POLSKA AKADEMIA NAUK
INSTYTUT BADAŃ SYSTEMOWYCH

PROCEEDINGS OF THE 3rd ITALIAN-POLISH CONFERENCE ON APPLICATIONS OF SYSTEMS THEORY TO ECONOMY, MANAGEMENT AND TECHNOLOGY

WARSZAWA 1977

Redaktor techniczny
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PREFACE

The present volume comprises the papers from the 3rd Italian-Polish Conference which was held in [illegible] (Poland) [illegible] from [illegible] to [illegible]. The conference organized by the Centre of Control Systems and Computation of the Italian National Research Council in Rome and the Institute for Organization, Management and Control Sciences of Polish Academy of Sciences and the Ministry of Science, Higher Education and Technology in Warsaw was the third conference of this type.

The first one took place in Kraków (Poland) in 1972, the second in Pugnochiuso (Italy) in 1974.

This conference, like both the previous ones, gathered the leading representatives of many scientific centres from Italy as well as from Poland, representing also the different aspects of the systems analysis applied to the economic management, data processing and technological systems.

From the past, such meetings enrich the exchange of views and experiences between the specialists from both countries what is the mutual advantage to both sides. Since the general goal, the summarization of the systems and control theory results with the economic practice concerns the essential development of the quantitative methods and the particular applications of large complex systems.

The meaning of such meetings, however, goes far beyond that: what is happening during the conference between the established contacts are also fruitful in form of common scientific research works and — "last but not least" — forms the friendly links between the scientists of both countries which last for years.

The proceedings of the conference were divided into three parts:

1. Optimization and Control Theory.
2. Systems Theory in Economics.
3. Technological, Managerial and Information Systems.

While the first two parts are of rather homogeneous character, the third part contains the papers concerning the different types of models — for the economic, technological, management and data processing systems.

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A SURVEY OF SOME RECENT APPLICATIONS OF THE THEORY OF BILINEAR SYSTEMS TO THE ANALYSIS OF OTHER CLASSES OF NONLINEAR SYSTEMS

1. INTRODUCTION

Most papers on Economy refer to time-invariant models. The motivation of such fact lies-more in the simplicity and in the wide development of the time invariant theory than in an actual presence of that property in economic processes. In order to obtain a better fitting to the properties of processes, time varying systems are to be used. Hence in the case of linear continuous time systems

$$\begin{aligned} \dot{x}(t) &= \boldsymbol{A}x(t)+\boldsymbol{B}u(t) \\ y(t) &= \boldsymbol{C}x(t)+\boldsymbol{D}u(t) \end{aligned} \tag{1}$$

$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ matrices depending upon the time t ought to be considered. Although natural, such approach is often rejected becouse of the difficulties which are involved. To this purpose, and without facing the differences between time-varying and time-invariant systems, it seems useful to outline some remarks which arise when the problem is considered from a system-theoretic viewpoint. It is always possible to look to the time-variance of the system as if due to further inputs acting on parameters; moreover a better economical analysis often shows the actual presence of a control acting on the process through an influence on its parameters.

Models which take account of these remarks have a lot of advantages. First of all the variability of the structure of the process is taken into account. Moreover this is accomplished through a time-invariant model, becouse the time-variance is replaced by the control. Further advantages are the flexibility of the variable structure models and the existence of a wide theory. It is useful to stress that often as a variable structure model a **bilinear model** is considered, that is a system described, in the discrete-time case, by equations of the form

$$\begin{aligned} \dot{x}(t) &= \boldsymbol{A}x(t)+\boldsymbol{B}u(t)+\sum_{i=1}^{p}\boldsymbol{N}_i x(t)u_i(t) \\ y(t) &= \boldsymbol{C}x(t) \end{aligned} \tag{2}$$

where u_i denotes the i-th component of the input and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{N}_1, \ldots, \boldsymbol{N}_p$ are constant matrices. In this case not only an additive control term $\boldsymbol{Bu}(t)$ is considered, but also a multiplicative control term, $\sum \boldsymbol{N}_i\, \boldsymbol{x}(t) \cdot u_i(t)$, which is bilinear in the input and in the state. The effect of the latter term on the system structure can be outlined rewriting (2) in the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= [\boldsymbol{A} + \sum_{i=1}^{p} \boldsymbol{N}_i u_i(t)]\, \boldsymbol{x}(t) + \boldsymbol{Bu}(t) \\ y(t) &= \boldsymbol{Cx}(t) \end{aligned} \tag{3}$$

This form can be seen as derived from (1) by replacing the matrix $\boldsymbol{A}$ with a matrix whose entries are a constant term and a linear in the input term. From this viewpoint the bilinear model looks as the simplest variable structure model, so that it plays with respect to the variable structure systems the same role as the linear model with respect to the fixed structure systems.

Due to these motivations, the bilinear systems constitute an approximate model which allows a relevant progress with respect to linear time-invariant model, althought it cannot describe the complexity of a general variable structure model. The success of bilinear systems is also due to the possibility of developing their theory in a complete manner, becouse of the simplicity of the equations (2).

In order to consider more complex models, recently cases were studied which are linear in the state but the control law is more general than the linear one. At the same time cases were studied in which the system is linear with respect to the input but not to the state, to take into account that feedback does not maintain the linearity in the state.

The purpose of this paper is to give a survey of the recent results in the above mentioned directions, showing how much these results are connected to the previous one on bilinear systems, both for its usefulness in establishing the results and for the suggestion of the operative techniques.

2. DESCRIPTIONS LINEAR WITH RESPECT TO THE STATE

We consider the state-space description given by the following differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}[u(t)]\, \boldsymbol{x}(t) + \boldsymbol{B}[u(t)] \tag{2.1'}$$

$$y(t) = \boldsymbol{Cx}(t) \tag{2.1''}$$

where $\boldsymbol{x}(t) \in R^n$, $u(t) \in R$, $y(t) \in R$ and $A(\cdot)$ (resp. $B(\cdot)$) are $n \times n$ (resp $n \times 1$) matrices of analytic functions of u. A simple manipulation shows that, by taking

$$z(t) = \begin{pmatrix} x(t) \\ 1 \end{pmatrix}, \qquad F(u) = \begin{pmatrix} A(u) & B(u) \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{H} = (\boldsymbol{C}\ \ 0) \tag{2.2}$$

the differential equation (2.1) can be reduced to the following

$$z(t) = \boldsymbol{F}[u(t)]\, z(t) \tag{2.3'}$$

$$y(t) = \boldsymbol{H} z(t) \tag{2.3''}$$

with the right-hand-side of (2.3′) linear function of z for each fixed u.

The interest in the knowledge of the properties of bilinear state-space descriptions follows from the possibility of reducing the state-space model described by eq. (2.3) to the cascade connection of a set of **non-linear memoryless** elements followed by a **bilinear** state-space model.

With the assumption

$$\boldsymbol{F}(u) = \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} \boldsymbol{N}_{ij} f_{ij}(u) \tag{2.4}$$

where the $\boldsymbol{N}_{ij}$ are $(n+1)\times(n+1)$ matrices such that

$$n_{rs} = 1 \quad \text{for } r=i \text{ and } s=j \tag{2.5'}$$

$$n_{rs} = 0 \quad \text{otherwise} \tag{2.5''}$$

eq. (2.3) can be rewritten as

$$z(t) = \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} \boldsymbol{N}_{ij} v_{ij}(t)\, z(t) \tag{2.6}$$

with

$$v_{ij}(t) = f_{ij}[u(t)] \tag{2.7}$$

The functions (2.7) associate with the input $u(t)$, by means of $(n+1)^2$ non-linear memoryless elements, a set of $(n+1)^2$ input functions $v_{ij}(t)$ to the bilinear state-space description (2.6) (*).

The equivalnce between a description linear in the state and a description resulting from the cascade connection of nonlinear memoryless elements and a bilinear description clearly shows the importance of the theory of bilinear descriptions in the investigation of the present (and more general) class of descriptions. As concerns **the input-output function** resulting from a given initial state z (t_0), it is possible to expand it into a series of functionals, with the aid of the formulae available for the Volterra series expansion of the input-output function of a bilinear description. One has

$$y(t) = w^{(0)}(t-t_0) + \sum \int_{t_0}^{t} w_{ij}^{(1)}(t-t_1) f_{ij}[u(t_1)]\, \mathrm{d}t_1 +$$

$$+ \sum \int_{t_0}^{t}\int_{t_0}^{t} w_{ijrs}^{(2)}(t-t_1, t-t_2) f_{ij}[u(t_1)]\, f_{rs}[u(t_2)]\, dt_1\, dt_2 + \dots \tag{2.8}$$

*) The number of inputs to the bilinear model, here taken as $(n+1)^2$ can be simply reduced — where possible — to the number of linearly independent functions, on R, of the set $\{f_{ij}(u)\}_i$, $_j = 1, \dots, n+1$

where the kernels $w^{(0)}(\cdot)$, $w^{(2)}(\cdot,\cdot)$, ... can be deduced from the expressions given in [1].

In order to reduce the (2.8) to a Volterra series expansion with respect to the input $u(\cdot)$, it is sufficient to perform a Taylor series expansion of the $f_{ij}(u)$, that are analytic by assumption.

In a similar way it is possible to perform the study of the **structure properties** (reachability, observability) and the **state-space decomposition;** this can be done by simply transferring the results of the theory of bilinear systems [2] to the only part of the description, i. e. the bilinear one, where dynamic effects take place. In this way it is possible to deduce some results already established in autonomous way [3].

In the present class of systems can be included those in which $A(u)$ and $B(u)$ are polinomial functions of u. These have been investigated in [4].

3. DESCRIPTIONS LINEAR WITH RESPECT TO THE CONTROL

We consider now the description given by the differential equation

$$\dot{x}(t) = f[x(t)] + g[x(t)]\,u(t) \tag{3.1'}$$

$$y(t) = Cx(t) \tag{3.1''}$$

where $f(\cdot)$ and $g(\cdot)$ are analytic functions of x. These desciiptions have been called "linear-analytic" and studied in [5].

Also in this case the knowledge of the theory of bilinear state-space descriptions is essential to the study. The main result concerns the computation of the Volterra series expansion of the input-output function resulting from the initial state $x(t_o)=0$. It has been shown (*) that with the description (3.1) it is possible to associate an infinite collection $\{S_n\}$ of bilinear descriptions

$$\dot{x}_n = A_n x_n + N_n x_n u + B_n u \tag{3.2}$$

having the following properties

(a) for all n, the first n Volterra kernels of the description S_n coincide with these of S_{n+1};

(b) the first n terms of the Volterra series expansion of the input-output function associated with (3.1) coincide with the first n terms of the Volterra series expansion of the input-output function associated with S_n.

Thanks to this property, the computation of the input-output function of a description linear with respect to the control is reduced to the use of formulae already etablished for bilinear descriptions.

The associated bilinear description (3.2) is determined in the following way. Denoting with $x^{[i]}$ the $\binom{n+i-1}{i}$-dimensional vector of homogeneous forms of order i in the n variables $x_1, \ldots, x_n$, we consider the expansions (note that $n = n^{[1]}$)

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*) Under the assumption that on any interval $[O, T]$ the solution of $\dot{x}(t) = f[x(t)]$ with $x(0) = 0$ exists.

$$f(x) = \sum_{i=1}^{\infty} F^{(i)} x^{[i]} \tag{3.3}$$

$$g(x) = \sum_{i=0}^{\infty} \Delta^{(i)} x^{[i]}$$

where $F^{(i)}$ and $G^{(i)}$ are constant matrices of proper dimensions and, without loss of generality, it has been assumed that $f(0)=0$. The matrices A_n, H_n, B_n that characterize (3.2) can be computed from the $F^{(i)}$ and $G^{(i)}$ in the following way

$$A_n = \begin{pmatrix} F^{(1)}_{[1,1]} & F^{(2)}_{[1,2]} & \dots & F^{(n)}_{[1,n]} \\ 0 & F^{(1)}_{[2,2]} & \dots & F^{(n-1)}_{[2,n]} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & F^{(1)}_{[n,n]} \end{pmatrix} \qquad N_n = \begin{pmatrix} G^{(1)}_{[1,1]} & G^{(2)}_{[1,2]} & \dots & G^{(n)}_{[1,n]} \\ G^{(0)}_{[2,1]} & G^{(1)}_{[2,2]} & \dots & G^{(n-1)}_{[2,n]} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & G^{(1)}_{[n,n]} \end{pmatrix}$$

$$B_n = \begin{pmatrix} G^{(0)}_{[1,0]} \\ 0 \\ \dots \\ 0 \end{pmatrix} \tag{3.4}$$

where the $F^{(i)}_{(r,s)}$ are computed recursively from $F^{(i)}_{(1,i)} \triangleq F^{(i)}$ and the $G^{(i)}_{(r,s)}$ are computed recursively from $G^{(i)}_{(1,i)} \triangleq G^{(i)}$ (*).

4. CONCLUDING REMARKS

The considerations presented in the previous sections can be easily extended to cover the case in which

$$\dot{x}(t) = f_0[x(t)] + g_0[u(t)] + \sum_{i=1}^{m} f_i[x(t)]\, g_i[u(t)] \tag{4.1}$$

In this case the description can be reduced to the cascade connection of some nonlinear memoryless elements and a description linear in the control, that can be analyzed in the way described in section 3.

(*) The rule is the following one. The matrix $F^{(i)}_{(i,\,i)}$ is introduced in the differential equation

$$\frac{d}{dt}\, x(t) = f_{i,\,i}\; x^{[i]} \tag{a}$$

Then both sides of (a) are multiplied by all the elements of $x^{[r]}$. Thr result can be rearranged for a differential equation with $d/dt[x^{[r+}{}_{i}{}^{]}]$ on the left and $x^{[i+r]}$ on the right, i. e. an equation of the form

$$\frac{d}{dt}\, x^{[r+i]}(t) = F_{(r+1,\; i+1)}\; x^{[i+r]}$$

In this class can be included these descriptions in which the right-hand-side of the differential equation is a polynomial function of x and u, whose interest has been recently exploited in [6] (in the case of discrete-time systems).

REFERENCES

[1] C. BRUNI, G. DI PILLO, G. KOCH: "Bilinear Systems. An Appealing Class of Nearly linear Systems in Theory and Applications" IEEE Trans. on A. C., (1975).
[2] P. d'ALESSANDRO, A. ISIDORI, A. RUBERTI: "Realization and Structure theory of bilinear dynamical systems", SIAM J. Control, 12 (1974) pp. 517—535.
[3] C. GORI-GIORGI, S. MONACO: "Structure Theory of Variable Structure Systems" Rapporto dell'Ist. di Automatica e del C.S.S. C.C.A. del C.N.R., 1976.
[4] E. D. SONTAG: "On the realizations of discrete-time nonlinear systems", Notices A.M.S. (to appear).
[5] R. W. BROCKETT: "Volterra series and geometric control theory", Automatica, 12 (1976), pp. 167—176.
[6] E. D. SONTAG, Y. ROUCHALEAU: "On discrete-time polynomial systems", (to appear).

SUMMARY

The bilinear models of systems allows a relevant progress with respect to linear time-invanant models, in the wide range of applications. The success of bilinear models is also due to the possibility of developing their theory in a complete manner becouse of the simplicity of the equations (2).

Recently some cases were studied which are linear in the state but the control law is more general than the linear one. Also there were investigated cases in which the system is linear with respect to the input but not to the state, allows to take into consideration that feedback does not maintain the linearity in the state.

The purpose of this paper is to give a survey of the recent results in the above mentioned directions, showing how much these results are connected with those for bilinear systems.

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