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ROLE OF INFORMATION STRUCTURES IN RISK-SENSITIVE DISCRETE-TIME NONCOOPERATIVE PROBLEMS

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Abstract: In the paper the linear, exponential-quadratic, Gaussian noncooperative problems are considered. The standard LQG assumptions of linear state constraints and Gaussian noises are preserved. The objective functionals in the form of expectation of the exponential of quadratic function with an added scalar allow to reflect decision makers' sensitivity to risk. For discrete-time problem Nash equilibrium solutions under different information structures are given. The impact on the presented solutions, of considered information structures is enlightened.

Keywords: Risk sensitivity, Nash games, information structures.

1. Introduction

Decision making processes under uncertainty possess several features that are not present in the case of the non-stochastic approach. One of the most important such characteristic that should be taken into account in mathematical formulation of decision problems is risk attitude of decision makers. It is obvious that decision makers when faced with different degrees of uncertainty will choose different strategies according to theirs risk attitudes.

Consider the linear, stochastic, discrete-time system

$$x(k+1) = Ax(k) + B_1u_1(k) + B_2u_2(k) + w(k) \qquad (1)$$

for $k = 0,1,\ldots,N-1$. Vectors x, u_1, u_2 denote state of the system and actions of decision makers and they take values in Euclidean spaces of fixed dimensions. Matrices A, B_1, B_2 are real and constant with appropriate dimensions. $x(o) - \bar{x}_o$, $\left(\bar{x}_o = E\ \{x(0)\}\right)$, and $w(k)$ for all k are zero-mean independent Gaussian random vectors with known covariances P and S.

Among a variety of approaches, those when decision makers minimize expectations of quadratic functionals

$$\psi_i = \sum_{k=0}^{N-1}\left(x^T(k)Q_ix(k) + u_i^T(k)u_i(k)\right) + x^T(N)Q_{iN}x(N) \qquad (2)$$

are the most commonly considered and applied. Theirs main advantages are relative simplicity and ease of computation of the corresponding optimal decision rules. However, such objective functionals are risk neutral and may be considered inappropriate to modeling decision processes.

To take into consideration the decision makers' attitudes to risk a fairly fruitful way is to define objective functionals in the following, exponential-quadratic form

$$J_i = E\left\{ \frac{2}{\theta_i}\left(e^{\frac{1}{2}\theta_i\Psi_i}-1\right)\right\}, \; i = 1,\, 2. \qquad (3)$$

If $-\psi_i$ is interpreted as a welfare then $-J_i$ can be considered as a decision maker utility function which is to be maximized. $\theta_i \in R$ may be interpreted as a measure of decision maker sensitivity to risk. It is positive for risk averse and negative for risk preferring decision maker. When θ_i tends to zero then J_i goes to ψ_i and this corresponds to the case of risk neutral decision maker.

The exponential-quadratic cost functions have already been applied in optimal control problems, see for example Jacobson (1977), Whittle (1981), and in team problems, Krainak et al. (1982a, 1982b, 1982c). But the notions of risk and risk sensitivity seem to be very important when non-cooperative decision problems are considered, see Caravani and Papavassilopoulos (1987), and Krajewski (1991a). This paper presents further aspects of risk sensitive decision processes.

Control actions $u_1(k)$, $u_2(k)$, for $k = 0,1,\ldots,N-1$ depend on available to decision makers information $I_1(k)$, $I_2(k)$ and are to be chosen to satisfy the Nash equilibrium conditions. Moreover, we assume that strategies of decision makers are affine functions of respective observations.

In the next section solutions of the posed Nash games are discussed under different information structures reflecting different degrees of decision makers' knowledge of the system state and their willingness to exchange observations made. All results are presented without proofs.

2. The Main Results

Open-loop pattern.

This is the simplest information structure we assume. The only observations are made over the initial state

$$y_i = C_i x(0) + v_i \quad , \qquad i=1,2 \ , \tag{4}$$

where v_1, v_2 are representing measurement noises, zero-mean, independent of $x(0)$, $w(k)$, $k=0,\dots,N-1$, Gaussian random vectors with covariances R_1, R_2. Additionally we assume that in (2) $Q_1 = Q_2 = 0$. Then at any stage k

$$u_i(k) = L_i(k) y_i + \mu_i(k) \tag{5}$$

and from Krajewski (1991) we obtain for $i,j = 1,2$, $j \neq i$

$$L_i(k) = -B_i (A \ )^{N-k-1} Q_i \tilde{L}_i \ , \tag{6}$$

$$\mu_i(k) = -B_i^T (A^T)^{N-k-1} Q_i \tilde{\mu}_i \ , \tag{7}$$

$$\left(I - \Theta_i F_i Q_i + \Phi(B_j B_j^T) Q_i\right) \tilde{L}_i = \left(A^N - \Phi(B_i B_i^T) Q_j \tilde{L}_j C_j\right) M_i \ , \tag{8}$$

$$\left(I - \Theta_i F_i Q_i + \Phi(B_j B_j^T) Q_i\right) \tilde{\mu}_i + \Phi(B_i B_i^T) Q_j \tilde{\mu}_j = \left(A^N - \Phi(B_i B_i^T) Q_j \tilde{L}_j C_j\right) M_{io} \bar{x}_o \tag{9}$$

$$\Phi(X) = \sum_{k=0}^{N-1} A^{N-k-1} \, X (A^T)^{N-k-1} \ , \tag{10}$$

$$F_i = \left(A^N - \Phi(B_i B_i^T) Q_j \tilde{L}_j C_j\right) \Sigma_i \left(A^N - \Phi(B_i B_i^T) Q_j \tilde{L}_j C_j\right)^T + $$

$$+ \Phi(B_i B_i^T) \tilde{L}_j R_j \tilde{L}_j^T \Phi(B_i B_i^T) + \Phi(S), \tag{11}$$

$$M_i = PC^T (\ CPC^T + R \ )^{-1}, \quad M_{io} = I - M_i C_i, \quad \Sigma_i = - M_{io} P, \tag{12}$$

$$F_i^{-1} - \Theta_i Q_i > 0. \tag{13}$$

Thus existence of a Nash affine solution depends on attitudes of decision makers, on the number of stages N and on stability properties of the system (1). In the case of unstable systems such solution exists only if decision makers are not too risk aversive that is when they are fairly daring.

Feedback perfect state pattern.

Now, assume that at every stage $k = 0,1,\dots,N$ both decision makers know the state vector $x(k)$, i.e. $y_i(k) = x(k)$. Then, for $i = 1,2$; $k = 0,1,\dots,N-1$, see Krajewski (1990)

$$u_i(k) = -B_i^T \Pi_i(k+1 \left(I - \theta_i S \Pi_i(k+1)\right)^{-1} \Lambda(k+1) A x(k), \tag{14}$$

$$\Lambda(k+1) = \left(I + \sum_{j=1}^{2} B_j B_j^T \Pi_j(k+1) \left(I - \theta_i S \Pi_i(k+1)\right)^{-1}\right)^{-1}, \tag{15}$$

$$\Pi_i(k) = Q_i(k) + \left(\left(I - \theta_i S \Pi_i(k+1)\right)^{-1} \Lambda(k+1) A\right)^T \left[\Pi_i(k+1) + \right.$$

$$\left. + \Pi_i(k+1)\left(B_i B_i^T - \theta_i S^2\right)\Pi_i(k+1)\right]\left(I - \theta_i S \Pi_i(k+1)\right)^{-1} \Lambda(k+1) A, \tag{16}$$

$$\Pi_i(N) = Q_i(N), \tag{17}$$

$$S^{-1} - \theta_i \Pi_i(k+1) > 0 \quad , \text{ if } \quad \theta_i > 0 \tag{18}$$

The structure of the above solution is the same as in the standard LQG problem. Note that this case admits the affine Nash solution for more cautious decision makers then the previous one.

Feedback imperfect state pattern.

At any moment $k = 0,1,\dots,N$ both decision makers observe the state vector through linear channels

$$y_i(k) = C_i x(k) + v_i(k), \quad i = 1,2. \tag{19}$$

$v_1(k)$, $v_2(k)$ are for any $k = 0,1,\dots,N$ zero-mean independent of $x(0)$, $w(k)$ Gaussian random vectors with covariances R_1, R_2. Moreover, we assume that decision makers exchange observations with no delay, i.e. their actions at any stage k are based on the same information

$$I_1(k) = I_2(k) = I(k) = \left\{\left(y_1(0), y_2(0)\right), \dots, \left(y_1(k), y_2(k)\right)\right\}. \tag{20}$$

Following Whittle (1981) we obtain risk sensitive versions of

Kalman filter, Krajewski (1991b), for $i,j = 1,2$, $j \neq i$

$$V_i(0) = \left(P^{-1}+ C_1^T R_1^{-1} C_1 + C_2^T R_2^{-1} C_2\right)^{-1}, \tag{21}$$

$$\hat{x}_i(0) = V_i(0) \left(P^{-1}\bar{x}_o + C_i^T R_i^{-1} y_i(0) + C_j^T R_i^{-1} y_j(0)\right), \tag{22}$$

$$V_i^{-1}(k) = S^{-1} + C_1^T R_1^{-1} C_1 + C_2^T R_2^{-1} C_2 - S^{-1}A(V_i^{-1}(k-1) - \theta_i Q_i + A^T S^{-1} A)^{-1} A^T S^{-1}, \tag{23}$$

$$V_i^{-1}(k)\hat{x}(k) = \left[S^{-1} - S^{-1}A\left(V_i^{-1}(k-1) - \theta_i Q_i + A^T S^{-1} A\right)^{-1} A^T S^{-1}\right]\left(A\hat{x}(k-1) + \right.$$
$$\left. + B_1 u_1(k-1) + B_2 u_2(k-1)\right) + S^{-1}A\left(V_i^{-1}(k-1) - \theta_i Q_i + A^T S^{-1} A\right)^{-1} \theta_i Q_i \hat{x}(k-1) +$$
$$+ C_1^T R_1^{-1} y_1(k) + C_2^T R_2^{-1} y_2(k). \tag{24}$$

Then combining the above recursive equations with (14),...,(18) we obtain for $k = 0,\dots,N-1$, $i = 1,2$

$$u_i(k) = C_i\left(V_i^{-1}(k) - \theta_i \Pi_i(k)\right)^{-1} V_i^{-1}(k)\hat{x}_i(k). \tag{25}$$

The structure of the above solution is the same as in the standard LQG problem. It can exists for less cautious players.

One-step delayed observation sharing pattern.

At any moment decision makers observe state vector as in (19) and available to them information consists of two parts

$$I_i(k) = I(k-1) \cup \left\{y_i(k)\right\}, \; i=1,2. \tag{26}$$

Risk sensitive versions of Kalman filter is, Krajewski (1991b), for $i,j = 1,2$, $j \neq i$

$$V_i^{-1}(0) = P^{-1} + C_i^T R_i^{-1} C_i, \tag{27}$$

$$\hat{x}_i(0) = \bar{x}_o + V_i(0) C_i^T R_i^{-1}\left(y_i(o) - C_i\bar{x}_o\right), \tag{28}$$

$$V_i^{-1}(k) = S^{-1} + C_i^T R_i^{-1} C_i - S^{-1}A\left(V_i^{-1}(k-1) - \theta_i Q_i + A^T S^{-1} A + C_j^T R_j^{-1} C_j\right)^{-1} A^T S^{-1}, \tag{29}$$

$$V_i^{-1}(k)\hat{x}_i(k) = \left(S^{-1} - S^{-1}A\left(V_i^{-1}(k-1) - \theta_i Q_i + A^T S^{-1} + C_j^{-1} R_j^{-1} C_j\right)^{-1} A^T S^{-1}\right) *$$

$$\cdot\left(A\hat{x}_i(k-1) + B_1u_1(k-1)+B_2u_2(k-1)\right) + S^{-1}A\left(V_i^{-1}(k-1)-\theta_iQ_i+A^TS^{-1}A + \right.$$

$$\left.+C_j^TR_j^{-1}C_j\right)^{-1}\left(C_j^TR_j^{-1}\left(y_j(k-1)-C_j\hat{x}_i(k-1)\right) + \theta_iQ_i\right)+C_i^TR_i^{-1}y_i(k). \qquad (30)$$

Next, with obtained state estimates and covariance matrices we may show as in Krajewski (1991a) that in the considered case the multistage problem is decomposable into a series of single-stage problems and for i =1,2, k = 0,1,...,N-1

$$u_i(k) = L_i(k)y_i(k) + \mu_i(k). \qquad (31)$$

For any k matrices $L_1(k)$, $L_2(k)$ satisfy matrix polynomial equations of order 3 and $\mu_1(k)$, $\mu_2(k)$ satisfy set of linear equations. Form of these equations is exactly as in Krajewski (1991a). Their solutions depend on decision makers' risk attitudes. Generally, they can exist for less cautious players too.

3. References

Caravani, P.,Papavassilopoulos, G. (1987) A class of risk sensitive games, IASI Techn.Report, R.193.

Jacobson D.H. (1977) Extensions of Linear-Quadratic Control, Optimization and Matrix Theory, Academic Press, New York.

Krainak, J.,Speyer, J., Marcus, S (1982a) Static team problems part I : Sufficient conditions and exponential cost criterion, IEEE Trans.Aut.Control, AC-27, 839-847.

Krainak, J.,Speyer, J., Marcus, S (1982b) Static team problem part II. Affine control laws, projection, algorithm and the LEGT problem. IEEE Trans.Aut.Control ,AC-27, 848-859.

Krainak, J.,Matchell, F.,Marcus, S.,Speyer, J. (1982c) The dynamic linear exponential gaussian team problem. IEEE Trans. Aut.Control , AC-27, 860-869.

Krajewski, W. (1991a) A Note on the Dynamic Risk Sensitive Nash Games. In: Dynamic Games in Economic Analysis, R.P. Hamalainen and H.K. Ehtamo (eds). Lecture Notes in Control and Information Sciences, 157, Springer-Verlag, 1991.

Krajewski, W. (1991b) Nash equilibrium solutions for a class of dynamic risk sensitive games. Report of Systems Research Institute ZTSW-7/A1217, (in polish).

Whittle, P. (1981) Risk-sensitive linear quadratic gaussian control. Adv. Appl. Prob., 13, 764-777.