_{123}\langle c_1 c_2\boldsymbol{\sigma}_3^d\rangle + \ldots]. \tag{4.7}$$

In the calculations, it should be borne in mind that the first part of the operator (3.14) yields the correlation functions at zero, for instance

$$\mathbf{a}\int \delta_{12}\langle c_1\boldsymbol{\sigma}_2^d\rangle dV_2 = \mathbf{a}\int \delta(\mathbf{r}_{12})f(\mathbf{r}_{12})dV_2 = \mathbf{a}f(0).$$

Passing to the centred functions — that is, assuming $C_0 = \bar{C}$ — we obtain the corresponding central statistic moments. The macro-strain (4.7) constitutes the solution of the problem if the series is convergent, which can be demonstrated. Discussion of the convergence is somewhat difficult owing to various types of probabilistic convergence and various definitions of norms of stochastic operators; that problem will be presented in a separate paper.

If the general expression for the n-point correlation moment is known from the theoretical hypotheses, then it suffices to apply the usual convergence criteria of numerical series. If, however, the correlation functions are determined experimentally, then on the basis of n-th order functions all the lower order functions can be determined, while nothing can be said about the functions of higher orders.

In order to perform the calculations from the series (4.7), the correlation moments must be expressed in a manner enabling us to isolate the influence of internal geometry. The general formula for the n-th order moment yields:

$$\langle c_1 c_2 \ldots \boldsymbol{\sigma}_n^d\rangle = \int\int \ldots \int c_1 c_2 \ldots \boldsymbol{\sigma}_n^d f(c_1, c_2 \ldots \boldsymbol{\sigma}_n^d; \mathbf{x}_1, \mathbf{x}_2 \ldots \mathbf{x}_n)dc_1 dc_2 \ldots d\boldsymbol{\sigma}_n^d, \tag{4.8}$$

where the integration is extended over the entire region of variability of random variables (at fixed points), and f represents the probability density function of the relevant n-dimensional vector (with tensorial components). The function is to be taken in the generalized sense — i.e., it can also describe the probability distributions of a discrete variable (by means of impulse functions).

To avoid possible ambiguities, let us denote the indices pertaining to phases by capital letters and consider first the N-phase medium consisting of isotropic phases. The random variables c, $\boldsymbol{\sigma}^d$ may then assume, at a given point, only one of the N possible values, and hence

$$f(c_1, c_2, \ldots, \boldsymbol{\sigma}^d) = \sum{}' p_{K_1,K_2\cdots,K_n}\delta(c_1 - c_{K_1})\delta(c_2 - c_{K_2})\ldots(\boldsymbol{\sigma}_n^d - \boldsymbol{\sigma}_{K_n}^d), \tag{4.9}$$

where the impulse (Dirac) functions appear and summing is extended over all possible combinations (combinatoric variations with repetitions) of n indices K_i selected from N; $p_{K_1K_2,\ldots,K_n}$ represents the probability that the point $\mathbf{x}_1$ will lie in the phase K_1, point $\mathbf{x}_2$ — in the phase K_2, etc. This probability is experimentally established by parallel shifting of the fixed ("rigid") set of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ within the medium in an arbitrary (random)

manner; this creates the statistics of events consisting in "hitting" the phase K_1 by the point $\mathbf{x}_1$, phase K_2 by the point $\mathbf{x}_2$, etc.(1)

Inserting (4.9) into (4.8), we obtain

$$\langle c_1 c_2 \dots \boldsymbol{\sigma}_n^d \rangle = \sum p_{K_1 K_2 \dots K_n}(\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_n) c_{K_1} c_{K_2} \dots \boldsymbol{\sigma}_{K_n}^d, \tag{4.10}$$

where $c_{K_1} c_{K_2} \dots \boldsymbol{\sigma}_{K_n}^d$ are tensorial constants in the regions of the corresponding phases.

Taking into account (4.10), we may write the n-th term of the expansion (4.7) as

$$\begin{aligned} \boldsymbol{\Lambda}_{12\dots n} \langle c_1 c_2 \dots \boldsymbol{\sigma}_n^d \rangle &= \sum [\boldsymbol{\Lambda}_{12\dots n} p_{K_1 K_2 \dots K_n}(\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_n)] c_{K_1} c_{K_2} \dots \boldsymbol{\sigma}_{K_n}^d \\ &= \sum \lambda^{(n)}_{K_1 K_2 \dots K_n} c_{K_1} c_{K_2} \dots \boldsymbol{\sigma}_{K_n}^d, \end{aligned} \tag{4.11}$$

where

$$\lambda^{(n)}_{K_1 K_2 \dots K_n} = \boldsymbol{\Lambda}_{12\dots n} p_{K_1 K_2 \dots K_n}(\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_n)$$

is a constant tensor coefficient (of order $4n$). It expresses the "secondary" interactions of the n-th order (that is of the n-th point "via" the preceding points) when the points are located in the phases indicated (combined effect). The probabilities p depend exclusively on the internal geometry of the medium (and not on the physical properties of the phases), and thus the coefficients λ depend on the geometry and on the reference medium assumed in the considerations. Their existence depends on the form of the function p — i.e. on the geometry.

Taking into account the Eq. (4.11), we write the solution (4.7) in the form

$$\bar{\boldsymbol{\epsilon}} = \mathbf{a}\left[\bar{\boldsymbol{\sigma}}^d + \sum \lambda^{(2)}_{K_1 K_2} c_{K_1} \boldsymbol{\sigma}_{K_2}^d + \sum \lambda^{(3)}_{K_1 K_2 K_3} c_{K_1} c_{K_2} \boldsymbol{\sigma}_{K_3}^d + \dots\right]. \tag{4.12}$$

In the case of anisotropic phases, the tensors c, $\boldsymbol{\sigma}^d$ may vary in a continuous manner within the phase owing to the variable orientation of the grains. Let us assume these orientations to be correlated neither with the internal geometry of grains nor with the properties of the neighbouring grains. Then the probability of hittings the phases $K_1, \dots, K_n$ with fixed orientations [in the sense of the formula (4.8)] by the respective points $\mathbf{x}_1, \dots, \mathbf{x}_n$ equals

$$p_{K_1 K_2 \dots K_n} f(Q_{K_1}) f(Q_{K_2}) \dots f(Q'_{K_n}) dQ_{K_1} dQ_{K_2} \dots dQ'_{K_n},$$

where Q_{K_i} denotes the orthogonal transformation of order 4 (while Q'_{K_i} — of order 2) grains belonging to the phase K_i. Substitution of that expression in the Eq. (4.8) yields

$$\begin{aligned} \langle c_1 c_2, \dots, \boldsymbol{\sigma}_n^d \rangle &= \sum p_{K_1 K_2, \dots, K_n} \int\int \dots \int Q_{K_1} c^0_{K_1} Q_{K_2} c^0_{K_2} \dots Q'_{K_n} \sigma^{d0}_{K_n} f(Q_{K_1}) \times \\ &\quad \times f(Q_{K_2}) f(Q_{K_3}) \dots f(Q'_{K_n}) dQ_{K_1} dQ_{K_2} \dots dQ'_{K_n} \\ &= \sum p_{K_1 K_2, \dots, K_n} \bar{Q}_{K_1} c^0_{K_1} \bar{Q}_{K_2} c^0_{K_2} \dots \bar{Q}'_{K_n} \boldsymbol{\sigma}^{d0}_{K_n} = \sum p_{K_1 K_2, \dots, K_n} \bar{c}_{K_1} \bar{c}_{K_2} \dots \bar{\boldsymbol{\sigma}}^d_{K_n}, \end{aligned} \tag{4.13}$$

where the index o denotes the corresponding quantities in the reference system, and dashes above the symbols — the phase quantities (averaged), explained earlier in Sec. 2.

(1) Certain assumptions concerning the ergodicity of the random field are needed here, and also the possibility of cross-sectional measurements. These assumptions and a detailed description of the measurements evaluation are given in [1].

Thus in this case it suffices to replace the isotropic tensors c by the averaged tensors (phase tensors) and, similarly to (4.12), we obtain

$$\boldsymbol{\epsilon} = \mathbf{a}\left[\boldsymbol{\sigma}^d + \sum \lambda^{(2)}_{K_1K_2}\bar{c}_{K_1}\bar{\boldsymbol{\sigma}}^d_{K_2} + \sum \lambda_{K_1K_2K_3}\bar{c}_{K_1}\bar{c}_{K_2}\bar{\boldsymbol{\sigma}}^d_{K_3} + \ldots\right]. \tag{4.14}$$

Now, it remains to determine the phase strains and the coefficients $\mathbf{a}_{ik}$ appearing in the formulae (2.7). To that end, the averaging procedure should be performed over the points $\mathbf{x}_0$ lying only in a definite phase which, for the sake of uniformity of notations, should be denoted by K_0. Analogously to the expansion (4.6), the phase strain $\bar{\boldsymbol{\epsilon}}_{K_0}$ is obtained in the form

$$\bar{\boldsymbol{\epsilon}}_{K_0} = \boldsymbol{\Lambda}_{01}\langle\sigma^d_1\rangle_{K_0} + \boldsymbol{\Lambda}_{012}\langle c_1\sigma^d_2\rangle_{K_0} + \ldots, \tag{4.15}$$

where $\langle\ \rangle_{K_0}$ is the expected value operator over the phase K_0. The correlation moments become conditional and, in accordance with (4.8), are expressed as follows:

$$\langle c_1c_2\ldots\boldsymbol{\sigma}^d_n\rangle_{K_0} = \int\int\ldots\int c_1c_2\ldots\boldsymbol{\sigma}^d_n f(c_1,\ldots\boldsymbol{\sigma}^d_n/(\mathbf{x}_0\subset D_{K_0});\mathbf{x}_0,\ldots\mathbf{x}_n)dc_1\ldots d\boldsymbol{\sigma}^d_n. \tag{4.16}$$

Here f is the corresponding conditional probability density given by

$$f = \sum p_{K_1\ldots K_n/K_0}\delta(c_1-c_{K_1})\delta(c_2-c_{K_2})\ldots\delta(\boldsymbol{\sigma}^d_n-\boldsymbol{\sigma}^d_{K_n}), \tag{4.17}$$

$p_{K_1\ldots K_n/K_0}$ is the conditional probability of "hitting" the respective phases by the points $\mathbf{x}_1,\ldots,\mathbf{x}_n$ if point $\mathbf{x}_0$ lies in the phase K_0, and summing is extended over the indices $K_1,\ldots,K_n$. This probability is expressed, as is known from the theory, in the following manner:

$$p_{K_1,\cdots,K_n/K_0} = \frac{p_{K_0K_1\cdots K_n}}{p_{K_0}} = \frac{p_{K_0\cdots K_n}}{\nu_{K_0}}, \tag{4.18}$$

where $p_{K_0} = \nu_{K_0}$ denotes, according to Sec. 2, the concentration of the phase K_0. The conditional probability estimator is then the ratio of the number of "hittings" of the respective phases by $\mathbf{x}_0,\ldots,\mathbf{x}_n$ to the number of hittings of K_0 by $\mathbf{x}_0$.

The above relations being taken into account, the n-th term of expansion is found to be (similarly to (4.11))

$$\boldsymbol{\Lambda}_{012\ldots n}\langle c_1c_2\ldots\boldsymbol{\sigma}^d_n\rangle_{K_0} = \sum\left[\frac{1}{\nu_{K_0}}\boldsymbol{\Lambda}_{012\ldots n}p_{K_0K_1\cdots K_n}\right]c_{K_1}c_{K_2}\ldots\boldsymbol{\sigma}^d_{K_n} = \frac{1}{\nu_{K_0}}\sum\lambda^{(n+1)}_{K_0K_1\ldots K_n}c_{K_1}c_{K_2}\ldots\boldsymbol{\sigma}^d_{K_n}, \tag{4.19}$$

where $= p_{K_0K_1\cdots K_n}p_{K_0K_1\cdots K_n}(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_n) = p_{K_0K_1\cdots K_n}(\mathbf{r}_{01},\mathbf{r}_{12}\ldots,\mathbf{r}_{n-1,n})$, so finally

$$\bar{\boldsymbol{\epsilon}}_{K_0} = \frac{1}{\nu_{K_0}}\left[\sum{}^{(2)}_{K_0K_1}\bar{\boldsymbol{\sigma}}^d_{K_1} + \sum\lambda^{(3)}_{K_0K_1K_2}c_{K_1}\boldsymbol{\sigma}^d_{K_2} + \sum\lambda^{(4)}_{K_0K_1K_2K_3}c_{K_1}c_{K_2}\boldsymbol{\sigma}^d_{K_3} + \ldots\right]. \tag{4.20}$$

It is seen that the difference, by comparison with the expression for the macro-strain, consists in the fact that the order of the geometric correlation at the corresponding terms increases. The coefficient a_{IL} in the Eq. (2.7) is readily expressed as

$$a_{IL} = \frac{1}{\nu_I}\left[\lambda^{(2)}_{IL} + \sum_{K_1}\lambda^{(3)}_{IK_1L}c_{K_1} + \sum_{K_1,K_2}\lambda^{(4)}_{IK_1K_2L}c_{K_1}c_{K_2} + \ldots\right], \tag{4.21}$$

which solves our problem. The method of calculation of the remaining coefficients and the generalization to anisotropic media are obvious.

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POLISH ACADEMY OF SCIENCES.

Received August 2, 1971.