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On the relation between stress and strain rates for elastic-plastic solids

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The form of relations between stress rate and strain rate for an elastic-plastic, hardening material has been modified in order to make it applicable for limiting cases: perfectly elastic and rigid-plastic. The stress rate is decomposed into normal and tangential components to the yield surface and both terms are related to respective strain rates. Two examples of a plate and a cylindrical shell illustrate applicability of the modified form.

1. Modified form of rate equations

Consider an elastic-plastic solid for which the equations relating strain rate to an objective stress rate take the form

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p = \mathbf{L}\dot{\boldsymbol{\tau}} + \frac{1}{h}\mathbf{m}(\mathbf{m}\dot{\boldsymbol{\tau}}), \quad (\mathbf{m}\dot{\boldsymbol{\tau}}) > 0, \tag{1.1}$$

where $\mathbf{L}$ denotes the elasticity matrix and h is the hardening function; $\mathbf{m}$ denotes the unit normal vector to the regular yield surface. The inverse relations to (1.1) are usually presented in the form, see HILL [1],

$$\dot{\boldsymbol{\tau}} = \mathbf{K}\dot{\boldsymbol{\epsilon}} - \mathbf{K}\mathbf{m}\frac{\mathbf{m}\dot{\boldsymbol{\tau}}}{h} \tag{1.2}$$

and using (1), we have

$$\dot{\boldsymbol{\tau}} = \mathbf{K}\dot{\boldsymbol{\epsilon}} - \mathbf{K}\mathbf{m}\frac{\mathbf{m}\mathbf{K}\dot{\boldsymbol{\epsilon}}}{\mathbf{h}+\mathbf{m}\mathbf{K}\mathbf{m}}, \tag{1.3}$$

where $\mathbf{K} = \mathbf{L}^{-1}$ denotes the matrix of instantaneous elastic stiffness. There are numerous cases where the relations (1.3) should be used rather than (1.1), for instance in studying bifurcation or instability problems for elastic-plastic bodies. The form (1.3), however, possesses certain disadvantages since the constitutive relation for rigid-plastic solids cannot be derived directly from (1.3). In fact, this case is obtained when all components of $\mathbf{K}$ tend to infinity and the form (1.3) becomes indefinite, since both terms tend to infinity

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simultaneously. It is the aim of this note to present the constitutive relations (1.3) in a somewhat different form that could be more convenient in discussing the limiting cases.

Let us note that the relations (1.3) can also be presented in the form

$$\dot{\tau} = \frac{h\mathbf{K}\dot{\boldsymbol{\epsilon}}}{h+\mathbf{mKm}} + \frac{\mathbf{K}\dot{\boldsymbol{\epsilon}}(\mathbf{mKm})-\mathbf{Km}(\mathbf{mK}\dot{\boldsymbol{\epsilon}})}{h+\mathbf{mKm}} \tag{1.4}$$

which may briefly be written as the sum of two terms

$$\dot{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}}^c + \dot{\boldsymbol{\tau}}^t, \tag{1.5}$$

where $\dot{\boldsymbol{\tau}}^t$ denotes the stress rate component that lies in the hyperplane tangential to the yield surface. In fact, calculating the inner product $\mathbf{m}\dot{\boldsymbol{\tau}}$, from (1.4) we have

$$\mathbf{m}\dot{\tau} = m\dot{\boldsymbol{\tau}}^c = \frac{h\mathbf{mK}\dot{\boldsymbol{\epsilon}}}{h+\mathbf{mKm}}, \quad \mathbf{m}\dot{\boldsymbol{\tau}}^t = 0. \tag{1.6}$$

It turns out that the decomposition (1.5) is more convenient in considering the behaviour of large **K**. We shall discuss this case when K is an isotropic tensor, thus

$$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.7}$$

where λ and μ are Lamé constants. Decomposing $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\tau}}$ into spherical and deviatoric parts, $\dot{\varepsilon}_{ij} = \dot{\varepsilon}_v\delta_{ij} + \dot{e}_{ij}$, $\dot{\tau}_{ij} = \dot{p}\delta_{ij} + \dot{s}_{ij}$, from (1.4) we have

$$\begin{aligned} \dot{p} &= M\dot{\varepsilon}_v \\ \dot{s}_{ij} &= \frac{2\mu h}{h+2\mu}\dot{e}_{ij} + \frac{4\mu^2}{h+2\mu}[\dot{e}_{ij} - m_{ij}(m_{kl}\dot{e}_{kl})], \end{aligned} \tag{1.8}$$

where $M = 3\lambda + 2\mu$ denotes the elastic bulk modulus. Let us introduce the normal and tangential components of the deviatoric strain rate

$$\dot{e}_{ij}^m = (\dot{e}_{kl}m_{kl})m_{ij}, \quad \dot{e}_{ij}^t = \dot{e}_{ij} - (\dot{e}_{kl}m_{kl})m_{ij}. \tag{1.9}$$

The second relation (1.8) now takes the form

$$\dot{s}_{ij} = 2\mu\alpha\dot{e}_{ij} + 2\mu(1-\alpha)\dot{e}_{ij}^t = 2\mu\alpha\dot{e}_{ij}^m + 2\mu\dot{e}_{ij}^t, \tag{1.10}$$

where $\alpha = h/(h+2\mu)$.

The elastic case can easily be obtained from (1.10) by setting $h = \infty$ or $\alpha = 1$. On the other hand, the rigid-plastic case is obtained for $2\mu = \infty$ or $\alpha = 0$. Then (1.10) takes the form

$$\dot{s}_{ij} = h\dot{e}_{ij} + \dot{s}_{ij}^t, \tag{1.11}$$

where $\dot{s}_{ij}^t = 2\mu\dot{e}_{ij}^t$ denotes the undeterminate tangential stress rate. Similar relation occurs for the total stress rate; in fact, from (1.4), for all components of K tending to infinity, we have

$$\dot{\tau}_{ij} = h\dot{\varepsilon}_{ij} + \dot{\tau}_{ij}^t. \tag{1.12}$$

The relation similar to (1.12) was also discussed by SEWELL [2].

Let us now discuss the scalar product $\dot{\boldsymbol{\tau}}\dot{\boldsymbol{\epsilon}}$ which can be expressed as follows:

$$\dot{\tau}_{ij}\dot{\varepsilon}_{ij} = 2\mu\alpha\dot{e}_{ij}^m\dot{e}_{ij}^m + 2\mu\dot{e}_{ij}^t\dot{e}_{ij}^t + 3M\dot{\varepsilon}_v^2. \tag{1.13}$$

It should be required that for $2\mu \to \infty$ and $M \to \infty$ both $\dot{e}^t_{ij}$ and $\dot{\varepsilon}_v$ tend to zero, since otherwise the stress rate would become infinite. An approximate evaluation of orders of the three terms of (1.13) can be obtained, for instance, by assuming that the stress rate remains fixed in magnitude and direction for increasing 2μ and M. Then, from (1.8) and (1.10) it follows that

$$\frac{\dot{e}^t_{ij}\dot{e}^t_{ij}}{e^m_{ij}e^m_{ij}} = A\alpha^2, \qquad \frac{\dot{\varepsilon}^2_v}{\dot{e}^m_{ij}\dot{e}^m_{ij}} = B\alpha^2, \tag{1.14}$$

where A and B are finite constants. Thus, the second and the third term of (1.13) vanish linearly with α and the rigid-plastic case is obtained by retaining only the first term, thus

$$\dot{\tau}_{ij}\dot{\varepsilon}_{ij} = h\dot{e}_{ij}\dot{e}_{ij}. \tag{1.15}$$

In the next section, we shall illustrate applicability of the form (1.10) by reexamining two cases of bifurcation of an elastic-plastic plate and a cylindrical shell, previously studied in [3, 4].

2. Examples

In considering bifurcation in elastic-plastic solids, it is convenient to start from the uniqueness condition [3, 4]

$$\int [\Delta\dot{\tau}_{ij}\Delta\dot{\varepsilon}_{ij} + \sigma_{ij}\Delta v_{k,i}\Delta v_{k,j} - 2\sigma_{ij}\Delta\dot{\varepsilon}_{ki}\Delta\dot{\varepsilon}_{kj}]dv > 0, \tag{2.1}$$

where $\dot{\tau}_{ij}$ denotes the Jaumann derivative of the Kirchhoff stress, σ_{ij} is the Cauchy stress, Δv_i denotes the difference of the two admissible velocity fields and comma preceding index denotes differentiation with respect to space variable. Using (1.13), we have

$$\int [2\mu\alpha\Delta\dot{e}^m_{ij}\Delta\dot{e}^m_{ij} + 2\mu\Delta\dot{e}^t_{ij}\Delta\dot{e}^t_{ij} + 3M\Delta\dot{\varepsilon}^2_v + \sigma_{ij}\Delta v_{k,i}\Delta v_{k,j} - 2\sigma_{ij}\Delta\dot{\varepsilon}_{ki}\Delta\dot{\varepsilon}_{kj}]dV > 0. \tag{2.2}$$

With the help of (1.14) and the relation $\dot{\varepsilon}_{ij} = \varepsilon_v\delta_{ij} + \dot{e}^t_{ij} + \dot{e}^m_{ij}$, we can rewrite (2.1) as

$$\int [(2\mu\alpha + 2\mu A\alpha^2 + 3MB\alpha^2)\Delta\dot{e}^m_{ij}\Delta\dot{e}^m_{ij} + \sigma_{ij}\Delta v_{k,i}\Delta v_{k,j} - 2\sigma_{ij}\Delta\dot{\varepsilon}_{ki}\Delta\dot{\varepsilon}_{kj}]dV > 0, \tag{2.3}$$

which is in a form convenient for approximation to rigid-plastic behaviour of the solid.

2.1. Stability of plate

Let us discuss first an elastic-plastic plate ($2a \times 2b \times 2c$) subjected to tensile stresses σ_1 and σ_2 on the faces $|x| = a$ and $|y| = b$, respectively, whereas the face $|z| = c$ is kept free of traction. We shall consider the field [3]

$$\begin{aligned} \Delta v_1 &= \sin\eta x\cos\beta y\,\mathrm{ch}\sqrt{\gamma z}, \\ \Delta v_2 &= (\beta/\eta)\cos\eta x\sin\beta y\,\mathrm{ch}\sqrt{\gamma z}, \\ \Delta v_3 &= (\sqrt{\gamma}/\eta)\cos\eta x\cos\beta y\,\mathrm{sh}\sqrt{\gamma z} \end{aligned} \tag{2.4}$$

with

$$\gamma = -\frac{h\nu + 2\mu\{k(k+l) + 2\nu l^2\}}{h(1-\nu) + 2\mu\{k^2 + l^2 + 2\nu kl\}}\eta^2 - \frac{h\nu + 2\mu\{l(k+l) + 2\nu k^2\}}{h(1-\nu) + 2\mu\{k^2 + l^2 + 2\nu kl\}}\beta^2, \tag{2.5}$$

where we have used k and l for m_{11} and m_{22}, respectively, and ν is the Poisson's ratio; $\eta = m\pi/a$, $\beta = n\pi/b$, m, n being positive integers.

Substituting in (2.3), we obtain

$$(2.6) \qquad (\eta^2+\beta^2)(\eta^2\sigma_1+\beta^2\sigma_2) < \frac{2\mu h(2\mu+h)\left(1+A\alpha+3B\alpha\dfrac{1+\nu}{1-2\nu}\right)}{[h(1-\nu)+2\mu(k^2+l^2+2\nu kl)]^2} \times \\ \times [(k+l\nu)\eta^2+(l+k\nu)\beta^2]^2$$

for uniqueness.

As a special case, let us consider an incompressible von Mises solid under equal all-round tension. Therefore, $\sigma_1 = \sigma_2 = \sigma$, $k = l = 1/\sqrt{6}$, and $\nu = 1/2$. Moreover, we assume that $2\mu \gg h$ and that the tangential components of the strain-rate vector is small in comparison to the normal component, which means that $\alpha \ll 1$. Hence, at the onset of bifurcation,

$$(2.7) \qquad \sigma/E = \frac{1}{1+\delta},$$

where $\delta = 2\mu/h$ and E is the Young's modulus. In deriving (2.7), we let $\dot{\varepsilon}_v = 0$, that is $B = 0$ before setting $\nu = 1/2$. It may be emphasized again that (2.6) and (2.7) are special forms of the uniqueness criterion (2.3) suitable only for the cases where $2\mu \gg h$.

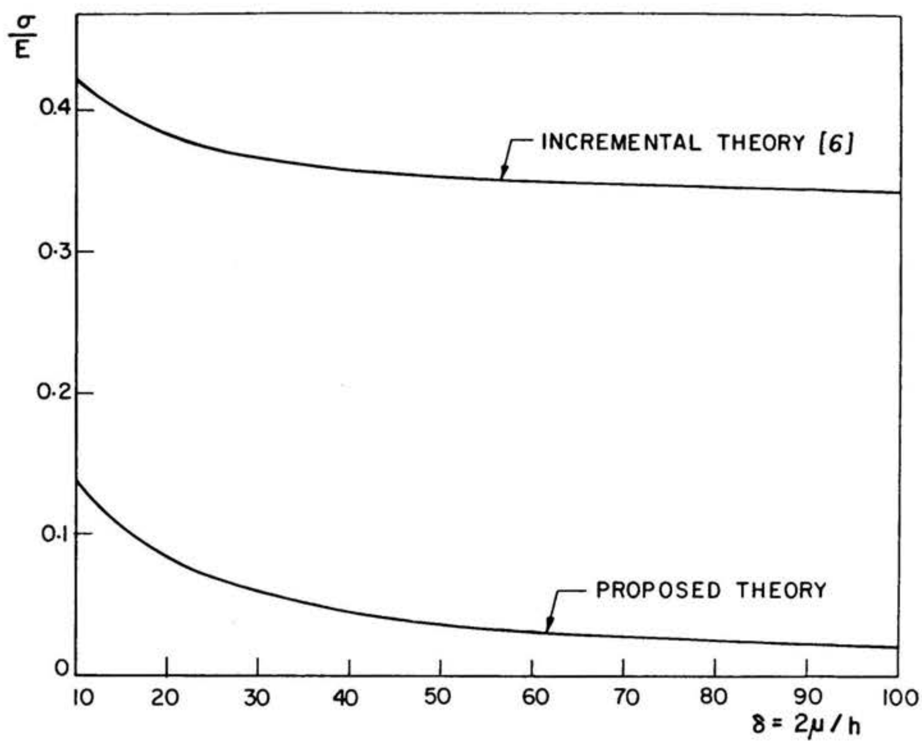

FIG. 1.

In Fig. 1, plot of σ/E against δ is shown for (2.7) and also for

$$\sigma/E = \frac{1}{3}\frac{4+\delta}{1+\delta}$$

obtained in [6] using the incremental theory.

2.2. Stability of cylindrical shell

Let us next consider a cylindrical shell of length L, radius a, and thickness t, under axial compressive stress σ. Let us assume that (a) the deformation is axi-symmetric, (b) the plane section normal to the middle surface remains plane and normal and (c) the stress rates normal to the middle surface are negligibly small. Under these assumptions and using the field [5]

$$\Delta u = \sin(n\pi x/L),$$
$$\Delta w = -(\beta C_{11}/C_{12})\cos(n\pi xL),$$

the uniqueness criterion (2.3) reduces to

$$(2.8)\qquad \beta^2(\beta^2C_{11}^2-C_{12}^2)\sigma < \frac{2\mu(1+\delta)\left(1+A\alpha+3B\alpha\dfrac{1+\nu}{1-2\nu}\right)}{[1-\nu+\delta(k^2+l^2+2\nu kl)]^2}\times$$
$$\times\left\{\beta^2[(k+l\nu)C_{12}-(l+k\nu)C_{11}]^2+\frac{t^2}{12a^2}[(k+l\nu)\beta^3-(l+k\nu)\beta]^2C_{11}^2\right\},$$

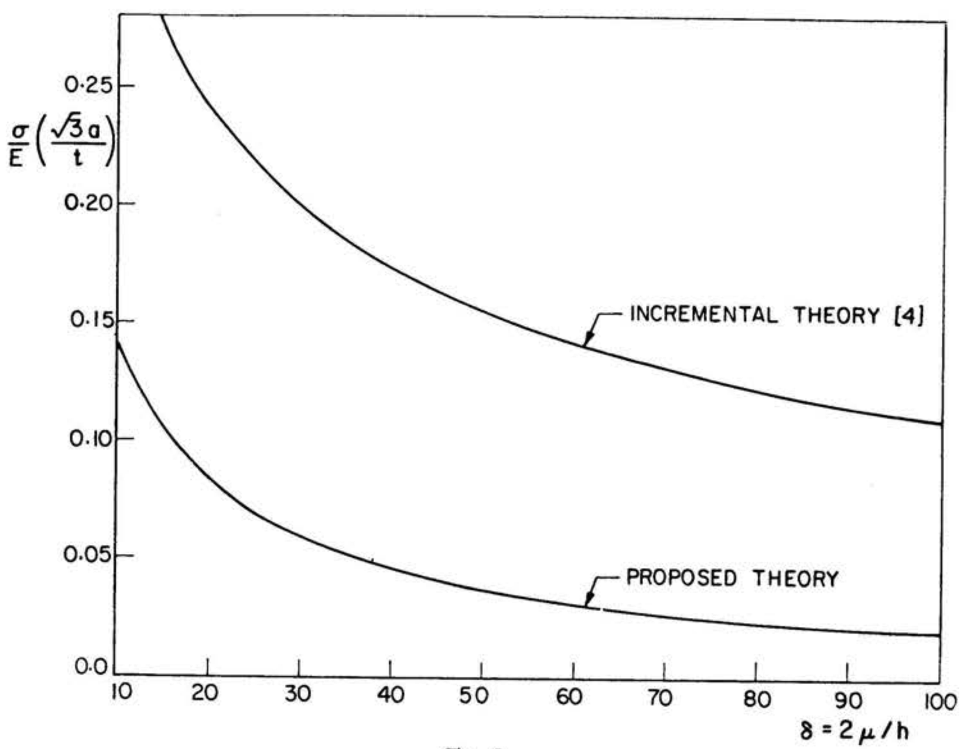

FIG. 2.

where

$$C_{11} = \frac{\delta(1+\nu)l^2}{(1-\nu)+\delta(k^2+l^2+2\nu kl)}, \quad C_{12} = \frac{\nu-\delta(1+\nu)kl}{(1-\nu)+\delta(k^2+l^2+2\nu kl)}, \tag{2.9}$$

$\beta = (n\pi a/L)$, n being a positive integer.

For further simplification, we consider the deformation to be predominantly plastic, so that $2\mu \gg h$. Moreover, we assume $\beta \gg 1$ and minimize σ with respect to β to obtain, for $\nu = 1/3$,

$$\frac{\sigma}{E}\Bigg(\frac{\sqrt{3}a}{t} = 90\frac{1+\delta}{(18+4\delta)(12+11\delta)} \tag{2.10}$$

at the onset of bifurcation.

The curve $\frac{\sigma}{E}\frac{\sqrt{3}a}{t}$ against δ corresponding to (2.10) is shown in Fig. 2, where we have also shown the corresponding plot for the bifurcation stress obtained from the incremental theory [4].

3. Concluding remarks

The proposed modified form of rate equations proves to be useful in discussing bifurcation phenomena for large values of 2μ and M. Thus it should be expected that (2.7) and (2.10) predict bifurcation stresses more accurately than the previous solutions, and σ/h tends to a finite value when $2\mu \to \infty$.

Starting from (2.3), the values of A and B can be determined for finite values of 2μ and M. Then, an approximate assumption on fixed direction of stress rate for increasing 2μ and M would imply that A and B are constant when $2\mu \to \infty$ and $M \to \infty$. Let us note that this assumption is based on the concept of statically admissible stress rate field when increasing tractions are applied to the boundary surface. The uniqueness condition can thus be evaluated for arbitrarily large 2μ and M. The corresponding bifurcation curves would then run between those presented in Figs. 1 and 2.

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