Modern Approaches in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
Volume I: Foundations

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Infinite distributivity in fuzzy mathematics

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Abstract

This paper concerns false applications of infinite distributivity in fuzzy mathematics. Some consequences of misuse of the empty set are also pointed out.

Keywords: complete lattice, complete homomorphism, infinite distributivity, complete distributivity, empty index set, empty function
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1 Introduction

From the beginning of fuzzy set theory we observe a misunderstanding with the lattice theory. In the paper of Goguen [8], instead of the name 'infinite distributive laws' (cf. [4], pp. 118–119) was used the name 'complete distributive laws', what has quite another meaning in lattice theory. The same name was used in the case of lattice ordered semigroups. Because of important results of [8] such terminology was applied in many papers and books.

Another misunderstanding is in using of these laws. Infinite distributivity appeared through passing from finite to infinite (arbitrary) index set. In some papers this name 'arbitrary' is used for introduction of empty index set, which leads to false results of distributivity.

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In this paper we shall explain correct application of infinite distributivity and describe some results of the above misunderstandings. In particular, Section 2 describes complete lattices as main domain of infnite distributivity; Section 3 compares five fundamental cases of distributivity in lattices; Sections 4 and 5 deal with compatibility properties of functions and binary operations with lattice order; Section 6 discuss controversial properties of empty subset in ordered sets. Finally, Section 7 describes fatal consequences of application of empty index set in definition of infinite distributivity.

2 Complete lattices

We remind here some properties of lattices. Lattice is an ordered set which has two binary operations: meet ($\wedge$) and join ($\vee$), and we write $(L, \vee, \wedge)$. Lattice with the greatest element (denoted by 1) and the least element (denoted by 0) is called bounded one and we write $(L, \vee, \wedge, 0, 1)$.

Definition 1 ([4], Chapter V). *Let $(L, \vee, \wedge)$ be a lattice.*
• *The lattice L is called complete if every its subset has infimum and supremum in L.*
• *The lattice L is called conditionally complete if every its nonempty bounded subset has infimum and supremum in L.*

Theorem 1 (cf.[18], pp. 59–65). *Let $(L, \vee, \wedge)$ be a lattice.*
• *The following conditions are equivalent:*
a) The lattice L is complete.
b) Every nonempty subset of L has infimum and supremum in L.
c) The lattice L is bounded above and every its nonempty subset has infimum in L.
d) The lattice L is bounded below and every its nonempty subset has supremum in L.
e) Every subset of L has infimum in L.
f) Every subset of L has supremum in L.
• *The following conditions are equivalent:*
a) The lattice L is conditionally complete.
b) Every nonempty bounded subset of L has infimum in L.
c) Every nonempty bounded subset of L has supremum in L.

From the above theorem it can be seen that Definition 1 can be formulated in many equivalent forms. In our further considerations it is important that the requirement of extremal bounds of empty set is not necessary and we can use case b). However, the formulations e) and f) are shorter than other.

3 Distributivity in complete lattices

In complete lattices we have generalized min-max inequality of the form $\sup(\inf()) \leqslant \inf(\sup())$. Under additional assumptions we obtain diverse kinds of distributivity.

Definition 2 ([4], Chapter V). *There are five fundamental cases of distributivity in a lattice* $(L, \vee, \wedge)$.
• *The lattice* L *is called distributive if binary operations* $\vee$ *and* $\wedge$ *are mutually distributive, i.e.*

$$\underset{a,b,c\in L}{\forall} a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \quad (1)$$

• *The complete lattice* L *is called infinitely* $\sup -$*distributive if operation* $\wedge$ *is distributive with respect to arbitrary supremum, i.e.*

$$\underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall} a \wedge (\sup_{t\in T} b_t) = \sup_{t\in T}(a \wedge b_t). \quad (2)$$

• *The complete lattice* L *is called infinitely* $\inf -$*distributive if operation* $\vee$ *is distributive with respect to arbitrary infimum, i.e.*

$$\underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall} a \vee (\inf_{t\in T} b_t) = \inf_{t\in T}(a \vee b_t). \quad (3)$$

• *The complete lattice* L *is called infinitely distributive if it is infinitely* $\sup -$ *distributive and* $\inf -$*distributive.*
• *The complete lattice* L *is called completely distributive if for arbitrary* $S, T_s \neq \emptyset$, $a_{s,t} \in L$ *for* $t \in T_s$, $s \in S$ *it fulfils*

$$\sup_{s\in S}(\inf_{t\in T_s} a_{s,t}) = \inf_{h\in H}(\sup_{s\in S} a_{s,h_s}), \quad (4)$$

where $H = \underset{s\in S}{X} T_s$.

Remark 1 ([17]). *Condition* (4) *has the equivalent dual form*

$$\inf_{s\in S}(\sup_{t\in T_s} a_{s,t}) = \sup_{h\in H}(\inf_{s\in S} a_{s,h_s}). \quad (5)$$

Remark 2. *It should be noted here, that property* (2) *is also called 'join infinite distributive identity', while property* (3) *is called 'meet infinite distributive identity' (cf. e.g. [9], p. 90). Conditions* (2) *and* (3) *appears also in definitions of the Brouwerian lattice (algebra) and the Heyting lattice (algebra), which need not be complete.*

Complete lattices from Definition 2 fulfil (1) *i.e. they are distributive lattices. However, the five described cases of distributivity are not mutually equivalent. Complete distributivity is the strongest one, while conditions* (2) *and* (3) *are incomparable. For example the family of open sets (e.g. on real line) is an infinitely* $\sup -$*distributive lattice, while family of closed sets is an infinitely* $\inf -$*distributive lattice but they are not infinitely distributive (cf. [4], p. 118). Similarly, every complete Boolean lattice is infinitely distributive, but need not be completely distributive (cf. [14], pp. 147-151).*

Thus the name 'complete distributive lattice' concerns the weakest distributivity assumption, while very similar name 'completely distributive lattice' concerns the strongest distributivity assumption. In the lattice theory, between these two bounds there are considered infinitely many intermediate conditions of the form (4) *called* $(\mathfrak{m},\mathfrak{n})-$*distributivity, if card* $S \leqslant \mathfrak{m}$*, card* $T_s \leqslant \mathfrak{n}$ *for* $s \in S$*, where* $\mathfrak{m}$*,* $\mathfrak{n} \geqslant 2$ *are given cardinal numbers. In particular* (2) *denotes the strongest* $(2, \mathfrak{n})-$*distributivity, while* (3) *denotes the strongest* $(\mathfrak{n}, 2)-$*distributivity. In the case of countable index set* T *we get* $\sigma-$*distributivity in* (2) *and* $\delta-$*distributivity in* (3) *(cf. [19], p. 62; [11]).*

Usually, the name 'completely distributive lattice' is used correctly in fuzzy set theory. For example in papers from fuzzy topology or concerning fuzzy functions (cf. [29], [5]). Moreover, completely distributive lattices with involutive complement are used under name 'fuzzy lattices' (cf. [1], [3]). However, the name 'completely distributive lattice' is sometimes used in another sense, contrapositive with Definition 2. The simplest case is a distributive lattice, which is also complete (cf. [16], Theorem 5.2).

The most important misunderstanding concerns replacing of infinite distributivity by complete distributivity. If this is made in an assumption, it is a too strong condition. If complete distributivity appears as result (with proving infinite distributivity), then usually such statement will be false (or the proof will be uncompleted). It was mentioned in Remark 2 that the complete distributivity is a particular case of infinite distributivity but the difference is considerable. In this case we have false definition (cf. [13], Definition 4.1) or hidden false definition 'completely distributive lattice with respect to the meet and the join' (cf. [12], Theorem 3.16, with proving conditions (2), (3)). Such errors can be motivated by a misprint in [4], p.128, Theorem 24, where condition (2) is read as 'the join operation is completely distributive on meets', while it should be read as: 'the meet operation is infinitely distributive on joins', i.e. it is infinitely $\sup -$distributive. This failed characterization of complete Brouwerian lattices was used in [8], what spread this misprint. Thus many authors did not refer to the definitions from [4], pp. 118–119, but are referred directly to [4], p. 128.

4 Lattice homomorphisms

Monotonic functions have many particular classes in the lattice theory.

Definition 3 ([4], pp. 24-26; [9], pp. 15-24)**.** *Let* $(L, \vee, \wedge)$ *be a complete lattice. The function* $h : L \to L$ *is called:*
• *isotone (order homomorphism) if* $a \leqslant b \Rightarrow h(a) \leqslant h(b)$ *for* $a, b \in L$*;*
• *antitone (order anti-homomorphism) if* $a \leqslant b \Rightarrow h(a) \geqslant h(b)$ *for* $a, b \in L$*;*
• *join-homomorphism if* $h(a \vee b) = h(a) \vee h(b)$ *for* $a, b \in L$*;*
• *meet-homomorphism if* $h(a \wedge b) = h(a) \wedge h(b)$ *for* $a, b \in L$*;*
• *lattice homomorphism if it is both join- and meet-homomorphism;*
• *join-anti-homomorphism if* $h(a \vee b) = h(a) \wedge h(b)$ *for* $a, b \in L$*;*
• *meet-anti-homomorphism if* $h(a \wedge b) = h(a) \vee h(b)$ *for* $a, b \in L$*;*
• *lattice anti-homomorphism if it is both join- and meet-anti-homomor- phism;*
• $\sup -$*homomorphism if*

$$\underset{T \neq \emptyset}{\forall} \ \underset{a_t \in L, t \in T}{\forall} \ h(\sup_{t \in T} a_t) = \sup_{t \in T} h(a_t); \tag{6}$$

• $\inf -$*homomorphism if*

$$\underset{T \neq \emptyset}{\forall} \ \underset{a_t \in L, t \in T}{\forall} \ h(\inf_{t \in T} a_t) = \inf_{t \in T} h(a_t); \tag{7}$$

• *complete homomorphism if it is both* $\sup -$ *and* $\inf -$*homomorphism;*
• $\sup -$*anti-homomorphism if*

$$\underset{T \neq \emptyset}{\forall} \ \underset{a_t \in L, t \in T}{\forall} \ h(\sup_{t \in T} a_t) = \inf_{t \in T} h(a_t); \tag{8}$$

• $\inf -$*anti-homomorphism if*

$$\underset{T \neq \emptyset}{\forall} \ \underset{a_t \in L, t \in T}{\forall} \ h(\inf_{t \in T} a_t) = \sup_{t \in T} h(a_t); \tag{9}$$

• *complete anti-homomorphism if it is both* $\sup -$ *and* $\inf -$ *anti-homomorphism.*

Remark 3. *Every homomorphism from the above definition is also order homomorphism (isotone function) and every anti-homomorphism is also order anti-homomorphism (antitone function).*

5 Lattice ordered groupoids

Next, we consider an additional binary operation $*$ in a lattice L.

Definition 4 ([4], Chapter XIV; [6]). *Let $(L, \vee, \wedge)$ be a complete distributive lattice and $* : L \times L \to L$.*

- *The operation $*$ is called join-distributive if it is distributive with respect to $\vee$.*
- *The operation $*$ is called meet-distributive if it is distributive with respect to $\wedge$.*
- *The operation $*$ is called infinitely (left-, right-)* $\sup -$*distributive if*

$$\underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall}\ a * (\sup_{t\in T} b_t) = \sup_{t\in T}(a * b_t), \quad \underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall}\ (\sup_{t\in T} b_t) * a = \sup_{t\in T}(b_t * a). \tag{10}$$

- *The operation $*$ is called infinitely (left-, right-)* $\inf -$*distributive if*

$$\underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall}\ a * (\inf_{t\in T} b_t) = \inf_{t\in T}(a * b_t), \quad \underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall}\ (\inf_{t\in T} b_t) * a = \inf_{t\in T}(b_t * a). \tag{11}$$

- *The operation $*$ is called infinitely distributive if it is both infinitely* $\sup -$ *and* $\inf -$*distributive.*
- *The operation $*$ is called infinitely (left-, right-)* $\sup - \inf -$*distributive if*

$$\underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall}\ a * (\sup_{t\in T} b_t) = \inf_{t\in T}(a * b_t), \quad \underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall}\ (\sup_{t\in T} b_t) * a = \inf_{t\in T}(b_t * a). \tag{12}$$

- *The operation $*$ is called infinitely (left-, right-)* $\inf - \sup -$*distributive if*

$$\underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall}\ a * (\inf_{t\in T} b_t) = \sup_{t\in T}(a * b_t), \quad \underset{T\neq\emptyset}{\forall}\ \underset{a,b_t\in L}{\forall}\ (\inf_{t\in T} b_t) * a = \sup_{t\in T}(b_t * a). \tag{13}$$

- *The operation $*$ is called infinitely mixed distributive if it is both infinitely* $\sup - \inf$ *and* $\inf - \sup -$*distributive.*

From infinite distributivity we obtain finite distributivity but the converse is possible in a finite lattice only.

Corollary 1. *Let L be a complete lattice with an additional binary operation $*$. If the operation $*$ is infinitely* $\sup -$*distributive (*$\inf -$*distributive), then it is distributive with respect to $\vee$ ($\wedge$). In both the above cases the operation $*$ is isotone.*

Remark 4. *A problem exists with terminology for distributivity in lattice ordered semigroups. If we generalize properties* (2), (3) *from meet and join onto other binary operations, it seems quite natural preserving of the name 'infinite distributivity' for binary operation $*$ (as it was stated in Definition 4). This argues with* *another tradition connected with Definition 3. If a mapping preservers completeness of order structure then it is called a complete homomorphism. Generalizing this property for mappings of two variables we get completeness of binary operations. Thus sometimes 'infinite distributivity" is hidden under names 'cl-groupoid', 'cl-monoid' or 'complete lattice ordered semigroup' (cf. [4], p. 327).*

According to [7], Theorem 1, in the case $L = [0, 1]$ we have

Lemma 1. *An operation* $* : [0, 1]^2 \to [0, 1]$ *is infinitely* $\sup -$*distributive if and only if it is increasing and left-continuous. Dually, it is infinitely* $\inf -$*distributive if and only if it is increasing and right-continuous.*

6 Properties of empty set

The empty set plays in mathematics similar role as number zero in arithmetic. As in arithmetic we frequently exclude value 0 (coefficient, denominator), similarly in mathematics we frequently exclude the empty set (domain of function, equivalence class). Fundamental algebraic structures such as groups, rings, fields or vector spaces are naturally nonempty because of assumed neutral elements. However, in the case of soft algebra (weak algebraic structures, tropical mathematics), there exists a possibility to admit empty algebraic structures (groupoids, semigroups, semirings, semimodules, semilattices or lattices). The second step relies on artificial extension of known notions to the case of the empty set, which leads to many contradictions and misunderstandings.

The empty set was introduced as a symbol for nothing, for a set of elements which do not exist. Formally we can write $\emptyset = \{x : x \neq x\}$. Empty set is useful in shortening of some formulas or considerations. For example in the case of complete lattices it is used in the proof of characterization Theorem 1 (cases e) and f)). It is very useful symbol in algebra of sets. In the case of disjoint sets A, B we can write $A \cap B = \emptyset$ (intersection is well defined). Similarly, for $A \subset B$ we can write $A \setminus B = \emptyset$ (difference is well defined). Moreover, for arbitrary set A we have (cf. [14], p. 9)

$$A \cap \emptyset = \emptyset \cap A = \emptyset,\ A \cup \emptyset = \emptyset \cup A = A,\ A \setminus \emptyset = A.$$

In particular we get

Proposition 1. *The empty set is included in arbitrary set.*

Because of antisymmetry of inclusion we also get

Proposition 2. *The empty set is unique (only one).*

Remark 5. *From the above facts we see that the empty set is very strange: it is the same in a family of elephants as in a family of computers.*

The empty subset of a Cartesian product is called the empty relation $\emptyset \subset X \times X$. In particular (cf.[14], p. 62]):

$$\emptyset \times X = X \times \emptyset = \emptyset = \emptyset \times \emptyset.$$

Thus we have

Proposition 3. *Reduction of arbitrary relation to the empty set is the empty relation.*

Since the inverse relation $\emptyset^{-1}$ is also empty, then we have

$$\emptyset^{-1} = \emptyset,\ \emptyset \cap \emptyset^{-1} = \emptyset,\ \emptyset \circ \emptyset = \emptyset.$$

As a result we get

Proposition 4. *The empty relation $\emptyset \subset X \times X$ in arbitrary set X is irreflexive, symmetric, asymmetric, antisymmetric and transitive.*

If $X \neq \emptyset$, then the empty relation need not be reflexive or connected. However, for $X = \emptyset$ the identity relation reduces to $\emptyset$ and $\emptyset \cup \emptyset^{-1} = \emptyset = \emptyset \circ \emptyset$, thus we get

Proposition 5. *The empty relation on the empty set is reflexive and connected.*

Directly from Propositions 4, 5 we get

Corollary 2. *The empty relation in arbitrary set is a strict order.*
The empty relation on the empty set is an equivalence, an order, and a linear order.

Remark 6. *From the above corollary arises that the empty set is ordered, strictly ordered and linearly ordered. In particular, the empty subset of an ordered set is ordered by empty reduction of given order relation. This is very strange situation that set without elements has legal linear ordering.*

Any function $F : X \to Y$ can be identified with its graph in the Cartesian product $X \times Y$, i.e. relation $F \subset X \times Y$, where $D_F = X$ and $F^{-1} \circ F \subset I_Y$. Since

$$D_\emptyset = \emptyset,\ \emptyset^{-1} \circ \emptyset = \emptyset \subset I_\emptyset,$$

then we get

Proposition 6. *Reduction of arbitrary function to the empty set is the empty relation.*
The empty relation is a function with empty domain.

Corollary 3. *The empty function F has no arguments and values (the expression $F(x)$ has no sense).*

Remark 7. *From the above corollary we can observe that admitting values of empty function F leads to contrapositive relations of the form*

$$\begin{aligned} x, y \in \emptyset \Rightarrow \quad & (F(x) = F(y), \quad F(x) \neq F(y), \\ & F(x) < F(y), \quad F(x) > F(y), \\ & F(x) \| F(y)). \end{aligned}$$

Since this implication is true, thus such notions as constant function, injection or monotonic function have no sense for empty function.

Now we discuss some additional questions about the empty set with more controversial properties.

Question 1. *Is the empty set bounded?*

Since boundedness is hereditary on subsets, then the empty subset (the same everywhere) is bounded on many ways as subset of diverse bounded sets. We discuss here the case of ordered sets. From Corollary 2 we know that restriction of order relation to the empty subset is still an order relation (empty).

Let P be an ordered set. If sets $A, B \subset P$ are bounded in P and $A \subset B$, then every upper (lower) bound of B is also upper (lower) bound of A. For arbitrary $x \in P$ the set $\{x\}$ is bounded above and below by x. Since $\emptyset \subset \{x\}$, then also the empty subset of P is bounded above and below by x. Thus we get

Proposition 7. *If P is an ordered set, then the empty subset of P is bounded above and below in P by every element of P.*

Corollary 4. *Let P be an ordered set.*
• *The empty subset has supremum in P if P has the least element and $sup\ \emptyset = \min P = \inf P$.*
• *The empty subset has infimum in P if P has the greatest element and $\inf\ \emptyset = \max P = \sup P$.*
• *The empty subset has no extremal bounds in P if and only if P is unbounded.*

Remark 8. *The above results are very strange, because the empty set is not only bounded by everything but also its upper and lower bounds are the same. From the* *one hand the empty set have no elements and is unique (Propositions 1), but from the other hand everything can be its supremum or infimum in suitable ordered set. For example, in the family FI of all fuzzy implications these extremal bounds are fuzzy implications (cf. [2], p. 5)*

$$I_0(x,y) = \begin{cases} 1, (x=0) \vee (y=1) \\ 0, \textit{ otherwise} \end{cases} , \quad I_1(x,y) = \begin{cases} 1, (x<1) \vee (y>0) \\ 0, \textit{ otherwise} \end{cases} ,$$

for $x, y \in [0,1]$.
Similarly, in the family FN of all fuzzy negations these extremal bounds are fuzzy negations (cf. [2], pp. 14-15)

$$N_0(x) = \begin{cases} 1, x=0 \\ 0, x>0 \end{cases} , \quad N_1(x) = \begin{cases} 1, x<1 \\ 0, x=1 \end{cases} , \; x \in [0,1].$$

Simultaneously, the empty set has no bounds in families of all continuous fuzzy implications or fuzzy negations, because these families are unbounded lattices (cf. [2], p. 184).

Usually universal bounds of bounded ordered set P are denoted by useful symbols 0 and 1, and by Corollary 4 we get

$$\sup\, \emptyset = 0, \quad \inf\, \emptyset = 1. \tag{14}$$

It should be remembered that these symbols denote elements from diverse sets, where numbers 0 and 1 are very particular case (mainly for $P = \{0,1\}$ or $P = [0,1]$).

Question 2. *Is the empty set completely ordered?*

According to Corollary 4 it is possible only in a bounded poset. Thus we have

Proposition 8. *The empty set is a complete subset in every bounded poset. In particular, it is a complete sublattice of every bounded poset.*
The empty set is not completely ordered in unbounded posets.

Considering of empty complete lattice is misleading. For example we have

Lemma 2 ([4], p. 115)**.** *Let L be a complete lattice. If $f : L \to L$ is an isotone mapping, then there exists element $s \in L$ such that $f(s) = s$.*

Such result is false for $L = \emptyset$ and notion of isotone mapping has no sense on the empty set (Remark 7). Thus we must correct assumptions of this result ($L \neq \emptyset$).

Question 3. *Is the empty set a lattice?*

It is very subtle problem and result depends from the method of solving. The standard solution is presented in Proposition 8. We cannot consider the answer in general but only relatively for empty subset of ordered sets. We have

Lemma 3 ([4], p. 7)**.** *The empty subset of a lattice is a lattice.*

Remark 9. *This result is obtained by restriction of functions* $F(x, y) = \inf\{x, y\}$, $G(x, y) = \sup\{x, y\}$ *from the given lattice to the empty subset. However, from Remark 7 we know that values of functions on the empty set have no sense, but what is impossible, that can be included in the empty set (i.e. the empty set is closed with respect to binary infimum and supremum). Thus such result is logically correct.*

Question 4. *Is the empty set finite?*

The positive answer seems to be obvious, because the cardinality of the empty set is equal 0. However, cardinality of sets use equivalence by bijections, what has no sense for the empty set (cf. Remark 7). This problem is more complicated, because many monographs and handbooks consider finite subsets beginning from one element. Thus some formulations of results can be false after admitting also empty set. For example we have

Lemma 4 ([4], p. 5)**.** *Any finite chain has the least element and the greatest element.*

This lemma is false in the case of empty chain (cf. Corollary 2), which is without elements. Thus in many statements the name ‘finite chain’ should be replaced by ’nonempty finite chain’ or ‘finite chain with positive cardinality’. We also have

Lemma 5 ([4], p. 7)**.** *Any finite lattice is complete.*

After Proposition 8 this statement can be false for the empty lattice (we need a sublattice of bounded poset).

Remark 10. *The empty set can be treated as a finite set, but its properties differs from that of other finite sets and such extension of the name can be very misleading.*

Question 5. *Is indexing by the empty set possible?*

The indexing was generalized from finite to infinite sets and from countable to uncountable case. This process did not begin from empty set and did not finish on empty index set (cf. e.g. [14], pp. 60, 107). However, in many formulations of definitions and results we meet the name 'arbitrary index set' (cf. [4], p. 118). As a rule: nonempty family can be indexed by nonempty index set and arbitrary family can be indexed by arbitrary index set, and in consequence the empty family can be indexed by the empty index set. Such generalization is another application of empty function, because an indexed family $(A_t)_{t\in T}$ subsets of X is equivalent to a function $A : T \to 2^X$. In the case $T = \emptyset$ we obtain function A on the empty set and values $A_t = A(t)$ have no sense for $t \in \emptyset$.

The most artificial case we meet in generalizations of distributivity. We start with two elements (binary distributivity) and by finite formulas (finite distributivity) we go to countable case ($\sigma-$distributivity) and uncountable case (infinite distributivity). Thus indexed set starts from two elements and increases to infinite set. None practical reasons lead to empty index set. Some consequences of application of empty index set will be discussed in the next section.

7 A history of one mistake

Now we put attention on certain misuse of empty set connected with papers [21] and [10]. We begin with reminding of definition of pseudo triangular norms in complete lattices.

Definition 5 ([27], Definition 3.1)**.** *Let* $(L, \vee, \wedge, 0, 1)$ *be a complete lattice.*
• *A binary operation* $T : L^2 \to L$ *is called a pseudo t-norm if it is isotone with respect to the second variable and*

$$T(0, y) = 0,\ T(1, y) = y \ \ \textit{for}\ y \in L. \tag{15}$$

The family of all infinitely left$-\sup-$*distributive pseudo t-norms will be denoted by* $T(L)$.
• *By residual of a binary operation* T *we call the operation*

$$I_T(x, y) = \sup\{u \in L : T(x, u) \leqslant y\} \ \ \textit{for}\ x, y \in L. \tag{16}$$

By Definition 4 if $T \in T(L)$, then

$$\underset{\emptyset\neq Y\subset L}{\forall}\ \underset{x\in L, y\in Y}{\forall}\ T(x, \sup_{y\in Y} y) = \sup_{y\in Y} T(x, y). \tag{17}$$

Paper [27] contains the following false result (Theorem 4.1 (2))

Theorem 2. *(False Theorem) If $T \in T(L)$, then operations T and I_T fulfils 'residuation principle', i.e.*

$$T(x,z) \leqslant y \Leftrightarrow I_T(x,y) \geqslant z \text{ for } x,y,z \in L. \tag{18}$$

It was pointed out by Han and Li [10] that the above theorem and many other results in papers [27], [28] were false. Falsity of the above theorem can be seen by the following

Example 1 ([27], Example 3.1; [10], Example 2.1)**.** *Let $L = [0,1]$. If we consider the greatest element T_M of $T(L)$, then we obtain suitable operation I_{T_M}, where*

$$T_M(x,y) = \begin{cases} y, \text{if } x = 1 \\ 0, \text{if } x = 0 \\ 1, \text{otherwise} \end{cases}, \quad I_{T_M}(x,y) = \begin{cases} y, \text{if } x = 1 \\ 0, \text{ if } x \in (0,1), y < 1 \\ 1, \text{otherwise} \end{cases},$$

for $x, y \in [0,1]$.
However, these operations do not fulfil principle (18)*, because for $x \in (0,1)$, $y < 1$ and $z = 0$ we get a contradiction*

$$I_{T_M}(x,y) = 0 \geqslant z, \ T_M(x,z) = T_M(x,0) = 1 > y.$$

If we look on the proof of Theorem 2, then we observe that

$$\{u \in L : T_M(x,u) \leqslant y\} = \emptyset \text{ for } x \in (0,1), \ y < 1$$

and elements of this set are used as arguments of operation T_M. According to Remark 7 we see that symbol $T_M(x,u)$ has no sense for $u \in \emptyset$. Thus a correction of such theorem should add such assumptions on T which provide that $\{u \in L : T(x,u) \leqslant y\} \neq \emptyset$.

After paper [10], an erratum [21] appeared with very artificial correction. The error was not corrected but accepted by join of such situation in the definition of infinite distributivity, i.e. admission of empty index set. This decision changes the family $T(L)$ onto $T^*(L)$ and now the operation T_M does not belong to $T^*(L)$, i.e. the above counterexample does not work (thus the paper needs only a correction of examples!). The author can introduce and examine new families of operations but without change of standard definitions, what can imply other false results. In particular we have

Lemma 6. *(Illusion Lemma, [21]) Let $(L, \vee, \wedge, 0, 1)$ be a complete lattice and $A : L^2 \to L$.*

• *If the operation A is infinitely left*$-\sup-$*distributive with arbitrary index set, then* $A(x,0)=0$ *for* $x\in L$.
• *If the operation A is infinitely left*$-\inf-$*distributive with arbitrary index set, then* $A(x,1)=1$ *for* $x\in L$.

Proof. We rewrite only the case of $\sup-$distributivity. Admitting $Y=\emptyset$ in (17) for arbitrary $x\in L$ we get (cf. (14))

$$A(x,0)=A(x,\sup\emptyset)=A(x,\sup_{y\in\emptyset}y)=\sup_{y\in\emptyset}A(x,y)=\sup\emptyset=0.$$

$\square$

As we see from the proof, besides the previous error with arguments from empty set we have new error with empty index set. Thus the obtained result is more contradictory than initial error, what we consider on examples.

Example 2. *Let us consider the useful complete lattice* $L=[0,1]$*, where are defined triangular norms. As authors have observed in [27], p. 115: 'every t-norm on* L *is also a pseudo-t-norm on* L*'. It is commonly known that every continuous triangular norm preserves suprema and infima, thus it is infinitely distributive* ($\sup-$*distributive and* $\inf-$*distributive, cf. e.g. Lemma 1). However, by Illusion Lemma we obtain the contradictory result* $T(0,1)=1$*. Thus, continuous triangular norms do not fulfil the assumption of this lemma, i.e. they are not infinitely* $\inf-$*distributive with arbitrary index set. It shows how drastic and unnatural is this assumption.*

Example 3. *Let us observe that Illusion Lemma concerns arbitrary binary operations with infinite distributivity (the only assumption). Let* $c\in(0,1)$ *and* $A(x,y)=c$ *for* $x,y\in[0,1]$*. It is evident that such constant operation is infinitely distributive, because on both sides we obtain the same constant value. But from Illusion Lemma we get* $A(x,0)=0\neq c$ *and* $A(x,1)=1\neq c$*. Thus such constant operations are not infinitely distributive with arbitrary index set. Since authors identify infinite distributivity with continuity (cf. [27], Definition 2.2), then constant functions are not continuous in such new sense.*

We see that this is quite artificial mathematics and therefore distributivity with arbitrary index set does not have sense. Unfortunately such Illusion Lemma is reproved also in [22], p. 400, [23], p. 2497, [24], p. 288, [25], p. 2089, [26], p. 24 and [20], p. 55, and used during proving of new results. Other authors use such artificial results or directly the Illusion Lemma (cf. e.g. [15], p. 74). It can be a hard work to indicate all such results, which leads to many new mistakes.

8 Concluding remarks

In a formulation of mathematical results a precise description of assumptions is very important. A little change of assumptions can falsify all statement. Especially dangerous is change of meaning of known names for needs of particular paper. If we cite single result from such paper we can obtain false conclusions. This was indicated by some examples.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.

It may be viewed as a result of fruitful discussions held during the Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) organized in Warsaw on October 11, 2013 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, Prof. Asen Zlatarov University, Burgas, Bulgaria, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Twelfth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2013) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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