Developments in Fuzzy Sets,
Intuitionistic Fuzzy Sets,
Generalized Nets and Related Topics.
Volume I: Foundations

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Dedicated to Professor Beloslav Riečan on his 75th anniversary

On a story inspired by IF-sets

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Abstract

Probability on IF-events is considered. Two basic theorems are discussed: IF-state representation theorem and MV-algebra embedding theorem.

Keywords: IF-events, IF-states, MV-algebras.

1 Introduction

There are two points of view on uncertainty studied with the help of mathematical theories in the present time. The first one is based on objective informations. From the mathematical point of view it can be described by the methods of probability and mathematical statistics. The second point of view on uncertainty is subjective and in the present time the theory of fuzzy sets seems to be adequate for study it.

In the paper the both directions are considered together. Namely, the probability theory is studied on families of fuzzy sets. More precisely, the special case, so-called intuitionistic fuzzy sets are studied in Section 2, and states on them in Section 3. Then the IF-probability representation theorem is presented in Section 4 as well as an embedding theorem in Section 5. We give a review of references for probability theory on some algebraic structures, since the corresponding scientific results can be applied to the IF-case.

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2 Set theories

Set theory presents one of the basic points of the modern mathematics. It may be characterized e.g. by the papers by Cantor ([2]). A subset A of a non-empty space Ω can be characterized by its characteristic function $\chi_A : \Omega \to \{0, 1\}$, where

$\chi_A(\omega) = 1$, if $\omega \in A$,

$\chi_A(\omega) = 0$, if $\omega \in \Omega - A$.

In the present time the set theory has general using, especially all mathematical theories are based on it. It is not the case of fuzzy sets introduced by L. A. Zadeh([18]), although it is a natural generalizatioon of the set theory and has many important and interesting applications so in mathematics as well as in a large space of sciences and useful practical problems. A fuzzy set is any mapping

$$A : \Omega \to [0, 1],$$

hence a set is a special case of a fuzzy set. The newest progress in this direction is probably the Atanassov theory of IF-sets (see [1]). An IF-set is a couple of fuzzy sets

$$A = (\mu_A, \nu_A), \mu_A, \nu_A : \Omega \to [0, 1],$$

such that

$$\mu_A + \nu_A \leq 1.$$

The function $\mu_A : \Omega \to [0, 1]$ is called the membership function, ν_A is called the non-membership function. A fuzzy set $f : \Omega \to [0, 1]$ is a special case of IF-set, where $\mu_A = f, \nu_A = 1 - f$. An IF-set can be considered as a mapping $A : \Omega \to \Delta$ where Δ is the triangle

$$\Delta = \{(u, v) \in R^2; u \geq 0, v \geq 0, u + v \leq 1\}.$$

Fuzzy set can be regarded as a special case, as a mapping $A : \Omega \to I$, where

$$I = \{(u, v); 0 \leq u \leq 1, v = 1 - u\}.$$

Of course, also a set can be regarded as a subset of Δ, namely

$$A : \Omega \to \{(0, 1), (1, 0)\}.$$

We shall denote by $\mathcal{G}$ the family of all IF-sets on a given space Ω. There are many possibilities for defining basic binary operations on $\mathcal{G}$. We shall use the Lukasiewicz's ones $\oplus, \odot$,

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1).$$

The operation $\oplus$ corresponds to the disjunction of statements (unions of sets), the operation $\odot$ corresponds to the conjunction of statements (intersection of sets). Indeed, if $A = \chi_A, 1 - \chi_A), B = (\chi_B, 1 - \chi_B)$, then

$$A \oplus B = (\chi_{A\cup B}, 1 - \chi_{A\cup B}), A \odot B = (\chi_{A\cap B}, 1 - \chi_{A\cap B}).$$

Recall the partial ordering

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Evidently the mapping $(0_\Omega, 1_\Omega)$ is the least element of $\mathcal{G}$, and $(1_\Omega, 0_\Omega)$ is the greatest element of $\mathcal{G}$.

3 Probability theories

One of the most important scientific results in the 20th century, and not only in mathematics, is the Kolmogorov probability (see [7]). In the theory

probability = measure,

random variable = measurable function,

mean value = integral.

It is interesting also that the concept of probability has been introduced shortly after the introducing of fuzzy sets and similarly after the introducing of IF-sets. The first one has been introduced by L. A. Zadeh in [19], the second by P. Grzegorzewski and E. Mrowka in [5]. The main aim of the paper is the presentation of some results concerned in the last model.The authors of [5] start with a classical Kolmogorovian probability space $(\Omega, \mathcal{S}, P)$. An IF-events is an IF-set $A = (\mu_A, \nu_A)$ such that the mappings $\mu_A, \nu_A : \Omega \to [0, 1]$ are measurable. Then the probability $\mathcal{P}(A)$ of A is the compact interval

$$\mathcal{P}(A) = [\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP].$$

Denote by $\mathcal{F}$ the family of all IF-events on Ω. Inspired by the definition we introduced probability axiomatically as a mapping

$$\mathcal{P} : \mathcal{F} \to \mathcal{J}$$

with some properties, where $\mathcal{J}$ is the family of all compact intervals in R.. Of course, $\mathcal{P}(A)$ is an interval

$$\mathcal{P}(A) = [P^\flat(A), P^\sharp(A)],$$

so we obtain two mappings

$$P^\flat : \mathcal{F} \to [0,1], P^\sharp : \mathcal{F} \to [0,1].$$

We shall call the mappings as states and define them axiomatically (see [13]).

Definition 1. A mapping $m : \mathcal{F} \to [0,1]$ is called a state, if it satisfies the following conditions:

1. $m((0_\Omega, 1_\Omega)) = 0, m((1_\Omega, 0_\Omega)) = 1$.
2. $A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus B) = m(A) + m(B)$.
3. $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A)$.

4 IF-probability representation theorem

The following theorem can be found in [3] (see also [14])

Theorem 1. To any state $m : \mathcal{F} \to [0,1]$ there exists $\alpha \in R$ and probability measures $P, Q : \mathcal{S} \to [0,1]$ such that

$$m(A) = \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dQ)$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

Of course, IF-probability $\mathcal{P}(A)$ of $A = (\mu_A, \nu_A)$ is an interval

$$\mathcal{P}(A) = [P^\flat(A), P^\sharp(A)],$$

where $P^\flat : \mathcal{F} \to [0,1], P^\sharp : \mathcal{F} \to [0,1]$ are such states that $P^\flat \leq P^\sharp$. Therefore as a consequence of Theorem 1 one can obtain the following result.

Theorem 2. If $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ is an IF-probability, then there exist $\alpha, \beta \in R$ and probability measures $P, Q, R, S : \mathcal{S} \to [0,1]$ such that

$$\mathcal{P}(A) = [\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dQ), \int_\Omega \mu_A dR + \beta(1 - \int_\Omega (\mu_A + \nu_A) dS)]$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

5 IF-embedding theorem

The second important result of probability on IF-events is an embedding theorem to an MV-algebra. Very instructive example of an MV-algebra is the unit interval $[0,1]$ with the usual ordering and two binary operations:

$$a \oplus b = \min(a + b, 1),$$

$$a \odot b = \max(a + b - 1, 0).$$

A similar situation works in the general situation.

Definition 2. An l-group is and algebraic structure $G(+, \leq)$ such that $(G, +)$ is a commutative group, $(G, \leq)$ is a lattice and the implication

$$a \leq b \Longrightarrow a + c \leq b + c$$

holds. Then MV-algebra is the algebraic system $(M, 0, u, \neg, \oplus, \odot)$, where 0 is the neutral element of G, u is a positive element of G,

$$M = \{x \in G; 0 \leq x \leq u\},$$

$\neg$ is a unary operation defined by the equality

$$\neg a = u - a,$$

and $\oplus, \odot$ are binary operations

$$a \oplus b = (a + b) \wedge u,$$

$$a \odot b = (a + b - u) \vee 0.$$

A mapping $m : M \to [0, 1]$ is called a state, if the following properties are satisfied:

1. $m(0) = 0, m(u) = 1$.
2. $a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b)$.
3. $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$.

A well developed probability theory on MV - algebras has been described in [16] (see also [17]). Therefore the following theorem is important from the point of view.

Theorem 3. To any family $\mathcal{F}$ of IF-events there exists and MV-algebra $\mathcal{M}$ containing $\mathcal{F}$ and such that the MV-algebra operations coincide on $\mathcal{F}$ with the IF-operations. Moreover, if $m : \mathcal{F} \to [0, 1]$ is a state, then there exists a state $\bar{m} : \mathcal{M} \to [0, 1]$ such that $\bar{m} \mid \mathcal{F} = m$.

Proof. Put $G = \{A = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \to R\}$. Of course,

$$(\mu_A, \nu_A) \leq (\mu_B, \nu_B) \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B,$$

and

$$(\mu_A, \nu_A) + (\mu_B, \nu_B) = (\mu_A + \mu_B, 1 - [1 - \nu_A, 1 - \nu_B]) = (\mu_A + \mu_B, \nu_A + \nu_B - 1).$$

Then $(G, +, \leq)$ is an l-group. Evidently $0 = (0_\Omega, 1_\omega)$. Put $u = (1_\Omega, 0_\Omega)$. Further

$$\mathcal{M} = \{A = (\mu_A, \nu_A); (0_\Omega, 1_\Omega) \leq A \leq (1_\Omega, 0_\Omega)\},$$

$$\neg A = (1_\Omega, 0_\Omega) - (\mu_A, \nu_A) = (1 - \mu_A, 1 - \nu_A),$$

$$A \oplus B = (A + B) \wedge (1_\Omega, 0_\Omega) = (\mu_A + \mu_B, \nu_A + \nu_B - 1) \wedge (1_\Omega, 0_\Omega) =$$

$$= ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$

$$A \odot B = (A + B - (1_\Omega, 0_\Omega)) \vee (0_\Omega, 1_\Omega) =$$

$$= [(\mu_A + \mu_B, \nu_A + \nu_B - 1) - (1_\Omega), 0_\Omega)] \vee (0_\Omega, 1_\Omega) =$$

$$= (\mu_A + \mu_B - 1, \nu_A + \nu_B - 1 - 0 + 1) \vee (0_\Omega, 1_\Omega) = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1).$$

We see that the MV-algebra operations coincide on $\mathcal{F}$ with the IF-operations. Now let $m : \mathcal{F} \to [0, 1]$ be a state, $A = (\mu_A, \nu_A) \in \mathcal{M}$. It is easy to see that

$$(\mu_A, \nu_A) \oplus (0_\Omega, 1_\Omega - \nu_A) = (\mu_A, 0_\Omega),$$

$$(\mu_A, \nu_A) \odot (0_\Omega, 1_\Omega - \nu_A) = (0_\Omega, 1_\Omega).$$

Therefore it is reasonable to define

$$\bar{m}((\mu_A, \nu_A)) = m((\mu_A, 0_\Omega)) - m(0_\Omega, 1_\Omega - \nu_A).$$

We see that $\bar{m} \mid \mathcal{F} = m$, and evidently $\bar{m} : \mathcal{M} \to [0, 1]$ is a state on the MV-algebra $\mathcal{M}$.

6 Conclusions

We have seen that the family of $\mathcal{F}$ can be embedded to an MV-algebra by such a way that all results summarized in [16] and [17] may be applied to $\mathcal{F}$. Namely, also results based on MV-algebra with product ([12], [11], [6]) can be applied to $\mathcal{F}$ using the product

$$A.B = (\mu_A.\mu_B, 1 - (1 - \nu_A).(1 - \nu_B)) = (\mu_A.\mu_B, \nu_A + \nu_B - \nu_A.\nu_B).$$

Recall the D-poset theory presented in [10] and equivalent with effect algebras theory [4]. MV-algebra is a special case of D-poset, hence all results from probability theory on D-posets can be applied to our IF-case, too. Recently also on D-posets an operation of product has been introduced ([8],[9]). Such D-posets has been called as Kôpka D-posets, also probability results on the Kôpka D-posets (see e.g. [15]) can be applied to the IF-events.

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) organized in Warsaw on September 30, 2011 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Tenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2011) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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