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A CLASS OF INTERACTIVE METHODS FOR DECISION MAKING UNDER CONFLICTING OBJECTIVES

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ABSTRACT: We discuss the use of a quadratic norm for departures from the bliss value of a decision problem under conflicting objectives. The use of a quadratic norm is, for example, of interest within the dynamic framework of optimal control. The possibility of tailoring the quadratic objective function to generate optimal policies which are acceptable to the policy maker is explored with two alternative interactive algorithms. One of these is for objective functions with diagonal weighting matrices and uses updates of the bliss values. The second algorithm proceeds by updating non-diagonal weights, while keeping the bliss values fixed. The equivalence of both algorithms are established.

KEYWORDS: Multiple objectives, quadratic programming, interactive algorithms.

1. INTRODUCTION

Let $\mathcal{B} \in \mathbb{R}^n$ be the bliss value of a multi-objective decision problem. Consider the determination of the relative importance of attaining the different elements of $\mathcal{B}$. The attainment of different elements of $\mathcal{B}$ generally represent conflicting objectives.

We adopt a quadratic objective for measuring deviations from the bliss value. The quadratic function is of interest, for example, within the framework of optimal control. In Section 2, we describe a way of avoiding, if required, the symmetric nature of the quadratic function. We consider two interactive methods for the specification of the quadratic objective. The purpose of both methods is to tailor the objective function such that the optimally generated solution of the decision problem is also politically acceptable to the decision maker. The first is a method for the specification of the quadratic objective with a diagonal Hessian. The second method is for the specification of the weights and generates a nondiagonal Hessian or weighting matrix.

Consider the linear-quadratic optimal decision problem

$$\min\left\{ < x - \mathfrak{B}, D (x - \mathfrak{B}) > \middle| N^T x = b \right\}. \tag{1.1}$$

where $x \in \mathbb{E}^n$, $b \in \mathbb{E}^m$ is a fixed vector, D is a diagonal matrix with nonnegative elements, $\mathfrak{B}$ is the bliss value of x or the unconstrained optimum of the quadratic objective. The columns of matrix $N \in \mathbb{E}^{n \times m}$ are assumed, without loss of generality, to be linearly independent. The diagonal weighting matrix in (1.1) occurs in economic decision making and in general in linear-quadratic optimal control problems. In the latter case, (1.1) can be regarded as the transcription of the dynamic problem into static form (see e.g. Polak, 1971). We can thus state the problem as follows:

Let $\Omega \subseteq \mathbb{E}^n$ denote the set of policies acceptable to the decision maker. We assume that there does not exist an analytical characterization of Ω and that Ω exists only in the mind of the decision maker. If an explicit characterization of Ω was possible, it could be used to augment the constraints of (1.1) and the resultant problem could easily be solved. An analytical characterization is both inherently difficult, or impossible, and sometimes also politically undesirable from the decision maker's point of view. The problem is to tailor an objective function for which the optimal solution of (1.1), x^*, also satisfies $x^* \in \Omega$.

We discuss two types of solutions to the above problem. The first involves modifications to $\mathfrak{B}$ and keeps D fixed as a diagonal matrix. The second involves modifications to D and generates a nondiagonal weighting matrix while keeping $\mathfrak{B}$ fixed. We discuss the equivalence of both methods.

The desirability of a diagonal D is due to computational reasons as well as the interpretation of the problem and the optimal solution. Non-diagonal weighting matrices can be diagonalized by appropriate transformations of the variables. However, such transformations lead to computational complications, particularly in relation to the constraints. A more important aspect is the difficulty of assigning an interpretation to the off-diagonal elements of a general symmetric weighting matrix. These elements are usually understood to represent trade-offs between achieving alternative objectives. However, they are difficult to assign an interpretation as distinct from the diagonal elements, which represent the relative importance of each objective. Also, for linear quadratic dynamic problems, solved via dynamic programming, maintaining the block diagonality of D, in terms of the time structure, is desirable at least for the physical interpretation of the resultant optimal linear feedback laws. This contrasts with the case when non-diagonal weighting matrices have to be factorized and the original variables have to be transformed in order to obtain feedback laws which may only apply to the transformed variables.

In most economic decision making problems under conflicting objectives, the precise value of the weighting matrix is not known. Even a judicious choice of the weighting matrix need not necessarily yield a solution of problem (1.1) that is in the set of acceptable policies, Ω. In Section 3, we discuss an iterative algorithm, involving interactions with the decision maker, to specify a diagonal quadratic objective function that generates a solution of (1.1) which is *acceptable* to the decision maker. The algorithm involves the modification of the bliss values. In Section 4, the method is shown to be equivalent to an alternative method involving the respecification of the general non-diagonal symmetric matrix (Rustem, Velupillai, Westcott, 1978; Rustem and Velupillai, 1988, a). In tbe latter method, the bliss values remain fixed but the weighting matrix is altered and does not maintain a diagonal structure. This correspondence is helpful computationally as diagonal matrices are simpler to compute. Conversely, the correspondence provides the non-diagonal weighting matrix that would generate the same optimal solution as the one obtained by modifying the bliss values. In addition, the correspondence can be used to discuss tbe complexity and polynomial time termination property of the algorithms by invoking the results in Rustem (1990) and Rustem and Velupillai (1988, b).

2. NONSYMMETRIC QUADRATIC FUNCTIONS

The quadratic function (1.1) assigns equal importance to deviations in either direction from the bliss value of a variable. Thus, if a variable could do better than the bliss value, the quadratic objective penalises such a departure just as much as it would penalise departures of the variable in the inferior direction. Consider, therefore, the problem

$$\min\left\{ \sum_{i=1}^{n} d^i \left(x^i - \mathfrak{B}^i \right)_+^2 \;\middle|\; N^T x = b \right\}$$

where the superscript i denotes the i th variable, d^i is the corresponding weight and

$$\left(\cdot \right) = \min\left\{ 0, x^i - \mathfrak{B}^i \right\}.$$

Extending a similar approach in linear programming, for each variable requiring non-symmetric penalisation, we define $x^i - \mathfrak{B}^i = x^i_+ - x^i_-$, where $x^i_+, x^i_- \geq 0$ and

$$x^i_+ : \quad x^i_+ = x^i - \mathfrak{B}^i \quad \text{if} \quad x^i - \mathfrak{B}^i \geq 0$$

$$= 0 \quad \text{if} \quad x^i - \mathfrak{B}^i < 0$$

$$x^i_- : \quad x^i_- = 0 \quad \text{if} \quad x^i - \mathfrak{B}^i \geq 0$$

$$x^i_-: \; x^i_- = -(x^i - \mathfrak{B}^i) \text{ if } \quad x^i - \mathfrak{B}^i < 0.$$

Thus, either $x^i_- = 0$ or $x^i_+ = 0$ or both. In addition to the nonnegativity restrictions, we therefore have to impose the constraint $(x^i_-)(x^i_+) = 0$. Since in vector notation we have, $x = x_+ - x_- + \mathfrak{B}$, we can express the above minimisation problem as

$$\min\left\{ \sum_{i=1}^{n} d^i \left(x^i_+ \right)^2_+ + \rho \sum_{i=1}^{n} (x^i_-)(x^i_+) \;\middle|\; [N^T : -N^T]\begin{bmatrix} x_+ \\ x_- \end{bmatrix} = b - N^T \mathfrak{B};\; x_+, x_- \geq 0 \right\}$$

where $\rho \geq 0$ is a sufficiently large penalty parameter chosen to ensure that, in view of the nonnegativity constraints, the minimum value of $(x^i_-)(x^i_+)$ is realised at either $x^i_- = 0$ or $x^i_+ = 0$. We have thus reduced the nonsymmetric problem to a symmetric problem. The extension of the methods discussed below to inequality constraints are discussed in Rustem and Velupillai (1988, b).

3. DIAGONAL QUADRATIC OBJECTIVE FUNCTIONS

We consider a solution to Problem 1. An alternative solution and the equivalence of both solutions are discussed in the next section. Let (1.1) be solved for a given weighting matrix D and a given initial vector of bliss values $\mathfrak{B}_0$. The solution is denoted by

$$x_0 = \arg\min\left\{ < x - \mathfrak{B}_0, D(x - \mathfrak{B}_0) > \;\middle|\; N^T x = b \right\}.$$

The solution is presented to the decision maker who is required to respond by either declaring that $x_0 \in \Omega$, or if $x_0 \notin \Omega$, the decision maker is required to specify the modified form of x_0 that is in Ω.

The decision problem we wish to consider can now be formulated as the computation of the policy, optimally determined via (1.1), but which also is acceptable to the policy maker and is hence in the set Ω. We assume that $\left\{ x \;\middle|\; N^T x = b \right\} \cap \Omega \neq \emptyset$. The decision maker's preferred alternative to x_0 is denoted by x_p. By definition, we have $x_p \in \Omega$ *but not necessarily* $x_p \in \{ x \mid N^T x = b \} \cap \Omega$. In case the latter is true, such a preferred alternative would conceptually solve the decision problem. Let $\delta_0 = x_p - x_0$, where δ_0 is the *correction* vector that needs to be added to x_0 in order to ensure that $x_0 + \delta_0 \in \Omega$. Using δ_0, we can revise the bliss value as $\mathfrak{B}_1 = \mathfrak{B}_0 + \alpha_0 \delta_0$ where $\alpha_0 \geq 0$ is a scalar. Using this new bliss value, problem (1.1) is solved once again to yield a new optimal solution, x_1. This solution is shown below to have desirable characteristics. However, as there is no guarantee that $x_1 \in \Omega$, the above procedure may need to be repeated. The resulting algorithm is summarized below. We discuss the complexity and termination properties of the algorithm in Rustem (1990) and Rustem and Velupillai (1988, b).

Algorithm : Updates of Bliss Values with a Fixed Diagonal Weighting Matrix

Step 0: Given D, $\mathfrak{B}_0$, the sequence $\{\alpha_k\}$ and the constraints, set $k = 0$.

Step 1: Compute the solution of the optimization problem

$$x_k = \arg\min \left\{ < x - \mathfrak{B}_k, D (x - \mathfrak{B}_k) > \,\middle|\, N^T x = b \right\}. \tag{3.1}$$

Step 2: Interact with the decision maker. If $x_k \in \Omega$, <u>stop</u>. Otherwise, the decision maker is required to specify the preferred value x_p, and hence,

$$\delta_k = x_p - x_k. \tag{3.2}$$

Step 3: Update the bliss values

$$\mathfrak{B}_{k+1} = \mathfrak{B}_k + \alpha_k \delta_k, \tag{3.3}$$

set $k = k + 1$ and go to Step 1.

The relationship of x_{k+1}, x_k and α_k, δ_k is summarised in the following results. The choice of the objective function may also be based on criteria other than $x_k \in \Omega$. For example, in the linear stochastic dynamical systems, the weighting matrices might be chosen to yield a stable minimum variance controller (see Engwerda and Otter, 1989). However, in the present study we consider the equivalence properties of the approach adopted in this section to respecification, discussed in Section 4, of nondiagonal weighting matrices while keeping $\mathfrak{B}$ fixed. This equivalence also provides the key to the complexity and convergence of the policy design process. In addition, as it is possible to decompose a symmetric matrix into a sequence of rank one updates (see Fiacco and McCormick, 1968), the method in this section and the equivalence result allow the possibility of expressing non diagonal weighting matrices in terms of diagonalized objective functions.

An alternative characterization of the problem would be in terms of finding a solution that simply satisfies the constraints and Ω, without any optimality requirement. One difficulty of this approach is that Ω needs to be explicitly specified. Let us *assume* that this was possible and that Ω was characterized by the intersection of a finite number of *linear inequalities*. It can then be shown that both the above algorithm and the algorithm in the next section are related to Khachian's (1979, 1980) algorithm for computing a feasible solution to a system of linear inequalities. In this framework, the vector δ is determined by one of the (linear) constraints bounding the set Ω. In particular, δ is related to the gradient of a constraint characterizing Ω, violated at x_k (Rustem and Velupillai, 1988,b; Rustem, 1989). By invoking Khachian's result, we can thereby obtain a polynomial time complexity for both algorithms. The added advantage of the two algorithms in this paper is that they relate each iteration with an objective function and optimality that is useful to the decision maker. Since the set of acceptable policies, Ω, does not exist anywhere except in the mind of the decision maker, the interpretation and specification of δ is aided by the optimality, at each iteration, of the quadratic objective function.

Proposition 1

Assume that D is positive semi-definite, the optimal solution and Lagrange multipliers can be written as

$$x_{k+1} = x_k + \alpha_k Z (Z^T D Z)^{-1} Z^T D \delta_k \tag{3.4}$$

where $Z \in E^{n \times (n-m)}$ is an orthogonal matrix[1] such that $Z^T N = 0$, and

$$\lambda_{k+1} = \lambda_k - \alpha_k (N^T N)^{-1} N^T D \left[I - Z (Z^T D Z)^{-1} Z^T D \right] \delta_k . \tag{3.5}$$

Proof (Rustem, 1991)

To establish (3.4), we note that $x_{k+1} - x_k \in \{ x \mid N^T x = 0 \}$ and, as $N^T Z = 0$, any such vector can be written as a linear combination of the columns of Z. Thus, we have $x_{k+1} - x_k = Z w$, for some vector $w \in E^{n-m}$ and from the first order optimality condition of (3.1) we can write

$$Z^T D \left[Z w + x_k - \mathcal{B}_k - \alpha_k \delta_k \right] = Z^T N \lambda_{k+1} = 0$$

and thus

$$x_{k+1} - x_k = Z w = - Z (Z^T D Z)^{-1} Z^T D \left[x_k - \mathcal{B}_k - \alpha_k \delta_k \right] . \tag{3.6}$$

From the optimality condition at iteration k, we have $Z^T D [x_k - \mathcal{B}_k] = Z^T N \lambda_k = 0$. Thus,

[1] The choice of the orthogonal matrix Z is discussed further in Rustem and Velupillai (1988, a). The numerically stable way of generating Z is by considering the QR decomposition of N. The matrix Z is given by the last n-m columns of the matrix Q of this decomposition (see, Gill, Murray and Wright, 1981).

expression (3.4) follows from (3.6).

For (3.5), we use the optimality condition at k+1 to yield

$$\lambda_{k+1} = (N^T N)^{-1} N^T D \left[x_{k+1} - x_k + x_k - \mathcal{B}_k - \alpha_k \delta_k \right] \tag{3.7}$$

Using (3.4) and the optimality condition at k, $D (x_k - x_k^d) = N \lambda_k$, leads to expression (3.5). □

4. THE DIAGONAL VERSION OF NON-DIAGONAL QUADRATIC FORMS

We consider the equivalence of the algorithm in Section 3 with a method generating nondiagonal weighting matrices, discussed in Rustem, Velupillai, Westcott (1978) and Rustem and Velupillai (1988, a). The complexity of former is discussed in Rustem (1989) and Rustem and Velupillai (1988, b), by exploiting this equivalence . The following algorithm uses the same δ_k as in the algorithm in Section 4. It keeps the bliss values fixed but updates the weighting matrix of the quadratic optimization problem .

Algorithm : Fixed Bliss Values and Non-diagonal Weighting Matrix

Step 0: Given a positive semi-definite weighting matrix Q_0, the sequence μ_k, the bliss values $\mathcal{B}^d$ and the constraints, set $k = 0$.

Step 1: Compute the solution of the optimization problem

$$x_k = \arg\min \left\{ < x - \mathcal{B}^d, Q_k (x - \mathcal{B}^d) > \;\middle|\; N^T x = b \right\}. \tag{4.1}$$

Step 2: Interact with the decision maker. If $x_k \in \Omega$, <u>stop</u>. Otherwise, the decision maker is required to specify the preferred value x_p, and hence, $\delta_k = x_p - x_k$.

Step 3: Update the weighting matrix

$$Q_{k+1} = Q_k + \mu_k \frac{Q_k \delta_k \delta_k^T Q_k}{< \delta_k, Q_k \delta_k >} \tag{4.2}$$

Set $k = k + 1$ and go to Step 1.

The matrix Q_k computed in the algorithm above is in general non-diagonal. It is shown below that starting with an initial diagonal matrix, the above algorithm and the algorithm in Section 3 are equivalent. At each stage, the non-diagonal version above has a constant diagonal equivalent in the algorithm of Section 4. The equivalent results to Proposition 1 related to the above algorithm are discussed in Rustem and Velupillai (1988, a). We summarize these results. When Q_k is positive semi-definite, each subsequent iterate of the above algorithm is given by

$$x_{k+1} = x_k + \hat{\alpha}_k Z (Z^T Q_k Z)^{-1} Z^T Q_k \delta_k \tag{4.3,a}$$

$$\lambda_{k+1} = \lambda_k - \hat{\alpha}_k (N^T N)^{-1} N^T Q_k (I - Z (Z^T Q_k Z)^{-1} Z^T Q_k) \delta_k \tag{4.3,b}$$

$$\hat{\alpha}_k = - \frac{\mu_k < \delta_k, Q_k (x_k - \mathcal{B}^d) >}{< \delta_k, Q_k \delta_k > + \mu_k < Q_k \delta_k, Z (Z^T Q_k Z)^{-1} Z^T Q_k \delta_k >} \tag{4.3,c}$$

(see, Rustem and Velupillai, 1988,a; Theorems 1, 2, Lemma 2).

We now show the equivalence of the solution of the two quadratic optimization problems: one with the diagonal Hessian fixed and only the bliss values modified, and the other with the bliss values fixed and only the diagonal matrix modified to a non-diagonal form.

Proposition 2

Let Q_k be nonsingular. Then, there exist μ_k, α_k and $\hat{\alpha}_k$ such that for

$$\mathcal{B}_{k+1} = \mathcal{B}_k + \alpha_k \delta_k; \quad Q_{k+1} = Q_k + \mu_k \frac{Q_k \delta_k \delta_k^T Q_k}{< \delta_k, Q_k \delta_k >} \tag{4.4}$$

we have

$$\arg\min\Big\{ < x - \mathcal{B}_{k+1}, Q_k(x - \mathcal{B}_{k+1}) > \Big| N^T x = b \Big\} = \arg\min\Big\{ < x - \mathcal{B}_k, Q_{k+1}(x - \mathcal{B}_k) > \Big| N^T x = b \Big\}. \tag{4.5}$$

Moreover, we have $\hat{\alpha}_k = \alpha_k$. If α_k , μ_k are restricted such that $\alpha_k, \mu_k \geq 0$, then the choice of δ_k is restricted by the inequality $< \delta_k, Q_k (x_k - \mathcal{B}_k) > \ \leq 0$.

Proof[2]

Consider the optimality conditions of the minimization problems on both sides of (4.5). With the solution and the Lagrange multipliers denoted respectively by x_{k+1}, λ_{k+1}, the left side yields

$$Q_k (x_{k+1} - \mathfrak{B}_{k+1}) - N \lambda_{k+1} = 0.$$

The right side yields

$$\left[Q_k + \mu_k \frac{Q_k \delta_k \delta_k^T Q_k}{< \delta_k, Q_k \delta_k >} \right] (x_{k+1} - \mathfrak{B}_k) - N \lambda_{k+1} = 0.$$

Equating both optimality conditions yields

$$Q_k \mathfrak{B}_{k+1} = Q_k \mathfrak{B}_k - \left[\mu_k \frac{Q_k \delta_k \delta_k^T Q_k}{< \delta_k, Q_k \delta_k >} \right] (x_{k+1} - \mathfrak{B}_k)$$

and hence $\mathfrak{B}_{k+1} = \mathfrak{B}_k + \alpha_k \delta_k$ where

$$\alpha_k = - \mu_k \frac{< \delta_k, Q_k (x_{k+1} - \mathfrak{B}_k) >}{< \delta_k, Q_k \delta_k >} \tag{4.6}$$

It can be shown that α_k in (4.6) and $\hat{\alpha}_k$ in (4.3,c) are equivalent (see Rustem, 1990).

The inequality $< \delta_k, Q_k (x_k - \mathfrak{B}_k) > \; \leq 0$ ensures that $\alpha_k, \mu_k \geq 0$. To demonstrate this when Q_{k+1} is given as above, we see that α_k given by (4.6) is equivalent to $\hat{\alpha}_k$ and this is nonnegative if the above inequality and $\mu_k \geq 0$ are satisfied. When the bliss value is being updated and an equivalent update to Q_k is being computed, then for $\alpha_k \geq 0$, and

$$\mu_k = - \alpha_k \frac{< \delta_k, Q_k \delta_k >}{< \delta_k, Q_k (x_{k+1} - \mathfrak{B}_k) >}. \tag{4.7}$$

[2]The above proposition clearly holds for $Q_k = D$ and Q_{k+1}, as given above, is D with a rank-one update and hence it is no longer, in general, diagonal.

When Q_k is *singular*, it can be shown that (4.4) can be written as $Q_k \mathfrak{B}_{k+1} = Q_k (\mathfrak{B}_k + \alpha_k \delta_k)$ from which the relevant parts of $\mathfrak{B}_{k+1}$ can be recovered. For example, when $Q_k = D$, a diagonal matrix, clearly only those elements of $\mathfrak{B}_{k+1}$ corresponding to nonzero diagonal elements of D can be recovered. The correspondence of $\hat{\alpha}_k$ given by (4.3,c) and α_k can be established in the same way as in the following proof except that (4.3,a) is used for x_{k+1}.

We now show that $< \delta_k, Q_k (x_k - \mathfrak{B}_k) > \; \leq 0 \Rightarrow < \delta_k, Q_k (x_{k+1} - \mathfrak{B}_k)> \; \leq 0$. The inequality $< x_{k+1} - x_k, Q_{k+1} (x_{k+1} - \mathfrak{B}_k) > \; \leq 0$ follows from the optimality of x_{k+1} for the quadratic optimization with Q_{k+1}. Using the expression for Q_{k+1}, we have

$$0 \geq < x_{k+1} - x_k, Q_{k+1} (x_{k+1} - \mathfrak{B}_k) >$$

$$= < x_{k+1}-x_k, Q_k(x_{k+1}-\mathfrak{B}_k) > + \frac{\mu_k}{< \delta_k, Q_k\delta_k >} < x_{k+1}-x_k, Q_k \delta_k > \; < \delta_k, Q_k(x_{k+1}-\mathfrak{B}_k) > .$$

Since $< x_{k+1} - x_k, Q_k (x_k - \mathfrak{B}_k) > \; \geq 0$ follows from the optimality of x_k with Q_k; and $< \delta_k, Q_k (x_{k+1} - x_k)> \; \geq 0$, holds if $< \delta_k, Q_k (x_k - \mathfrak{B}_k) > \; \leq 0$ (see Rustem and Velupillai, 1988,b; Lemma 2) then we have $< \delta_k, Q_k(x_{k+1} - \mathfrak{B}_k) > \; \leq 0 \Rightarrow \mu_k \geq 0$ and the corresponding μ_k is given by (4.7). □

The extension of the above result to nonlinear constraints is straightforward. The method can be extended to the nonlinear constrained case when the diagonal equivalent of a non-diagonal quadratic function is desired. The useful analytical equivalence of α and $\hat{\alpha}$ cannot be established exactly in the nonlinear case. Clearly, if the departure of x_{k+1} from x_k is sufficiently small, then this equivalence can also be established by invoking a mean value theorem and thereby using a local linear representation of the constraints (see e.g. Rustem and Velupillai, 1988, b; Theorem 5). The following corollary summarizes the straightforward extension.

Corollary [The Extension to Nonlinear Constraints]

Let $\mathfrak{B}_{k+1}$ and Q_{k+1} be defined by (4.4) and let the constraints be given by

$$G = \left\{ x \in E^n \;\middle|\; g(x) = 0 \right\} \tag{4.8}$$

where g is twice differentiable and $g : E^n \rightarrow E^m$. Then the equivalence

$$\arg\min \left\{ < x - \mathfrak{B}_{k+1}, Q_k(x - \mathfrak{B}_{k+1}) > \;\middle|\; x \in G \right\} = \arg\min \left\{ < x - \mathfrak{B}_k, Q_{k+1}(x - \mathfrak{B}_k) > \;\middle|\; x \in G \right\}$$

holds for α_k given by (4.6).

Proof

The proof follows from the equivalence of the first order optimality conditions □

The extension to nonlinear constraints is thus easily implementable as the basic ingredients that enter α_k are δ_k and x_{k+1}. Both of these vectors are known when any one of the two quadratic problems have already been solved.

Proposition 2 relates the effect of a single update of Q_k that yields Q_{k+1} or a single update of $\mathfrak{B}_k$ that yields $\mathfrak{B}_{k+1}$. As a corollary, we consider the sequence $\{Q_k\}$ generated by the algorithm in this section and the corresponding sequence $\{\mathfrak{B}_k\}$ generated by the algorithm in Section 1.

Theorem 1 [The Diagonalizibility of Quadratic Forms]

Let the sequence $\{Q_k\}$ be generated by (4.2) and $\{\mathfrak{B}_k\}$ be generated by (3.3), let $Q_0 = D$ then the equivalence between the diagonal and non-diagonal quadratic optimizations

$$\arg\min \left\{ < x-\mathfrak{B}_k, D\,(x-\mathfrak{B}_k) > \;\middle|\; N^T x = b \right\} = \arg\min \left\{ < x-\mathfrak{B}_0, Q_k\,(x-\mathfrak{B}_0) > \;\middle|\; N^T x = b \right\} \tag{4.9,a}$$

holds if

$$D\,\mathfrak{B}_k = D\,\mathfrak{B}_0 - \sum_{i=0}^{k-1} \mu_i \frac{< \delta_i, Q_i\,(x_k - \mathfrak{B}_0) >}{< \delta_i, Q_i\,\delta_i >} Q_i\,\delta_i . \tag{4.9,b}$$

Proof

(4.9, b) follows from the following equivalence of the optimality conditions of both problems

$$D\,(x_k - \mathfrak{B}_k) = Q_k\,(x_k - \mathfrak{B}_0)$$

$$= \left[D + \sum_{i=0}^{k-1} \mu_i \frac{Q_i\,\delta_i\,\delta_i^T\,Q_i}{< \delta_i, Q_i\,\delta_i >} \right] (x_k - \mathfrak{B}_0). \qquad \Box$$

The complexity of the algorithm in Section 3 can be discussed, for Ω characterized by linear inequalities, by invoking its equivalence to the algorithm in this section and the relation of the latter to Khachian's (1979, 1980) algorithm for linear programming. As Khachian's algorithm is known to be convergent in polynomial time, its equivalence to the present algorithm would ensure the same convergence rate for the latter. It is shown in Rustem and Velupillai (1988, b, Theorems 1 and 7) that the algorithm in this section is equivalent to Khachian's algorithm *provided* that $\mathfrak{B}_0$ on the right of the equivalence (4.9,a) is shifted, at every k, in a way that will only affect the stepsize α_k or $\hat{\alpha}_k$ above (see Rustem, 1990).

CONCLUDING REMARKS

The possibility of constructing quadratic objective functions with diagonal weighting matrices, or Hessians, is desirable both in terms of computational convenience and interpretability. The extension to nonlinear constraints permits the wider applicability of the results.

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REFERENCES

Engwerda, J.C. and P.W. Otter (1989). "On the Choice of Weighting Matrices in the Minimum Variance Controller", *Automatica*, 25, 279-285.

Fiacco, A.V. and G. P. McCormick (1968). *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*. J. Wiley, New York.

Gill, P.E., W. Murray, M.H. Wright (1981). *Practical Optimization*. Academic Press, London and New York.

Khachian, L.G. (1979). "A polynomial Algorithm in Linear Programming", *Soviet Mathematics Doklady*, 20, 191-194.

Khachian, L.G. (1980). "Polynomial Algorithm in Linear Programming", *USSR Computational Mathematics and Mathematical Physics*, 20, 53-72.

Polak, E. (1971). *Computational Methods in Optimization*. Academic Press, New York and London.

Rustem, B. (1990). "On the Diagonalizibility of Quadratic Forms and the Arbitrariness of Shadow Prices", in N. Christodoulakis (Ed.) *Modelling and Control of National Economies*. Pergamon Press, Oxford.

Rustem, B. (1991). "The Diagonalizability of Quadratic Functions and the Arbitrariness of Shadow Prices", *Automatica*, 573-578.

Rustem, B., K. Velupillai, J. Westcott (1978). "Respecifying the Weighting Matrix of a Quadratic Objective Function", *Automatica*, 14, 567-582.

Rustem, B. and K. Velupillai (1988,a). "Constructing Objective Functions for Macroeconomic Decision Models: A Formalization of Ragnar Frisch's Approach", *PROPE Discussion Paper No. 69*, Imperial College, London.

Rustem, B. and K. Velupillai (1988,b). "Constructing Objective Functions for Macroeconomic Decision Models: On the Complexity of the Policy Design Process" *PROPE Discussion Paper No. 84*, Imperial College, London.