Modern Approaches in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
Volume I: Foundations

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α-properties of fuzzy relations and aggregation procedure

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Abstract

In the paper the problem of preservation of properties of fuzzy relations during aggregation process is considered. It means that properties of fuzzy relations $R_1, \ldots, R_n$ on a set X are compared with properties of the aggregated fuzzy relation $R_F = F(R_1, \ldots, R_n)$, where F is a function of the type $F : [0,1]^n \to [0,1]$. There are discussed α-properties (which may be called graded properties) as reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness and transitivity, where $\alpha \in [0,1]$. Fuzzy relations with a given graded property are considered (there may be diverse grades of the same property) and the obtained grade of the aggregated fuzzy relation is provided. There is also discussed the „converse" problem. Namely, relation $R_F = F(R_1, \ldots, R_n)$ is assumed to have a graded property and the properties of relations $R_1, \ldots, R_n$ are examined (possibly with some assumptions on F).

Keywords: fuzzy relations, properties of fuzzy relations, aggregation functions

1 Introduction

Since Zadeh has introduced the definition of fuzzy relations [23, 24], the theory of them was developed by several authors. Thanks to the „fuzzy environment" we

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may discuss diverse types of fuzzy relation properties. For example, graded properties of fuzzy relations were observed in [16] and α-properties were introduced in [4]. These properties may be understood as properties to some grade α, where $\alpha \in [0, 1]$.

Aggregation functions, including means [17], are now widely investigated and there are a few monographes devoted to this topic, e.g. [1, 2, 15]. Aggregation is a fundamental process in multi-criteria decision making and in other scientific disciplines where the fusion of different pieces of information for obtaining the final result is important. For example, in the multi-criteria decision making a finite set of alternatives $X = \{x_1, \ldots, x_m\}$ and a finite set of criteria on the base of which the alternatives are evaluated $K = \{k_1, \ldots, k_n\}$ may be considered. Fuzzy relations $R_1, \ldots, R_n$ on a set X corresponding to each criterion are provided. With the use of a function F the aggregated fuzzy relation $R_F = F(R_1, \ldots, R_n)$ is obtained and it is supposed to help decision makers to make up their mind. It is useful to know which properties of fuzzy relations $R_1, \ldots, R_n$ are transposed to the relation R.

There are several works contributed to the problem of preservation of properties of fuzzy relations during aggregation process, e.g. [14, 20, 21, 22]. In this paper the problem of preservation of graded properties of fuzzy relations (cf. [8, 10, 12]) is examined. A finite number of fuzzy relations having a given graded property is considered (there can be diverse grades of the same property) and the obtained grade of the aggregated fuzzy relation is provided. There are discussed several graded properties: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness and transitivity. There is also considered another problem. Namely, relation $R_F = F(R_1, \ldots, R_n)$ is assumed to have a graded property and relations $R_1, \ldots, R_n$ are examined whether they have the same property. Appropriate assumptions on F to fulfill the required property are proposed.

In Section 2, useful definitions are collected. In Section 3, motivation from real-life situations to consider such theoretical problem is presented. Finally, in Section 4 graded properties: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness and transitivity are examined one by one.

2 Preliminaries

Now we recall some definitions which will be helpful in our investigations.

Definition 1 ([23]). *A fuzzy relation in $X \neq \emptyset$ is a function $R : X \times X \to [0, 1]$. The family of all fuzzy relations in X is denoted by FR(X).*

The notation FR(X) will be used in the sequel and to make the statements shorter the notion of a „fuzzy relation" will be sometimes replaced with the notion of a „relation". It will not be ambiguous since only fuzzy relations are considered in the paper. With the use of n-argument functions F we aggregate given fuzzy relations $R_1, \dots, R_n$ for a fixed $n \in \mathbb{N}$.

Definition 2 ([18]). *Let $F : [0,1]^n \to [0,1]$, $R_1, \dots, R_n \in FR(X)$. By aggregated fuzzy relation we call $R_F \in FR(X)$,*

$$R_F(x,y) = F(R_1(x,y), \dots, R_n(x,y)), \;\; x, y \in X.$$

A function F preserves a property of fuzzy relations if for every relation $R_1, \dots, R_n \in FR(X)$ having this property, R_F also has this property.

Example 1. *Projections $P_k(t_1, \dots, t_n) = t_k$, $k \in \{1, \dots, n\}$ preserve each property of fuzzy relations because for $F = P_k$ we get $R_F = R_k$.*

Definition 3 ([2]). *Let $n \geqslant 2$. A function $F : [0,1]^n \to [0,1]$ is called an aggregation function, if it is increasing with respect to any variable*

$$\underset{s_1,\dots,s_n,t_1,\dots,t_n \in [0,1]}{\forall} \Big(\underset{1 \leqslant k \leqslant n}{\forall} s_k \leqslant t_k \Big) \Rightarrow F(s_1, \dots, s_n) \leqslant F(t_1, \dots, t_n) \tag{1}$$

and $F(0, \dots, 0) = 0$, $F(1, \dots, 1) = 1$.

Triangular norms and conorms are examples of binary aggregation functions.

Definition 4 ([19]). *A triangular norm $T : [0,1]^2 \to [0,1]$ (a triangular conorm $S : [0,1]^2 \to [0,1]$) is an arbitrary associative, commutative, increasing in both variables function having a neutral element $e = 1$ ($e = 0$).*

Basic triangular norms and conorms are presented below.

Example 2 ([19], p. 6). *For arbitrary $s, t \in [0,1]$ we have functions:*
- *lattice, $T_M(s,t) = \min(s,t)$, $\quad S_M(s,t) = \max(s,t)$,*
- *Łukasiewicz, $T_L(s,t) = \max(s+t-1, 0)$, $\quad S_L(s,t) = \min(s+t, 1)$,*
- *product, $T_P(s,t) = st$, $\quad S_P(s,t) = s + t - st$,*
- *drastic, $T_D(s,t) = \begin{cases} 0, & s,t < 1 \\ s, & t = 1 \\ t, & s = 1 \end{cases}$, $\quad S_D(s,t) = \begin{cases} 1, & s,t > 0 \\ s & t = 0 \\ t, & s = 0 \end{cases}$.*

Thanks to the associativity property triangular norms and conorms may be extended to n-argument functions. Special case of aggregation functions are the ones which are idempotent.

Lemma 1 ([18], Proposition 5.1)**.** *Every function $F : [0,1]^n \to [0,1]$ increasing in each variable and idempotent*

$$\underset{t\in[0,1]}{\forall} \quad F(t,\dots,t) = t \tag{2}$$

fulfils

$$\underset{t_1,\dots,t_n\in[0,1]}{\forall} \quad \min(t_1,\dots,t_n) \leqslant F(t_1,\dots,t_n) \leqslant \max(t_1,\dots,t_n). \tag{3}$$

Here we present examples of functions which fulfil (3).

Example 3. *Let $\varphi : [0,1] \to \mathbb{R}$ be a continuous, strictly monotonic function. A quasi-linear mean (cf. [18], p. 112) is the function*

$$F(t_1,\dots,t_n) = \varphi^{-1}(\sum_{i=1}^{n} w_i\varphi(t_i)),\ t_1,\dots,t_n \in [0,1], \sum_{i=1}^{n} w_i = 1, w_i \in [0,1].$$

Particularly, we obtain weighted arithmetic means

$$F(t_1,\dots,t_n) = \sum_{i=1}^{n} w_i t_i,\ t_1,\dots,t_n \in [0,1], \sum_{i=1}^{n} w_i = 1, w_i \in [0,1].$$

Median (cf. [2], p. 21) is the function

$$med(t_1,\dots,t_n) = \begin{cases} \frac{s_k+s_{k+1}}{2}, & \text{if } n = 2k \\ s_{k+1}, & \text{if } n = 2k+1 \end{cases},\ t_1,\dots,t_n \in [0,1],$$

where $(s_1,\dots,s_n)$ is the increasing permutation of the sequence $(t_1,\dots,t_n)$, such that $s_1 \leqslant \dots \leqslant s_n$. An aggregation function

$$F(t_1,\dots,t_n) = p \max_{1\leqslant k\leqslant n} t_k + (1-p) \min_{1\leqslant k\leqslant n} t_k \tag{4}$$

is idempotent, where $p \in (0,1)$ is a parameter.

There are some connections between functions. For example, we may consider domination of one function over another.

Definition 5 (cf. [22], Definition 2.5)**.** *Let $m, n \in \mathbb{N}$. A function $F : [0,1]^m \to [0,1]$ dominates a function $G : [0,1]^n \to [0,1]$ ($F \gg G$), if for arbitrary matrix $[a_{ik}] = A \in [0,1]^{m\times n}$ we have*

$$F(G(a_{11},\dots,a_{1n}),\dots,G(a_{m1},\dots,a_{mn})) \geqslant$$
$$G(F(a_{11},\dots,a_{m1}),\dots,F(a_{1n},\dots,a_{mn})). \tag{5}$$

Lemma 2. *Let* $G : [0,1]^n \to [0,1]$ *be increasing,* $m = 2$ *(cf.* (5)*). Thus* $\min \gg G$ *([22], p. 16) and* $G \gg \max$ *(cf. [5], Theorem 2), so for* $s_1, ..., s_n, t_1, ..., t_n \in [0,1]$ *we have respectively*

$$\min(G(s_1, ..., s_n), G(t_1, ..., t_n)) \geqslant G(\min(s_1, t_1), ..., \min(s_n, t_n)). \quad (6)$$

$$G(\max(s_1, t_1), ..., \max(s_n, t_n)) \geqslant \max(G(s_1, ..., s_n), G(t_1, ..., t_n)). \quad (7)$$

Theorem 1. *An increasing in each variable function* $F : [0,1]^n \to [0,1]$ *dominates minimum (*$F \gg \min$*) if and only if*

$$F(t_1, \ldots, t_n) = \min(f_1(t_1), \ldots, f_n(t_n)), \ \ t_1, \ldots, t_n \in [0,1], \quad (8)$$

where functions $f_k : [0,1] \to [0,1]$ *are increasing for* $k = 1, \ldots, n$ *(cf. [22], Proposition 5.1).*
An increasing in each variable function $F : [0,1]^n \to [0,1]$ *is dominated by maximum* ($\max \gg F$*) if and only if*

$$F(t_1, \ldots, t_n) = \max(f_1(t_1), \ldots, f_n(t_n)), \ \ t_1, \ldots, t_n \in [0,1], \quad (9)$$

where functions $f_k : [0,1] \to [0,1]$ *are increasing for* $k = 1, \ldots, n$*.*

Example 4 (cf. [20])**.** *Here are examples of functions fulfilling* (8)*:*
if $f_k(t) = t$*,* $k = 1, \ldots, n$*, then* $F = \min$,
if for some $k \in \{1, \ldots, n\}$*,* $f_k(t) = t$*,* $f_i(t) = 1$ *for* $i \neq k$*, then* $F = P_k$*,*
if $f_k(t) = \max(1 - v_k, t)$*,* $v_k \in [0,1]$*,* $k = 1, \ldots, n$*,* $\max\limits_{1 \leqslant k \leqslant n} v_k = 1$*, then* F *is the weighted minimum*

$$F(t_1, \ldots, t_n) = \min_{1 \leqslant k \leqslant n} \max(1 - v_k, t_k), \quad (10)$$

where $t = (t_1, \ldots, t_n) \in [0,1]^n$*.*
Here are examples of functions fulfilling (9)*:*
if $f_k(t) = t$*,* $k = 1, \ldots, n$*, then* $F = \max$,
if for some $k \in \{1, \ldots, n\}$*,* $f_k(t) = t$*,* $f_i(t) = 0$ *for* $i \neq k$*, then* $F = P_k$*,*
if $f_k(t) = \min(v_k, t)$*,* $v_k \in [0,1]$*,* $k = 1, \ldots, n$*,* $\max\limits_{1 \leqslant k \leqslant n} v_k = 1$*, then* F *is the weighted maximum*

$$F(t_1, \ldots, t_n) = \max_{1 \leqslant k \leqslant n} \min(v_k, t_k), \quad (11)$$

where $t = (t_1, \ldots, t_n) \in [0,1]^n$*.*

Lemma 3 (cf. [12]). *If a function $F : [0,1]^n \to [0,1]$ is increasing in each variable and has a neutral element $e = 1$, i.e.*

$$\underset{t\in[0,1]}{\forall} \quad \underset{1\leqslant k\leqslant n}{\forall} F(1,\ldots,1,t,1,\ldots,1) = t, \tag{12}$$

where t is at the k-th position, then $F \leqslant \min$.
If a function $F : [0,1]^n \to [0,1]$ is increasing in each variable and has a neutral element $e = 0$, i.e.

$$\underset{t\in[0,1]}{\forall} \quad \underset{1\leqslant k\leqslant n}{\forall} F(0,\ldots,0,t,0,\ldots,0) = t, \tag{13}$$

where t is at the k-th position, then $F \geqslant \max$.

Here are recalled definitions of concepts connected with fuzzy relations.

Definition 6 (cf. [23]). *Let $R \in FR(X)$, $\alpha \in [0,1]$. The α-cut of a fuzzy relation R is the relation*

$$R_\alpha = \{(x,y) \in X \times X : R(x,y) \geqslant \alpha\}. \tag{14}$$

The strict α-cut of a fuzzy relation R is the relation

$$R^\alpha = \{(x,y) \in X \times X : R(x,y) > \alpha\}. \tag{15}$$

Definition 7 (cf. [23]). *Let $R, S \in FR(X)$. The composition of relations R and S is called the relation*

$$(R \circ S)(x,z) = \sup_{y\in X} \min(R(x,y), S(y,z)), \quad (x,z) \in X \times X. \tag{16}$$

The power of a relation R is called the sequence $R^1 = R$ and $R^{n+1} = R^n \circ R$ for $n \in \mathbb{N}$.

Remark 1. *If $card\ X = n$, $X = \{x_1,\ldots,x_n\}$, then a relation $R \in FR(X)$ may be presented by a matrix $R = [r_{ik}]$, where $r_{ik} = R(x_i,x_k)$, $i,k = 1,\ldots,n$.*

3 Motivation

In this section the idea of multicriteria decision making is recalled. Presented problem is related to considerations provided in this paper.

Let $card\ X = m$, $m \in \mathbb{N}$, $X = \{x_1,\ldots,x_m\}$ be a set of alternatives.

A decision maker has to:
— choose among alternatives („choice problem"),
— rank („ranking problem"),
— part („cluster problem").
Let $K = \{k_1, \ldots, k_n\}$ be a set of criteria on the base of which the alternatives are evaluated. $R_1, \ldots, R_n$ be fuzzy relations corresponding to each criterion represented by matrices, where $R_k : X \times X \to [0,1]$, $k = 1, \ldots, n$, $n \in \mathbb{N}$, $R_k(x_i, x_j) = r_{ij}^k$, $1 \leqslant i, j \leqslant m$. We assume that for example:
r_{ij}^k – an intensity with which x_i is better than x_j under $k \in K$,
$r_{ij}^k = 1$ – „x_i is absolutely better than x_j under criterion k",
$r_{ij}^k = 0$ – „x_j is absolutely better than x_i under criterion k",
$r_{ij}^k = 0.5$ – „x_i is equally good as x_j under criterion k".

Relation $R_F = F(R_1, \ldots, R_n)$ is supposed to help the decision makers to make up their mind. Some functions F maybe more adequate for aggregation than the others since they may (or not) preserve the required properties of individual fuzzy relations $R_1, \ldots, R_n$. According to some experimental works [25] weighted arithmetic mean and function (4) are the aggregation functions which occur the most often in the process of human decision making.

Application of such considerations is presented by a numerical example in [21] where the choice or ranking problems of a set of alternatives evaluated by fuzzy preference relations using the aggregation functions are considered. It is shown how the properties of the aggregated fuzzy relation $R_F = F(R_1, \ldots, R_n)$, depending on the properties of the individual fuzzy relations $R_1, \ldots, R_n$, help to solve the given problem.

4 α-properties of fuzzy relations

Now, dependencies related to α-properties in the context of aggregation process, between relations $R_1, \ldots, R_n$ on a set X and the aggregated fuzzy relation $R_F = F(R_1, \ldots, R_n)$ will be investigated. Moreover, some previous results will be recalled.

Definition 8 ([4], p. 75, [12]). *Let $\alpha \in [0,1]$. A relation $R \in FR(X)$ is:*

- *α-reflexive, if* $\forall_{x \in X} \; R(x,x) \geqslant \alpha$*,*
- *α-irreflexive, if* $\forall_{x \in X} \; R(x,x) \leqslant 1 - \alpha$*,*

- *totally* α*-connected, if* $\underset{x,y\in X}{\forall}\ \max(R(x,y),R(y,x))\geqslant\alpha$,
- α*-connected, if* $\underset{x,y,x\neq y\in X}{\forall}\ \max(R(x,y),R(y,x))\geqslant\alpha$,
- α*-asymmetric, if* $\underset{x,y\in X}{\forall}\ \min(R(x,y),R(y,x))\leqslant 1-\alpha$,
- α*-antisymmetric, if* $\underset{x,y,x\neq y\in X}{\forall}\ \min(R(x,y),R(y,x))\leqslant 1-\alpha$,
- α*-symmetric, if* $\underset{x,y\in X}{\forall}\ R(x,y)\geqslant 1-\alpha\Rightarrow R(y,x)\geqslant R(x,y)$,
- α*-transitive, if for all* $x,y,z\in X$
  $\min(R(x,y),R(y,z))\geqslant 1-\alpha\Rightarrow R(x,z)\geqslant\min(R(x,y),R(y,z))$.

Let us notice that conditions for α-symmetry and α-transitivity may be written in a more convenient way.

Corollary 1 ([9])**.** *Let* $\alpha\in[0,1]$. *A relation* $R\in FR(X)$ *is* α*-symmetric if and only if*

$$\underset{x,y\in X}{\forall}\ R(x,y)\geqslant 1-\alpha\Rightarrow R(y,x)=R(x,y). \tag{17}$$

Corollary 2 (cf. [7], Theorem 10)**.** *Let* $R\in FR(X)$. *Relation* R *is* α*-transitive if and only if*

$$\underset{x,y\in X}{\forall}\ R^2(x,y)\geqslant 1-\alpha\Rightarrow R(x,y)\geqslant R^2(x,y). \tag{18}$$

When a fuzzy relation R is not e.g. asymmetric, then the greatest value of α for which it is α–asymmetric one can find in the following way

Corollary 3 ([9])**.** *Let* $R\in FR(X)$,

$$\alpha_0=1-\sup_{x,y\in X}\min(R(x,y),R(y,x)),\quad \beta_0=1-\sup_{x\neq y}\min(R(x,y),R(y,x)),$$

$$\gamma_0=\inf_{x,y\in X}\max(R(x,y),R(y,x)),\quad \delta_0=\inf_{x\neq y}\max(R(x,y),R(y,x)),$$

$$\mu_0=\inf_{x\in X}R(x,x),\quad \nu_0=\inf_{x\in X}(1-R(x,x))=1-\sup_{x\in X}R(x,x).$$

Thus a relation R *is:* α*–asymmetric for* $\alpha\in[0,\alpha_0]$, β*–antisymmetric for* $\beta\in[0,\beta_0]$, *totally* γ*–connected for* $\gamma\in[0,\gamma_0]$, δ*–connected for* $\delta\in[0,\delta_0]$, μ*–reflexive for* $\mu\in[0,\mu_0]$ *and* ν*–irreflexive for* $\nu\in[0,\nu_0]$.

Example 5. *Let card $X = 2$, $R \in FR(X)$, where*

$$R = \begin{bmatrix} 0.7 & 0.2 \\ 0.5 & 0.4 \end{bmatrix}.$$

The relation R is totally α–connected and α-reflexive for $\alpha \in [0, 0.4]$ and α–connected for $\alpha \in [0, 0.5]$. It is α-asymmetric and α-irreflexive for $\alpha \in [0, 0.3]$ and α-antisymmetric for $\alpha \in [0, 0.8]$.

For checking the α-transitivity of a fuzzy relation R the composition of R by itself will be useful.

Corollary 4. *Let $R \in FR(X)$,*

$$\alpha_0 = 1 - \sup_{x \neq y \in X} \max(R(x,y), R(y,x)), \quad \text{if } R(x,y) \neq R(y,x),$$

$$\beta_0 = 1 - \sup_{x,y \in X} R^2(x,y), \quad \text{if } R(x,y) < R^2(x,y).$$

Thus a relation R is α-symmetric for $\alpha \in [0, \alpha_0)$ and β-transitive for $\beta \in [0, \beta_0)$.

For α-symmetry and α-transitivity properties see Examples 7 and 8. The presented α-properties (graded properties) for $\alpha = 1$ become the basic properties of fuzzy relations [24]. Graded properties are „fuzzy versions" of properties introduced by Zadeh. It means that, if a fuzzy relation, e.g. is not reflexive, it may be reflexive to some grade α, where $\alpha \in [0, 1]$. Taking into account $\alpha = 0$, each fuzzy relation is 0-reflexive, 0-irreflexive, 0-asymmetric, 0-antisymmetric, 0-connected and totally 0-connected. However, it is not true for graded symmetry and transitivity.

Example 6. *Let card $X = 3$, relations $R, S \in FR(X)$ be presented by matrices:*

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The relation R is not 0-transitive because $\min(r_{13}, r_{31}) = 1$ but $0 = r_{11} < \min(r_{13}, r_{31}) = 1$. The relation S is not 0-symmetric because $r_{31} = 1$ and $0 = r_{13} < r_{31} = 1$.

Theorem 2 ([9])**.** *Let $\alpha \in [0, 1]$, $R \in FR(X)$. A fuzzy relation R is totally α-connected (α-connected, α-reflexive) if and only if relation R_α is totally connected (connected, reflexive). A fuzzy relation R is α-asymmetric (α-antisymmetric, α-irreflexive) if and only if relation $R^{1-\alpha}$ is asymmetric (antisymmetric, irreflexive). If a fuzzy relation R is α-transitive, then relation $R_{1-\alpha}$ is transitive ([11]). If a fuzzy relation R is α-symmetric, then relation $R_{1-\alpha}$ is symmetric.*

Proof. We will show the property for asymmetry. Let $\alpha \in [0,1]$, $R \in FR(X)$, $x, y \in X$. Let us see that

$$R \text{ is asymmetric } \Leftrightarrow \underset{x,y\in X}{\forall} \sim (\min(R(x,y), R(y,x)) > 1-\alpha) \Leftrightarrow$$

$$\underset{x,y\in X}{\forall} \sim (R(x,y) > 1-\alpha \text{ and } R(y,x) > 1-\alpha) \Leftrightarrow$$

$$\underset{x,y\in X}{\forall} \sim ((x,y) \in R^{1-\alpha} \text{ and } (y,x) \in R^{1-\alpha}) \Leftrightarrow$$

$$\underset{x,y\in X}{\forall} ((x,y) \notin R^{1-\alpha} \text{ or } (y,x) \notin R^{1-\alpha}) \Leftrightarrow$$

$$\underset{x,y\in X}{\forall} ((x,y) \in R^{1-\alpha} \Rightarrow (y,x) \notin R^{1-\alpha}) \Leftrightarrow R^{1-\alpha} \text{ is asymmetric.}$$

As a result a relation $R \in FR(X)$ is α–asymmetric if and only if $R^{1-\alpha}$ is asymmetric. Proofs for the remaining properties are analogous. □

Similar characterizations for other properties for fuzzy relations one may find in [6] (Theorem 1). The conditions for α-symmetry and α-transitivity are only the sufficient ones.

Example 7 ([9]). *Let card $X = 2$, $R \in FR(X)$,*

$$R = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.4 \end{bmatrix}.$$

The cuts R_β are symmetric for $\beta \in [0,0.5] \cup (0.7,1]$, so the cuts $R_{1-\alpha}$ have this property for $\alpha \geqslant 0.5$ and $\alpha < 0.3$. Relation R is α-symmetric for $\alpha \in [0,0.3)$, as a result for $\alpha = 0.5$, the cut $R_{0.5}$ is symmetric, while R is not 0.5-symmetric.

Example 8 ([9]). *Let $R \in FR(X)$, card $X = 3$,*

$$R = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.8 & 0.9 & 0 \\ 0.6 & 0.9 & 0.8 \end{bmatrix}, \quad S = R^2 = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.8 & 0.9 & 0 \\ 0.8 & 0.9 & 0.8 \end{bmatrix}.$$

The cuts R_β are transitive for $\beta \in [0,0.6] \cup (0.8,1]$, so the cuts $R_{1-\alpha}$ have this property for $\alpha \in [0,0.2) \cup [0.4,1]$. Since $0.8 = s_{32} \geqslant 1-\alpha$ for $\alpha \in [0.4,1]$ and $s_{32} = 0.8 > 0.6 = r_{32}$, relation R is not α-transitive for $\alpha \in [0.4,1]$ (it is α-transitive for $\alpha \in [0,0.2)$, see Corollary 4).

Other results describing graded properties one can find in [4] (p. 78–79).

4.1 Reflexivity

Graded reflexivity was considered by many authors, e.g. [3, 4].

Theorem 3 ([10]). *Let $\alpha \in [0,1]$. $F : [0,1]^n \to [0,1]$ preserves α-reflexivity of fuzzy relations, if and only if*

$$F|_{[\alpha,1]^n} \geqslant \alpha.$$

Theorem 4 ([10]). *$F : [0,1]^n \to [0,1]$ preserves α-reflexivity of fuzzy relations for arbitrary $\alpha \in [0,1]$ if and only if $F \geqslant \min$.*

By Lemma 1 we get

Corollary 5. *Quasi-linear means preserve α-reflexivity of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Theorem 5. *Let $\alpha_1, \dots, \alpha_n \in [0,1]$, a function $F : [0,1]^n \to [0,1]$ be increasing in each variable. If relations $R_i \in FR(X)$ are α_i-reflexive for $i = 1, \dots, n$, then relation $R_F = F(R_1, \dots, R_n)$ is α-reflexive for $\alpha = F(\alpha_1, \dots, \alpha_n)$.*

Proof. Let $\alpha_1, \dots, \alpha_n \in [0,1]$, a function $F : [0,1]^n \to [0,1]$ be increasing in each variable, $R_i \in FR(X)$ be α_i-reflexive for $i = 1, \dots, n$, $x \in X$. Then

$$R(x,x) = F(R_1(x,x), \dots, R_n(x,x)) \geqslant F(\alpha_1, \dots, \alpha_n),$$

so relation $R_F = F(R_1, \dots, R_n)$ is α-reflexive for $\alpha = F(\alpha_1, \dots, \alpha_n)$. □

Each aggregation function is increasing, so we get

Corollary 6. *Let $\alpha_1, \dots, \alpha_n \in [0,1]$, $F : [0,1]^n \to [0,1]$ be an aggregation function. If relations $R_i \in FR(X)$ are α_i-reflexive for $i = 1, \dots, n$, then relation $R_F = F(R_1, \dots, R_n)$ is α-reflexive for $\alpha = F(\alpha_1, \dots, \alpha_n)$.*

Theorem 6. *Let $\alpha \in [0,1]$ and $F \leqslant \min$. If a fuzzy relation $R_F = F(R_1, ..., R_n)$ is α-reflexive, then all relations $R_1, \dots, R_n$ are α-reflexive.*

Proof. Let $\alpha \in [0,1]$, $F \leqslant \min$, $R_F = F(R_1, \dots, R_n)$ be α-reflexive, $x \in X$, $k \in \{1, \dots, n\}$. Then

$$R_k(x,x) \geqslant \min_{1 \leqslant i \leqslant n} R_i(x,x) \geqslant F(R_1(x,x), \dots, R_n(x,x)) \geqslant \alpha.$$

As a result relation R_k is α-reflexive. □

In virtue of Lemma 3 we get

Corollary 7. *Let $\alpha \in [0,1]$, F be a triangular norm. If a fuzzy relation $R_F = F(R_1, ..., R_n)$ is α-reflexive, then all relations $R_1, \dots, R_n$ are also α-reflexive.*

The next example shows that the condition presented in Theorem 6 is only sufficient.

Example 9. *Let $card\, X = 2$. We consider fuzzy relations with matrices:*

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$W_1 = \max(R,S) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, W_2 = \frac{R+S}{2} = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

Relation W_1 is α-reflexive for $\alpha \in [0,1]$, W_2 for $\alpha \in [0,0.5]$, but relations R, S do not have this property for any $\alpha \in (0,1]$.

4.2 Irreflexivity

For irreflexivity we get dual results to reflexivity.

Theorem 7 ([10])**.** *Let $\alpha \in [0,1]$. A function $F : [0,1]^n \to [0,1]$ preserves α-irreflexivity of fuzzy relations if and only if*

$$F|_{[0,1-\alpha]^n} \leqslant 1 - \alpha.$$

Theorem 8 ([10])**.** *A function $F : [0,1]^n \to [0,1]$ preserves α-irreflexivity of fuzzy relations for arbitrary $\alpha \in [0,1]$ if and only if $F \leqslant \max$.*

Corollary 8. *Quasi–arithmetic means preserve α-irreflexivity of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Definition 9 (cf. [2])**.** *A function $F : [0,1]^n \to [0,1]$ is super additive, if for all $i = 1, \dots, n$ and all $x_i, y_i, x_i + y_i \in [0,1]$*

$$F(x_1 + y_1, \dots, x_n + y_n) \geqslant F(x_1, \dots, x_n) + F(y_1, \dots, y_n). \tag{19}$$

Example 10. *Weighted arithmetic means and minimum are super additive functions.*

Theorem 9. *Let $\alpha_1, \dots, \alpha_n \in [0,1]$, $F : [0,1]^n \to [0,1]$ be a super additive aggregation function. If relations $R_i \in FR(X)$ are α_i-irreflexive for $i = 1, \dots, n$, then relation $R_F = F(R_1, \dots, R_n)$ is α-irreflexive for $\alpha = F(\alpha_1, \dots, \alpha_n)$.*

Proof. Let $\alpha_1, \ldots, \alpha_n \in [0,1]$, $F : [0,1]^n \to [0,1]$ be a super additive aggregation function, $R_i \in FR(X)$ be α_i-irreflexive for $i = 1, \ldots, n$, $x \in X$. Then $R_i(x,x) + \alpha_i \leqslant 1$, so

$$F(R_1(x,x), \ldots, R_n(x,x)) + F(\alpha_1, \ldots, \alpha_n)$$

$$\leqslant F(R_1(x,x) + \alpha_1, \ldots, R_n(x,x) + \alpha_n) \leqslant F(1, \ldots, 1) = 1.$$

As a result

$$F(R_1(x,x), \ldots, R_n(x,x)) \leqslant 1 - F(\alpha_1, \ldots, \alpha_n),$$

so $R_F = F(R_1, \ldots, R_n)$ is α-irreflexive for $\alpha = F(\alpha_1, \ldots, \alpha_n)$. □

Corollary 9. *Let $\alpha_1, \ldots, \alpha_n \in [0,1]$. If relations $R_i \in FR(X)$ are α_i-irreflexive for $i = 1, \ldots, n$, then relation $R = \sum_{i=1}^{n} w_i R_i$ is α-irreflexive, where $\sum_{i=1}^{n} w_i = 1$, $w_i \in [0,1]$ and $\alpha = \sum_{i=1}^{n} w_i \alpha_i$.*

Analogously to reflexivity we obtain the following result.

Theorem 10. *Let $\alpha \in [0,1]$ and $F \geqslant \max$. If a fuzzy relation $R_F = F(R_1, ..., R_n)$ is α-irreflexive, then all relations $R_1, ..., R_n$ are also α-irreflexive.*

In virtue of Lemma 3 we get

Corollary 10. *Let $\alpha \in [0,1]$, F be a triangular conorm. If a fuzzy relation $R_F = F(R_1, ..., R_n)$ is α-irreflexive, then all relations $R_1, \ldots, R_n$ are also α-irreflexive.*

The next example shows that the condition given in Theorem 10 is only sufficient.

Example 11. *Let $\mathrm{card}\, X = 2$. We consider fuzzy relations with matrices:*

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$T_1 = \min(R,S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, T_2 = \frac{R+S}{2} = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

Relation T_1 is α-irreflexive for $\alpha \in [0,1]$, T_2 for $\alpha \in [0, 0.5]$, but relations R, S do not have this property for any $\alpha \in (0,1]$.

4.3 Connectedness

Here graded connectedness and total connectedness will be examined. The total 0.5-connectedness was regarded in [21] (p. 619). In that paper this property is called weak comparability. It was shown there that maximum preserves the total 0.5-connectedness ([21], Table 1).

Theorem 11 ([10]). *Let $\alpha \in [0,1]$, $card\ X \geqslant 2$. A function $F : [0,1]^n \to [0,1]$ preserves total α-connectedness (α-connectedness) of fuzzy relations, if and only if*

$$\underset{s,t\in[0,1]^n}{\forall} \big(\underset{1\leqslant k\leqslant n}{\forall} \max(s_k,t_k) \geqslant \alpha\big) \Rightarrow \max(F(s),F(t)) \geqslant \alpha.$$

Theorem 12 ([10]). *Let $card\ X \geqslant 2$. A function $F : [0,1]^n \to [0,1]$ preserves total α-connectedness (α-connectedness) of fuzzy relations for arbitrary $\alpha \in [0,1]$, if and only if*

$$\underset{s,t\in[0,1]^n}{\forall} \max(F(s),F(t)) \geqslant \min_{1\leqslant k\leqslant n} \max(s_k,t_k).$$

Corollary 11. *Maximum and the median preserve total α-connectedness (α-connectedness) of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Theorem 13. *Let $\alpha_1,\dots,\alpha_n \in [0,1]$, a function $F : [0,1]^n \to [0,1]$ be increasing in each variable and $\max \gg F$. If relations $R_i \in FR(X)$ are totally α_i-connected (α_i-connected) for $i = 1,\dots,n$, then relation $R_F = F(R_1,\dots,R_n)$ is totally α-connected (α-connected) for $\alpha = F(\alpha_1,\dots,\alpha_n)$.*

Proof. Let $\alpha_1,\dots,\alpha_n \in [0,1]$, a function $F : [0,1]^n \to [0,1]$ be increasing in each variable, $\max \gg F$ and $R_i \in FR(X)$ be α_i-connected for $i = 1,\dots,n$, $x,y \in X$, $x \neq y$. Then by Lemma 2 and by the fact that $\max \gg F$ we obtain

$$\max(R(x,y),R(y,x)) =$$
$$\max(F(R_1(x,y),\dots,R_n(x,y)),F(R_1(y,x),\dots,R_n(y,x))) \geqslant$$
$$F(\max(R_1(x,y),R_n(x,y)),\dots,\max(R_n(x,y),R_n(y,x))) \geqslant$$
$$F(\alpha_1,\dots,\alpha_n) = \alpha.$$

It means that a fuzzy relation $R_F = F(R_1,\dots,R_n)$ is α-connected for $\alpha = F(\alpha_1,\dots,\alpha_n)$. Proof for total α-connectedness is analogous. □

We can also compute the value of α for which a fuzzy relation $R_F = F$ $(R_1,\dots,R_n)$ is α-connected (totally α-connected) for concrete functions F in another way than it is presented in Theorem 13. It is shown in the following example.

Example 12. *Let $\alpha_1, \dots, \alpha_n \in [0,1]$. If relations $R_i \in FR(X)$ are α_i–connected (totally α_i–connected) for $i = 1, \dots, n$, then relation $R \in FR(X)$ is α-connected (totally α-connected), where*

$$R = \frac{1}{n}\sum_{i=1}^{n} R_i, \qquad \alpha = \frac{1}{n} \max_{1 \leqslant i \leqslant n} \alpha_i.$$

Theorem 14. *Let $\alpha \in [0,1]$ and $F \leqslant \min$. If a fuzzy relation $R_F = F(R_1, \dots, R_n)$ is totally α-connected (α-connected), then all fuzzy relations $R_1, \dots, R_n$ are totally α-connected (α-connected).*

Proof. Let $\alpha \in [0,1]$, $F \leqslant \min$ and a fuzzy relation $R_F = F(R_1, \dots, R_n)$ be α-connected, $x, y \in X$, $x \neq y$, $k \in \{1, \dots, n\}$. As a result we have $\max(R(x,y), R(y,x)) \geqslant \alpha$, so $F(R_1(x,y), \dots, R_n(x,y)) = R(x,y) \geqslant \alpha$ or $F(R_1(y,x), \dots, R_n(y,x)) = R(y,x) \geqslant \alpha$. Let us consider the first case. Since $F \leqslant \min$, we get

$$R_k(x,y) \geqslant \min_{1 \leqslant i \leqslant n} R_i(x,y) \geqslant F(R_1(x,y), \dots, R_n(x,y)) \geqslant \alpha.$$

It means that $\max(R_k(x,y), R_k(y,x)) \geqslant \alpha$. Similarly we may consider the second case, i.e. $R(y,x) \geqslant \alpha$. Thus relations R_i are α-connected for $i \in \{1, \dots, n\}$. The proof for total α-connectedness is analogous. □

By Lemma 3 we get

Corollary 12. *Let $\alpha \in [0,1]$, F be a triangular norm. If a fuzzy relation $R_F = F(R_1, \dots, R_n)$ is totally α-connected (α-connected), then all fuzzy relations $R_1, \dots, R_n$ are totally α-connected (α-connected).*

Example 13. *The condition given in Theorem 14 is only sufficient. For total α-connectedness it is enough to consider relations from Example 9. Relation W_1 is totally α-connected for $\alpha \in [0,1]$, W_2 for $\alpha \in [0, 0.5]$, but relations R, S do not have this property for any $\alpha \in (0,1]$. For α-connectedness let us take $R = [r_{ij}]$, with $r_{ij} = 1$ and $S = [s_{ij}]$, with $s_{ij} = 0$ for $i, j = 1, \dots, n$. Then relation $W = \max(R, S) = R$ and R, W are α-connected for $\alpha \in [0,1]$, while S is not α-connected for any $\alpha \in (0,1]$.*

4.4 Asymmetry

Now graded asymmetry and antisymmetry will be discussed. The obtained results are dual to the ones obtained for total α-connectedness and α-connectedness, respectively. It is worth mentioning that in [21] (p. 619) the 0.5-asymmetry was regarded. However, in that paper this property is called weak asymmetry. It was shown there that minimum preserves the 0.5-asymmetry ([21], Table 1).

Theorem 15 ([10]). *Let $\alpha \in [0,1]$, card $X \geqslant 2$. A function $F : [0,1]^n \to [0,1]$ preserves α-asymmetry (α-antisymmetry) of fuzzy relations, if and only if*

$$\forall_{s,t\in[0,1]^n} \left(\forall_{1\leqslant k\leqslant n} \min(s_k, t_k) \leqslant 1-\alpha \right) \Rightarrow \min(F(s), F(t)) \leqslant 1-\alpha.$$

Theorem 16 ([10]). *Let card $X \geqslant 2$. A function $F : [0,1]^n \to [0,1]$ preserves α-asymmetry (α-antisymmetry) of fuzzy relations for arbitrary $\alpha \in [0,1]$, if and only if*

$$\forall_{s,t\in[0,1]^n} \min(F(s), F(t)) \leqslant \max_{1\leqslant k\leqslant n} \min(s_k, t_k).$$

Corollary 13. *The median function and minimum preserve α-asymmetry (α-antisymmetry) of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Dually to graded connectedness properties, by Lemma 2, similarly to the proof of Theorem 9 we may prove

Theorem 17. *Let $\alpha_1, \ldots, \alpha_n \in [0,1]$, a function $F : [0,1]^n \to [0,1]$ be a super additive increasing in each variable function and $F \gg \min$. If relations $R_i \in FR(X)$ are totally α_i-asymmetric (α_i-antisymmetric) for $i = 1, \ldots, n$, then relation $R_F = F(R_1, \ldots, R_n)$ is α-asymmetric (α-antisymmetric) for $\alpha = F(\alpha_1, \ldots, \alpha_n)$.*

In Theorem 1 we have the characterization of increasing functions which dominate minimum. Appropriate examples are presented in Example 4 and among them minimum is a super additive function (because, by Lemma 2, it dominates any increasing function which coincides with the inequality (19)).

We can also compute the value of α for which a fuzzy relation $R_F = F(R_1, \ldots, R_n)$ is α-asymmetric (α-antisymmetric) for concrete functions F in another way than it is presented in Theorem 17. It is shown in the following example.

Example 14. *Let $\alpha_1, \ldots, \alpha_n \in [0,1]$. If relations $R_i \in FR(X)$ are α_i–asymmetric (α_i–antisymmetric) for $i = 1, \ldots, n$, then relation $R \in FR(X)$ is α-asymmetric (α-antisymmetric), where*

$$R = \frac{1}{n}\sum_{i=1}^{n} R_i, \qquad \alpha = \frac{1}{n}\min_{1\leqslant i\leqslant n} \alpha_i.$$

Dually to Theorem 14 we may prove

Theorem 18. *Let $\alpha \in [0,1]$ and $F \geqslant \max$. If a fuzzy relation $R_F = F(R_1, \ldots, R_n)$ is α-asymmetric (α-antisymmetric), then also all relations $R_1, \ldots, R_n$ are α-asymmetric (α-antisymmetric).*

By Lemma 3 we obtain

Corollary 14. *Let $\alpha \in [0,1]$, F be a triangular conorm. If a fuzzy relation $R_F = F(R_1, \ldots, R_n)$ is α-asymmetric (α-antisymmetric), then all relations $R_1, \ldots, R_n$ are α-asymmetric (α-antisymmetric).*

Example 15. *The condition given in Theorem 18 is only sufficient. For α-asymmetry it is enough to consider relations from Example 11. The relation T_1 is α-asymmetric for $\alpha \in [0,1]$, T_2 for $\alpha \in [0,0.5]$, but relations R, S do not have this property for any $\alpha \in (0,1]$. For α-antisymmetry let us take $R = [r_{ij}]$, with $r_{ij} = 1$ and $S = [s_{ij}]$, with $s_{ij} = 0$ for $i,j = 1,\ldots,n$. Then relation $W = \min(R,S) = S$ and S, W are α-antisymmetric for $\alpha \in [0,1]$, while R is not α-antisymmetric for any $\alpha \in (0,1]$.*

4.5 Symmetry

Now graded symmetry will be discussed.

Theorem 19 ([12])**.** *Let $\alpha \in [0,1]$. If a function $F : [0,1]^n \to [0,1]$ fulfils*

$$F|_{[0,1]^n \setminus [1-\alpha,1]^n} < 1 - \alpha,$$

then it preserves α-symmetry of relations $R_1, \ldots, R_n \in FR(X)$.

Theorem 20 ([12])**.** *If a function $F : [0,1]^n \to [0,1]$ fulfils condition $F \leqslant \min$, then it preserves α-symmetry of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Corollary 15. *Any triangular norm preserves α-symmetry of fuzzy relations for arbitrary $\alpha \in [0,1]$.*

Example 16. *Since any projection P_k, $k \in \mathbb{N}$, preserves the α-symmetry for each $\alpha \in [0,1]$ but it is not true that $P_k \leqslant \min$, then Theorem 20 gives only a sufficient condition for preservation of the α-symmetry for any $\alpha \in [0,1]$.*

Theorem 21. *Let $\alpha_1, \ldots, \alpha_n \in [0,1]$, $F \leqslant \min$. If relations $R_i \in FR(X)$ are α_i-symmetric for $i = 1, \ldots, n$, then relation $R_F = F(R_1, \ldots, R_n)$ is α-symmetric for $\alpha = F(\alpha_1, \ldots, \alpha_n)$.*

Proof. Let relations R_i be α_i-symmetric for $i = 1, \ldots, n$ and $x, y \in X$. If $R(x,y) = F(R_1(x,y), \ldots, R_n(x,y)) \geqslant 1 - \alpha$ and $F \leqslant \min$, then for $k = 1, ..., n$

$$R_k(x,y) \geqslant \min(R_1(x,y), \ldots, R_n(x,y)) \geqslant 1 - \alpha = 1 - F(\alpha_1, \ldots, \alpha_n).$$

Moreover,

$$1 - F(\alpha_1, \ldots, \alpha_n) \geqslant 1 - \min(\alpha_1, \ldots, \alpha_n) \geqslant 1 - \alpha_k \text{ for } k = 1, ..., n.$$

As a result $R_k(x,y) \geqslant 1 - \alpha_k$ for $k = 1, ..., n$. It means that $R_k(x,y) = R_k(y,x)$ for $k = 1, ..., n$, so $R(x,y) = R(y,x)$ and R is α-symmetric for $\alpha = F(\alpha_1, \ldots, \alpha_n)$. □

If it comes to the „converse problem" for α-symmetry we have only counter-examples. Observe that diverse functions were applied for aggregation of fuzzy relations, namely greater (smaller) than or equal to minimum (maximum).

Example 17. *Let* $\text{card}\, X = 2$. *We consider fuzzy relations with matrices:*

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$W_1 = \min(R,S) = R \cdot S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$W_2 = \max(R,S) = R + S - R \cdot S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$W_3 = \frac{R+S}{2} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}.$$

Relations W_1, W_2, W_3 *are* α*-symmetric for* $\alpha \in [0,1]$*, but relations* R, S *do not have this property for any* $\alpha \in (0,1]$.

4.6 Transitivity

In [20] a special case of the graded transitivity is considered. Namely, this is the 0.5-transitivity (there this property is called moderate transitivity). However, the problem of preservation of this property during aggregation process is not discussed. The property of the 0.5-transitivity is also known as one of the types of a stochastic transitivity (e.g. [13]).

Theorem 22 ([12]). *Let $\alpha \in [0,1]$. If an increasing function $F : [0,1]^n \to [0,1]$ fulfils*

$$F|_{[0,1]^n \setminus [1-\alpha,1]^n} < 1-\alpha,$$

and $F \gg \min$, then it preserves α-transitivity of fuzzy relations.

Example 18 ([12]). *Let $a \in (0,1]$ and $F : [0,1]^2 \to [0,1]$ be of the form*

$$F(s,t) = \begin{cases} 0, & (s,t) \in [0,a) \times [0,a) \\ \min(s,t), & \textit{otherwise} \end{cases}$$

F is a t–norm and $F|_{[0,1]^n \setminus [1-\alpha,1]^n} < 1-\alpha$ but it does not dominate minimum. However, the function F preserves the α-transitivity for each $\alpha \in [0,1)$ and $\alpha \leqslant 1-a$. As a result conditions for preservation of the α-transitivity stated in Theorem 22 are only sufficient.

Theorem 23 ([12]). *If a function $F : [0,1]^n \to [0,1]$ is increasing in each variable, fulfils $F \gg \min$ and $F \leqslant \min$, then it preserves α-transitivity of fuzzy relations for any $\alpha \in [0,1]$.*

Corollary 16. *Minimum and the aggregation function*

$$A_w(t_1,\dots,t_n) = \begin{cases} 1, & (t_1,\dots,t_n) = (1,\dots,1) \\ 0, & \textit{otherwise} \end{cases}$$

preserve the α-transitivity of fuzzy relations for any $\alpha \in [0,1]$ (because both functions fulfil assumptions of Theorem 23).

Theorem 24. *Let $\alpha_1,\dots,\alpha_n \in [0,1]$, $F \leqslant \min$, $F \gg \min$ and a function F be increasing. If relations $R_i \in FR(X)$ are α_i-transitive for $i = 1,\dots,n$, then relation $R_F = F(R_1,\dots,R_n)$ is α-transitive for $\alpha = F(\alpha_1,\dots,\alpha_n)$.*

Proof. Let relations R_i be α_i-transitive for $i = 1,\dots,n$ and $x,y,z \in X$. If

$$\min(R(x,y), R(y,z)) =$$

$$\min(F(R_1(x,y),\dots,R_n(x,y)), F(R_1(y,z),\dots,R_n(y,z))) \geqslant 1-\alpha$$

and $F \leqslant \min$, then by the monotonicity of minimum we get

$$\min(R_k(x,y), R_k(y,z)) \geqslant$$

$$\min(\min(R_1(x,y),\dots,R_n(x,y)), \min(R_1(y,z),\dots,R_n(y,z))) \geqslant$$

$$1 - \alpha = 1 - F(\alpha_1, \ldots, \alpha_n)$$

for $k = 1, ..., n$. Moreover,

$$1 - F(\alpha_1, \ldots, \alpha_n) \geqslant 1 - \min(\alpha_1, \ldots, \alpha_n) \geqslant 1 - \alpha_k \text{ for } k = 1, ..., n.$$

As a result $\min(R_k(x, y), R_k(y, z)) \geqslant 1 - \alpha_k$ for $k = 1, ..., n$. By assumptions it means that $\min(R_k(x, y), R_k(y, z)) \leqslant R_k(x, z)$ for $k = 1, ..., n$. Since $F \gg \min$ and F is increasing, one obtains

$$\min(R(x, y), R(y, z)) =$$

$$\min(F(R_1(x, y), ..., R_n(x, y)), F(R_1(y, z), ..., R_n(y, z))) \leqslant$$
$$F(\min(R_1(x, y), R_1(y, z)), ..., \min(R_n(x, y), R_n(y, z))) \leqslant$$
$$F(R_1(x, z), ..., R_n(x, z)) = R(x, z)$$

which proves the α-transitivity of a relation R_F for $\alpha = F(\alpha_1, \ldots, \alpha_n)$. □

If we look for functions F which fulfil both conditions $F \gg \min$ and $F \leqslant \min$ we see that $F = \min$ which is an aggregation function, fulfils these conditions. Moreover, we have the following property

Corollary 17 ([12]). *For a function $F : [0, 1]^n \to [0, 1]$ which has a neutral element $e = 1$ the following holds true: F is increasing in each variable, $F \gg$ min and $F \leqslant \min$ if and only if $F = \min$.*

If it comes to the „converse problem” for α-transitivity we have only counter-examples. In the following example diverse functions were applied to aggregate fuzzy relations, namely greater (smaller) than or equal to minimum (maximum).

Example 19. *Let card $X = 3$. For fuzzy relations described by matrices*

$$R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

we have the following aggregated fuzzy relations

$$\min(R, S) = R \cdot S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\max(R,S) = R + S - R \cdot S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\frac{R+S}{2} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix},$$

which are α-transitive for each $\alpha \in [0,1]$, while relations R and S do not have this property. For example for $\alpha = 1$ and relation R we have $\min(r_{12}, r_{21}) = 1 \geqslant 0$, *but* $0 = r_{11} < \min(r_{12}, r_{21}) = 1$.

5 Conclusion

In this paper preservation of graded properties of fuzzy relations in the context of aggregation process was discussed. Mutual dependencies related to graded properties, between relations $R_1, \ldots, R_n$ on a set X and the aggregated fuzzy relation $R_F = F(R_1, \ldots, R_n)$ were examined. Sufficient conditions for functions $F : [0,1]^n \to [0,1]$ to fulfill the given property were provided. Diverse „regularities" for α-properties were observed.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.

It may be viewed as a result of fruitful discussions held during the Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) organized in Warsaw on October 11, 2013 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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