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Basic concepts of the mechanics of discretized bodies with an introduction to discrete element calculus

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BY A DISCRETIZED body we mean an approximate model of a continuous body, obtained by the process of discretization. The aim of this note is to present a general approach to the mechanics of discretized bodies in terms of the laws of motion (which have the same form for each discretized body) and the constitutive equations. The characteristic feature of the mechanics formulated for discretized bodies is the simple form of the basic equations and their formal resemblance to the known equations of the mechanics of continuous media. The equations obtained form also the theoretical basis of the known finite element method [7].

Ciałem dyskretyzowanym nazywamy przybliżony model ośrodka ciągłego, otrzymany w procesie dyskretyzacji. W pracy przedstawiono ogólne podejście do mechaniki ciał dyskretyzowanych za pomocą równań ruchu (takich samych dla każdego ciała dyskretyzowanego) oraz równań konstytutywnych. Cechą szczególną tych równań jest ich prosta budowa oraz formalne podobieństwo do znanych równań mechaniki ośrodków ciągłych. Otrzymane równania tworzą jednocześnie teoretyczną podbudowę dla znanej metody elementów skończonych [7].

Дискретизуемым телом называем приближенную модель сплошной среды, получаемую в результате процесса дискретизации. В работе изложен общий подход к проблемам механики дискретизуемых тел, основанный на уравнениях движения (одинаковых для всех дискретизуемых тел) и на определяющих уравнениях. Отличительной чертой этих уравнений является простое строение и формальное подобие с известными уравнениями механики сплошной среды. Выведенные уравнения образуют одновременно теоретический базис известного метода конечных элементов [7].

1. Discretized bodies

TO OBTAIN a discretized body from a given continuous body we have to separate the continuous body by imaginary surfaces into a finite or a countable set of disjointed elements. Such elements are said to be the finite elements. We assume that the motion of an arbitrary finite element can be described, with sufficient accuracy, by the motion of the finite set of material elements not belonging to any finite elements and having only a finite number of degrees of freedom. These material elements are said to be the particles of the discretized body. Finite elements are assumed to be connected only by particles. For a particle we can take a free point, a finite set of free points or a finite set of points subjected to constraints. An arbitrary particle is denoted by d, and D is the set of all particles. Moreover, we assume that:

1. The mass distribution in the continuous body is approximated by concentrated masses assigned only to the particles of the discretized body,

2. Each particle is a separate holonomic dynamic system with n degrees of freedom, where n is constant for an arbitrary $d \in D$.

3. The generalized coordinates $q^a(d, \tau)$, $a = 1, 2, \ldots, n$, of the particle $d \in D$ (τ is the time coordinate) are, for each $d \in D$, the components of the vector in one n-th dimensional vector space V^n which is said to be the configuration space(¹).

We see that to each particle $d \in D$ we can assign the kinetic energy

$$(1.1) \qquad T = T(d, \ldots) = \frac{1}{2} a_{ab}(d, \ldots)\dot{q}^a(d, \tau)\dot{q}^b(d, \tau) + a_a(d, \ldots)\dot{q}^a_i(d, \tau) + a(d, \ldots),$$

where $a_{ab}(d, \ldots)$, $a_a(d, \ldots)$, $a(d, \ldots)$ are, for each $d \in D$, the known functions of the time coordinate τ and generalized coordinates $q^c(d, \tau)$; in what follows, the indices a, b, c involve the sequence $1, 2, \ldots, n$.

The set of particles which determines the motion of the given finite element is said to be the discrete element. Each discrete element will be denoted by E and the set of all such elements by $\mathscr{E}$, $E \in \mathscr{E}$. We see that $\mathscr{E}$ is a covering of D, and particles interact only in subsets $E \in \mathscr{E}$.

From the point of view of geometry, the discretized body is a pair $(D, \mathscr{E})$, where D is a countable or finite set of particles $d \in D$, and $\mathscr{E}$ is the given covering of D by the discrete elements $E \in \mathscr{E}$, and where to each particle d is assigned the n-dimensional configuration vector space V^n_d. In the present paper, we restrict our considerations to the case in which an arbitrary space V^n_d is the same configuration space V^n.

The discretization process given above is also used, in more special form, in the well known finite element method [7]. Thus we conclude that the mechanics of discretized bodies can also be used as the theoretical basis of the finite element method. At the same time, the mechanics of discretized bodies yields many general theorems, analytical solutions (in the case of elastic discretized bodies [4]) and deals with problems in which the particles of the discretized body may have a very complicated structure. Moreover, we shall prove that the form of the basic equations of discretized bodies is independent of the number of finite elements of the continuous body, and is similar to the form of the known equations of the mechanics of continuous media. It is obvious that there are many discretization processes which can be applied to the same continuous body, but this problem is not considered in the mechanics of a discretized body, where we assume that such body is given a priori.

Now, we shall present some very simple examples of discretized bodies. On Fig. 1a is given the extensible rod (the continuous body), which can be approximated by a discretized body given on Fig. 1b. The particles of this discretized body are concentrated masses $d_1, d_2, \ldots, d_K$, the finite elements are weightless rod segments between particles d_i and d_{i+1}, $i = 1, 2, \ldots, K-1$, and the discrete elements are the sets $\{d_1, d_2\}, \ldots, \{d_{K-1}, d_K\}$. If the particles are free material points, then $n = 3$, and if they are rigid bodies, then $n = 6$; the positive integer n is said to be the number of local degrees of freedom of the discretized body. Into the pair $(D, \mathscr{E})$ we can introduce the allowable difference structure given on Fig. 1c; the function $f_I : D_I \to D_{-I}$ has the form $d_{i+1} = f_I(d_i)$,$i = 1, 2, \ldots, K-1$, where $D_I = \{d_1, d_2, \ldots, d_{K-1}\}$, $D_I = \{d_2, d_3, \ldots, d_K\}$ and $m = 1$ (cf. Appendix). On Fig. 2a

(¹) This assumption can be formulated in a weaker form; we can introduce a configuration space V^n_d for each $d \in D$ and define the connexion in the bundle of spaces V^n_d, $d \in D$ [5], cf. Sec. 5.

is given a rectangular plate which is divided into a set of finite elements. As the particles of the discretized body we may take straight-line segments, normal to the middle plane of the plate; if they are rigid we have $n = 5$. Any discrete element is a set of four particles

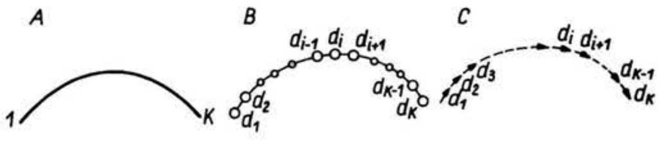

FIG. 1.

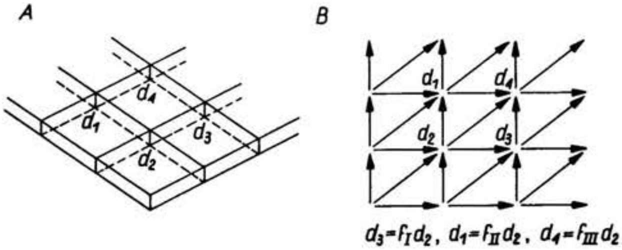

FIG. 2.

interconnected by the single finite element (for example particles d_1, d_2, d_3, d_4 on Fig. 2). The allowable difference structure on $(D, \mathscr{E})$ (on the discretized body) is given on Fig. 2b in the form of an oriented graph. The set of all "horizontal" vectors on Fig. 2b is represented by the function $f_{\rm I}: D_{\rm I} \to D_{-\rm I}$, the set of all "veitical" vectors is represented by the function $f_{\rm II}: D_{\rm II} \to D_{-\rm II}$, and all other vectors are represented by the function $f_{\rm III}: D_{\rm III} \to D_{-\rm III}$ (cf. Appendix). Another example of the pair $(D, \mathscr{E})$ is given on Fig. 3a,

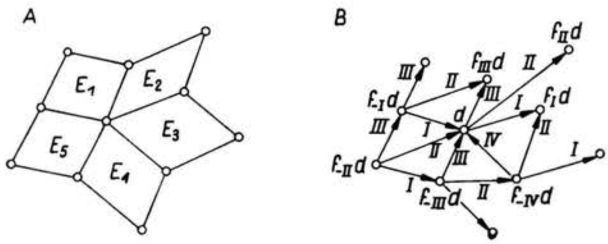

FIG. 3.

where D is a set of 11 points and $\mathscr{E}$ is a set of 5 discrete elements; each of them has 4 points. On Fig. 3b the allowable difference structure (cf. Appendix) is given in the form of a graph, where $m = 4$ and the functions $f_{\rm I}, f_{\rm II}, f_{\rm III}, f_{\rm IV}$ are represented by vectors denoted by I, II, III, IV, respectively.

2. Equations of motion

Let us denote by $Q_a = Q_a(d, \tau)$ the generalized force acting at the particle d, $d \in D$. Lagrange's equations of the second kind have the known form:

$$Q_a = r_a, \quad r_a \stackrel{\mathrm{df.}}{=} \frac{d}{d\tau}\frac{\partial T}{\partial \dot{q}^a} - \frac{\partial T}{\partial q^a}, \quad d \in D. \tag{2.1}$$

In what follows we assume that the argument d, not given explicitly in the equation, is defined in the set referred to on the right side of this equation. In order to write the equations of motion of the discretized body (i.e., the equations of motion of an arbitrary particle of this body), we have to introduce an allowable difference structure on each $(D_d, \mathscr{E}_d)$, $d \in D$, where $\mathscr{E}_d$ is a set of all discrete elements which contain the particle d, and D_d is a set of particles which is covered by $\mathscr{E}_d$ (cf. Appendix). Throughout this Section we assume that the indices $\Lambda, \ldots$ involve the sequence I, II, ..., m_d, where $m_d \stackrel{\mathrm{df}}{=} \max(\overline{\overline{D}}_d, \overline{\overline{\mathscr{E}}}_d)-1$. Let us denote by $F_a(d', \tau)$ the generalized force acting on the particle $d' \in D_d$ in the discrete element $E_{d'} \in \mathscr{E}_d$, and by $-T_a{}^{\Lambda}(d', \tau)$ the generalized force acting on the particle $f_\Lambda d'$ in the same discrete element $E_{d'}$. (In Sec. 1 we have assumed that the particles $d \in D$ interact only in subsets $E \in \mathscr{E}$); these denotations hold only when the discrete element $E_{d'}$ exists, and when $d' \in D_d^{\Lambda}$. To define the symbols $F_a(d, \tau)$, $T_a^{\Lambda}(d, \tau)$, $T_a{}^{\Lambda}(f_{-\Lambda}d, \tau)$, for an arbitrary $\Lambda = \mathrm{I}, \mathrm{II}, \ldots, m_d$, we put $F_a(d, \tau) \stackrel{\mathrm{df}}{=} 0$, $T_a^{\Lambda}(d, \tau) \stackrel{\mathrm{df}}{=} 0$ when the discrete element E_d does not exist (in this case $d \sim \in \bigcup_{\Lambda} D_d^{\Lambda}$); we also put $T_a{}^{\Lambda}(d, \tau) \stackrel{\mathrm{df}}{=} 0$ when $d \sim \in D_d^{\Lambda}$, and $T_a{}^{\Lambda}(f_{-\Lambda}d, \tau) \stackrel{\mathrm{df}}{=} 0$ when $d \sim \in D_d^{-\Lambda}$. All these denotations can be introduced in an arbitrary allowable difference structure on $(D_d, \mathscr{E}_d)$, $d \in D$; in many special cases, allowable difference structures on $(D_d, \mathscr{E}_d)$ can be induced by a single global difference structure on $(D, \mathscr{E})$ (cf. Appendix). Because at each $d \in D$ there acts also an external force, we can write:

$$Q_a(d, \tau) = F_a(d, \tau) - \sum_{\Lambda=I}^{m_d} T_a{}^{\Lambda}(f_{-\Lambda}d, \tau) + f_a(d, \tau), \tag{2.2}$$

where $f_a(d, \tau)$ is an external generalized force acting at d. Let us denote by $-t_a(d, \tau)$ the sum of generalized internal forces acting at the discrete element E_d, which are produced by the corresponding finite element—i.e., by interaction among the particles of this discrete element:

$$t_a(d, \tau) \stackrel{\mathrm{df.}}{=} F_a(d, \tau) - \sum_{\Lambda=I}^{m_d} T_a{}^{\Lambda}(d, \tau). \tag{2.3}$$

Combining (2.2) and (2.3), we obtain:

$$Q_a(d, \tau) = \sum_{\Lambda=I}^{m_d} [T_a{}^{\Lambda}(d, \tau) - T_a{}^{\Lambda}(f_{-\Lambda}d, \tau)] + t_a(d, \tau) + f_a(d, \tau). \tag{2.4}$$

Substituting (2.4) into (2.1) and using the symbol $\bar{\Delta}_\Lambda$ (cf. Appendix), we arrive at:

$$\bar{\Delta}_\Lambda T_a{}^{\Lambda} + t_a + f_a = r_a, \quad d \in D, \tag{2.5}$$

where the summation convention for the index Λ holds. The form of Eqs. (2.5) is independent of the material and structural properties of the finite elements $E \in \mathscr{E}_d$,(²); hence we see that Eqs. (2.5) may be called the equations of motion of the discretized body. The functions $T_a^{\Lambda}(d, \tau)$, $t_a(d, \tau)$ and the differences $\bar{\Delta}_\Lambda$ in (2.5) are determined only in a given allowable difference structure on $(D_d, \mathscr{E}_d)$; in any other allowable difference structure on $(D_d, \mathscr{E}_d)$, they have to be replaced by $T_a^{\Lambda'}(d, \tau)$, $t'_a(d, \tau)$, $\bar{\Delta}_{\Lambda'}$, respectively. Since the expression $r_a(d, \ldots)-f_a(d, \ldots)$ is independent of the difference structure on $(D_d, \mathscr{E}_d)$, and

$$\bar{\Delta}_{\Lambda'} T_a^{\Lambda'}(d, \tau)+t'_a(d, \tau) = \bar{\Delta}_\Lambda T_a^{\Lambda}(d, \tau)+t_a(d, \tau) = r_a(d, \ldots)-f_a(d, \tau),$$

then the sum $\Delta_\Lambda T_a^\Lambda + t_a$ is an invariant under arbitrary transformation of the allowable difference structure on $(D_d, \mathscr{E}_p)$.

Let us denote by $\delta L = \delta L(E)$ the elementary work performed by internal forces in the given discrete element $E \in \mathscr{E}$. Let on E be prescribed the local coordinate system (cf. Appendix) in which the internal forces are $F_a(d, \tau)$, $T_a^{\Lambda}(d, \tau)$. Denoting by $\delta q^a(d, \tau)$ an arbitrary variation of the functions $q^a(d, \tau)$, we arrive at:

$$\text{(2.6)} \qquad \delta L(E) = F_a(d, \tau)\delta q^a(d, \tau)+\sum_\Lambda T_a^{\Lambda}(d, \tau)\delta q^a(f_\Lambda d, \tau)$$
$$= -T_a(d, \tau)\delta q^a(d, \tau)+\sum_\Lambda T_a^{\Lambda}(d, \tau)\delta q^a(d, \tau)+\sum_\Lambda T_a^{\Lambda}(d, \tau)\delta\Delta_\Lambda q^a(d, \tau).$$

By virtue of (2.3), we obtain from (2.6):

$$\text{(2.7)} \qquad \delta L = T_a^{\Lambda}\delta\Delta_\Lambda q^a - t_a \quad \text{or} \quad \dot{L} = T_a^{\Lambda}\Delta_\Lambda \dot{q}^a - t_a\dot{q}^a.$$

The form δL is an invariant under a group of transformations of the local coordinate system (cf. Appendix). Hence we see that m_d+1 numbers $-t_a$, T_a^{Λ}, given for each a, τ, d, are components of a vector at E_d and with respect to the group U^{m+1}. The m_d+1 numbers $q^a, \Delta_\Lambda q^a$, given for each a, d, τ, are components of a covector at E_d and with respect to the same group U^{m+1}. The components of the discrete gradient $\Delta_\Lambda q^a$ (a and d, τ are fixed) are components of the covector at E_d with respect to the group U_m. Now, let the index Λ and the parameter τ be fixed. The n numbers $T_a^{\Lambda}(d, \tau)$ and the n numbers $t_a(d, \tau)$ are components of the covectors in the vector space V_d^{*n} dual to V^n. The external force acting at d, and the inertial force, are also represented by covectors in V_d^{*n}, and we may call the vector space V_d^{*n} the space of forces.

3. Constitutive equations

Now, we are to establish the relation between the motion of an arbitrary discrete element E and the internal forces in such element. Prescribing at E the coordinate system $f\colon E \to (d, f_1 d, \ldots, f_m d)$, $m \overset{\text{df}}{=} \overline{\overline{E}}-1$, we can describe the motion of E by the functions $q^a(d, \tau)$, $\Delta_\Lambda q^a(d, \tau)$; it is a motion localized at $d \in E$. The internal forces in E can be given

(²) The term "finite element" in the mechanics of discretized bodies has a meaning different from that in the discretization process, denoting only the external field by means of which the particles of the given discrete element interact.

now by the functions $t_a(d, \tau)$, $T_a{}^A(d, \tau)$. According to the principle of determinism, we postulate that the internal forces in E and at the time instant τ are produced by the history of motion of E. It follows that the constitutive equations of the discretized body can be written, for each $E \in \mathscr{E}$, in the form:

$$(3.1)\qquad \begin{aligned} T_a{}^A(d, \tau) &= \mathop{\mathbf{S}_a^A}_{\sigma=-\infty}^{\tau} \big(d, q^b(d, \sigma), \Delta_\Phi(d, \sigma)\big),\\ t_a(d, \tau) &= \mathop{\mathbf{S}_a}_{\sigma=-\infty}^{\tau} \big(d, q^b(d, \sigma), \Delta_\Phi q^b(d, \sigma)\big); \qquad A, \Phi = I, II, \ldots, m, \end{aligned}$$

where $\mathbf{S}_a^A$ and $\mathbf{S}_a$ are said to be the constitutive functionals. The argument d in these functionals denotes that the motion of E is localised at $d \in E$. The constitutive functionals in the mechanics of discretized bodies characterizes the material, kinematical and geometrical properties of the finite part of a deformable body (of a given finite element), while the constitutive functionals in the theory of continuous media describe only local (i.e. material) properties of this body.

If for each $E \in \mathscr{E}$ we have $\overline{\overline{E}} = m+1 = \text{const}$, and if we can prescribe at each E the local coordinate system in which the form of the constitutive functionals is the same for each $E \in \mathscr{E}$, then the discretized body will be called uniform. If $\overline{\overline{E}} = m+1 = \text{const}$ for each $E \in \mathscr{E}$, $\overline{\overline{\mathscr{E}}}_d \leqslant m+1$ for each $d \in D$, and all the local coordinate systems mentioned are induced by the sole difference structure on $(D, \mathscr{E})$, then the discretized body is said to be homogeneous. An arbitrary discrete element E will be called homogeneous if the form of the constitutive functionals of E is independent of $d \in E$.

Equations (3.1) represent the general form of the constitutive equations in the mechanics of discretized bodies, and together with the equations of motion (2.5) form the basic system of equations of discretized bodies. We see that the equations of discretized bodies have a form similiar to the known equations of mechanics of continuous media. The main problem of the discretized body theory is to determine the form of the constitutive functionals; this problem has been solved only for certain special kinds of discretized bodies and examples of the constitutive equations will be given in separate papers.

Suppose we are given now the special case in which the potential $e\big(d, q^a(d, \tau), q^a(f_A d, \tau)\big)$ of the discrete element $E = E_d$ exists, where we have tacitly assumed that the coordinate system is prescribed at E. From the definition of the potential it follows that:

$$(3.2)\qquad F_a(d, \tau) = -\frac{\partial e(d, \ldots)}{\partial q^a(d, \tau)}, \qquad T_a{}^A(d, \tau) = \frac{\partial e(d, \ldots)}{\partial q^a(f_A d, \tau)}.$$

Let us define the function $\varepsilon(d, \ldots)$ putting

$$(3.3)\qquad \varepsilon\big(d, q^a(d, \tau), \Delta_A q^a(d, \tau)\big) \overset{\text{df}}{=} e\big(d, q^a(d, \tau), q^a(d, \tau) + \Delta_A q^a(d, \tau)\big).$$

By virtue of (3.2), (3.3), (2.3) and the relations

$$(3.4)\qquad \frac{\partial e(d, \ldots)}{\partial q^a(f_A d, \tau)} = \frac{\partial \varepsilon(d, \ldots)}{\partial \Delta_A q^a(d, \tau)}, \qquad \frac{\partial \varepsilon(d, \ldots)}{\partial q^a(d, \tau)} = \frac{\partial \varepsilon(d, \ldots)}{\partial q^a(d, \tau)} + \sum_{A=I}^{m_d} \frac{\partial e(d, \ldots)}{\partial q^a(f_A d, \tau)},$$

we arrive at

$$(3.5)\qquad T_a{}^{\Lambda}(d,\tau) = \frac{\partial\varepsilon(d,\tau)}{\partial\Delta_\Lambda q^a(d,\tau)}, \qquad t_a(d,\tau) = -\frac{\partial\varepsilon(,\dots)}{\partial q^a(d,\tau)}.$$

The discrete element for which Eqs. (3.5) hold is said to be elastic. If the relations (3.5) hold for each $E \in \mathscr{E}$, then the discretized body is called elastic, and Eqs. (3.5) and (2.5) are equations of discrete elasticity, [4]. These equations form a system of the second order ordinary differential equations for the unknown functions $q^a(d, \tau)$, $d \in D$, and have to be considered together with the initial conditions

$$(3.6)\qquad q^a(d,\tau_0) = q_0^a(d), \qquad \dot{q}^a(d,\tau_0) = \dot{q}_0^a(d), \qquad d \in D,$$

where $q_0^a(d)$, $\dot{q}_0^a(d)$ are given.

Now, let us suppose that for a given discrete element E there exists a real-valued differentiable function, which in a given local coordinate system can be represented in the form $\Phi = \Phi(d, T_a{}^{\Lambda}(d,\tau), t_a(d,\tau))$. We assume that $\Phi \leqslant 0$; if $\Phi < 0$, then the discrete element E can be treated as elastic and if

$$(3.7)\qquad \Phi = 0, \quad \delta\Phi = \frac{\partial\Phi}{\partial T_a{}^{\Lambda}}\delta T_a{}^{\Lambda} + \frac{\partial\Phi}{\partial t_a}\delta t_a = 0,$$

then appear the plastic quantities $\Delta_\Lambda \tilde{q}^a$, $\tilde{q}^a$. These quantities are not determined uniquely but within an arbitrary rigid motion of E (cf. Sec. 4). Using $(2.7)_2$, we can write:

$$(3.8)\qquad \dot{\delta}L = \delta T_a{}^{\Lambda}\Delta_\Lambda \dot{\tilde{q}}^a - \delta t_a \dot{\tilde{q}}^a = 0.$$

By virtue of (3.8) and $(3.7)_2$, we obtain:

$$(3.9)\qquad \Delta_\Lambda \dot{\tilde{q}}^a = \lambda\frac{\partial\Phi}{\partial T_a{}^{\Lambda}}, \qquad \dot{\tilde{q}}^a = -\lambda\frac{\partial\Phi}{\partial t_a}; \qquad d \in D,$$

where λ is a constant. The function $\Phi(d, \dots)$ is said to be the plastic potential at the discrete element E_d, the condition $\Phi(d, \dots) = 0$ is called the condition of plasticity, and Eqs. (3.9) are the laws of plastic flow at E_d. If the function Φ is determined for each discrete element of the discretized body, then this body is said to be the elastic-plastic discretized body.

The form of the constitutive functionals depends on the choice of the local coordinate system. Using the transformation formulas given in the Appendix, we can write:

$$(9,10)\qquad \begin{aligned} \mathbf{S}_a(d, q^b, \Delta_\Lambda q^b) &= {}'\mathbf{S}_a(d', {}'q^b, \Delta_\Lambda, {}'q^b) = {}'\mathbf{S}_a(d, q^b + {}'a^{\Lambda}\Delta_\Lambda q^b, B_{\Lambda'}{}^{\Lambda}\Delta_\Lambda q^b),\\ \mathbf{S}_a(d, q^b, \Delta_\Phi q^b) &= B_{\Lambda'}{}^{\Lambda}{}'\mathbf{S}_a^{\Lambda'}(d', {}'q^b, \Delta_{\Phi'}{}'q^b) + {}'a^{\Lambda}{}'\mathbf{S}_a(d', {}'q^b, \Delta_{\Phi'}{}'q^b)\\ &= B_{\Lambda'}{}^{\Lambda}{}'S_a^{\Lambda'}(d, q^b + {}'a^{\Phi}\Delta_\Phi \; B\; {}^{\Phi}\Delta_\Phi q\;) + {}'a^{\Lambda}{}'S_a(d, q^b + {}'a^{\Phi}\Delta_\Phi q^b, B_{\Phi'}{}^{\Phi}\Delta_\Phi q^b) \end{aligned}$$

$$\mathbf{S}_{a]}^{\Lambda} \overset{\text{df}}{=} \overset{\tau}{\underset{\sigma=-\infty}{\mathbf{S}}}{}_a, \qquad \mathbf{S}_a^{\Lambda} \overset{\text{df}}{=} \overset{\tau}{\underset{\sigma=-\infty}{\mathbf{S}}}{}_a^{\Lambda}, \qquad q^b \overset{\text{df}}{=} q^b(d,\sigma), \qquad {}'q^b \overset{\text{df}}{=} q^b(d',\sigma),$$

where $f: E \to (d, f_{\mathrm{I}}d, \dots, f_m d)$ and $f': E \to (d', \dots,)$, $m = m' = \overline{\overline{E}} - 1$, are two arbitrary coordinate systems at E. In what follows, we shall interpret the transformation $f' \circ f^{-1}$ in (3.10) (given by ${}'a^{\Phi}$ and $B_{\Phi'}{}^{\Phi}$) as the point transformation which maps the particle d on the particle d', and the particle $f_\Lambda d$ on the particle $f_{\Lambda'}d'$ for $\Lambda' = \Lambda = \mathrm{I}, \mathrm{II}, \dots, m$. We can prove that there always exists a subgroup I^{m+1} of the group U^{m+1} (cf. Appendix) such that $\mathbf{S}_a(\dots) = \mathbf{S}_a(\dots)$ and $'S_a^{\Lambda'}(\dots) = \delta_\Lambda^{\Lambda'} S_a^\Lambda(\dots)$, where the arguments denoted by points are the same in each functional. In this case, by virtue of (3.10), we conclude that the relations

$$\begin{aligned}\mathbf{S}_a(d, q^b, \Delta_\Phi q^b) &= \mathbf{S}_a(d, q^b + 'a^\Phi \Delta_\Phi q^b, B_{\Phi'}{}^\Phi \Delta_\Phi q^b),\\ \mathbf{S}_a^\Lambda(d, q^b, \Delta_\Phi q^b) &= B_{\Lambda'}{}^\Lambda \delta_\Xi^{\Lambda'} \mathbf{S}_a^\Xi(d, q^b + 'a^\Phi \Delta_\Phi q^b, B_{\Theta'}{}^\Phi \Delta_\Phi q^b)\\ &\quad + 'a^\Lambda \mathbf{S}_a(d, q^b + 'a^\Phi \Delta_\Phi q^b, B_{\Phi'}{}^\Phi \Delta_\Phi q^b),\end{aligned} \tag{3.11}$$

hold for an arbitrary element of the group I^{m+1}, given by the sequence $('a_\Phi)$and by the matrix $(B_{\Phi'}{}^\Phi)$. The group I^{m+1} is said to be the isotropy group of the constitutive functionals $\mathbf{S}_a$, $\mathbf{S}_a^\Lambda$ at the discrete element $E \in \mathscr{E}$ and in the localization of motion in the particle $d \in E$. In the special case, the isotropy group is a trivial group—i.e., $'a^\Phi = 0$, $B_{\Phi'}{}^\Phi = \delta_{\Phi'}^\Phi$ is the only element of this group. If the group I^{m+1} contains the group 0^{m+1} (cf. Appendix) for all $m+1$ localizations of the motion of E, then the discrete element E is said to be isotropic. If the discrete element is elastic, then Eq. (3.10) has to be replaced by:

$$\varepsilon(d, q^b, \Delta_\Phi q^b) = \varepsilon'(d', 'q^b, \Delta_{\Phi'} 'q^b) = \varepsilon'(d, q^b + 'a^\Phi \Delta_\Phi q^b, \quad B_{\Phi'}{}^\Phi \Delta_\Phi q^b). \tag{3.12}$$

Let us suppose that $'a^\Phi$, $B_{\Phi'}{}^\Phi$ in (3.12) determine the element of the isotropy group J^{m+1}. In this case, we obtain $\varepsilon'(\dots) = \varepsilon(\dots)$ and we can write

$$\varepsilon(d, q^b, \Delta_\Phi q^b) = \varepsilon(d, q^b + 'a^\Lambda \Delta_\Lambda q^b, B_{\Lambda'}{}^\Lambda \Delta_\Lambda q^b). \tag{3.13}$$

If the elastic discrete element is isotropic, then (3.13) must be satisfied for $'a^\Phi = 0$, for each $(B_{\Lambda'}{}^\Lambda) = (Q_{\Lambda'}{}^\Lambda)$ and each $d \in E$. In isotropic elastic discretized bodies, the elastic potential is an invariant under the group O^m.

4. Stresses and strains

Let there be given an arbitrary discrete element E and a local coordinate system $f: E \to \to (d, f_I d, \dots, f_m d)$. Let the elementary work $dL = dL(E)$ be an invariant under the k-parameter infinitesimal group of transformations $q^a(d', \tau) \to q^a(d', \tau) + \lambda\big(d', q^b(d', \tau)\big) \epsilon^\alpha$ $d' \in E$, where ϵ^α, $\alpha = 1, 2, \dots, k$, are independent infinitesimal parameters. Moreover, we assume that $dL(E)$ is not invariant under any $k+1$ parameter group of transformations. Substituting $\delta q^a(d, \tau) = \lambda_\alpha^a\big(d, q^b(d, \tau)\big) \epsilon^\alpha$ and $\delta \Delta_\Lambda q^a(d, \tau) = \Delta_\Lambda \lambda_\alpha^a\big(d, q^b(d, \tau)\big) \epsilon^\alpha$ into Eqs. (2.7), we obtain $\delta L = 0$ for arbitrary $\epsilon^1, \epsilon^2, \dots, \epsilon^k$. It follows that

$$T_a{}^\Lambda(d, \tau) \Delta_\Lambda \lambda_\alpha^a\big(d, q^b(d, \tau)\big) - t_a(d, \tau) \lambda_\alpha^a\big(d, q^b(d, \tau)\big) = 0, \qquad \alpha = 1, \dots, k. \tag{4.1}$$

Let us denote by ϵ^μ, $\mu = 1, 2, 3$, three independent infinitesimal translations of the physical space and by $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, $\nu = 1, 2, 3$, the parameters which determine three independent infinitesimal rotations of this space. Using the parameters ϵ^μ, $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ in place of $\epsilon^1, \epsilon^2, \dots, \epsilon^6$, and assuming that $q^a(d', \tau) \to q^a(d', \tau) + C_a^\mu \epsilon_\mu + C_{b\mu\nu}^a q^b(d', \tau) \epsilon^{\mu\nu}$, where C_a^μ, $C_{b\mu\nu}^a = -C_{b\nu\mu}^a$ are constants, we can write Eqs. (4.1) in the form:

$$t_a C_\mu^a = 0, \qquad (T_a{}^\Lambda \Delta_\Lambda q^b - t_a q^b) C_{b\mu\nu}^a \varepsilon^{\mu\nu}{}_\pi = 0, \tag{4.2}$$

where $\varepsilon^{\mu\nu}{}_\pi$, $\pi = 1, 2, 3$, is the permutation symbol. It is easy to prove that the left-hand sides of (4.2) represent the components of the resultant force and the resultant moment acting on the finite element from the corresponding discrete element; as we assumed in

Eq. (2.6), the finite element is loaded only by forces acting from the discrete element. In view of (4.2), we have $\min k = 6$. If $k = 6$, then the finite element (cf. footnote on p. 6) is said to be stable, and if $k > 6$ then it is said to be unstable.

Now let us introduce the sequence of K differentiable functions

$$\gamma_A(d, \tau) = \varphi_A\big(d, q^a(d, \tau), \Delta_A q^a(d, \tau)\big), \quad A = 1, 2, \ldots, K, \tag{4.3}$$

defined at each $E \in \mathscr{E}$ in an arbitrary local coordinate system. Moreover, we assume that the form of the functions φ_A is the same in each local coordinate system at E. It follows that the numbers $\gamma_1, \gamma_2, \ldots, \gamma_K$ are, for each E, the components of the concomitant of the covectors with components q^a, $\Delta_A q^a$ (cf. Appendix). We also assume that $\delta\gamma_A = 0$, $A = 1, 2, \ldots, K$, are necessary and sufficient conditions for $\delta L = 0$. By virtue of

$$\delta\gamma_A = \frac{\partial\varphi_A}{\partial q^a}\delta q^a + \frac{\partial\varphi_A}{\partial\Delta_A q^a}\delta\Delta_A q^a = \left(\frac{\partial\varphi_A}{\partial q^a}\lambda^a_\alpha + \frac{\partial\varphi_A}{\partial\Delta_A q^a}\Delta_A\lambda^a_\alpha\right)\epsilon^\alpha = 0,$$

we arrive at

$$\frac{\partial\varphi_A}{\partial\Delta_A q^a}\Delta_A\lambda^a_\alpha + \frac{\partial\varphi_A}{\partial q^a}\lambda^a_\alpha = 0, \quad \alpha = 1, 2, \ldots, k, \quad A = 1, 2, \ldots, K. \tag{4.4}$$

Now, let us introduce the sequence of K functions $p^A = p^A(d, \tau)$ and let us suppose that the variation δL can always be prescribed by the expression $\delta L = p^A\delta\gamma_A$ (the summation convention for the index A holds). In view of $(2.7)_1$ we have

$$\delta L = p^A\delta\gamma_A = p^A(\Phi^A_{Aa}\delta\Delta_A q^a + \Phi_{Aa}\delta q^a) = T_a{}^A\delta\Delta_A q^a - t_a\delta q^a, \tag{4.5}$$

where we have denoted

$$\Phi^A_{Aa} \overset{\text{df}}{=} \frac{\partial\varphi_A}{\partial\Delta_A q^a}, \quad \Phi_{Aa} \overset{\text{df}}{=} \frac{\partial\varphi_A}{\partial q^a}, \quad N = 1, 2, \ldots, K. \tag{4.6}$$

From (4.5) we conclude that

$$T_a{}^A = p^A\Phi^A_{Aa}, \quad t_a = p^A\Phi_{Aa}. \tag{4.7}$$

Since $n(m+1)$ the internal forces $T_a{}^A$, t have to satisfy the k relations (4.1), then we can assume that $K = n(m+1)-k$. Using the K functions $\gamma_A(d, \tau)$ and the K functions $p^A(d, \tau)$, we may disregard the condition (4.1), which will be satisfied as an identity. The functions $\gamma_A(d, \tau)$ are said to be the strains, and the functions $p^A(d, \tau)$ are said to be the stresses in a given discrete element of the discretized body. Some examples of stresses and strains in discretized bodies are given in separate papers.

2. Alternative forms of basic equations

By virtue of (4.7), we can transform the equations of motion (2.5) to the following:

$$\bar{\Delta}_A(p^A\Phi^A_{Aa}) + p^A\Phi_{Na} + f_a = r_a, \tag{5.1}$$

The constitutive equations (3.1) may be replaced by the following:

$$p^A(d, \tau) = \mathop{\mathbf{P}^A}_{-\infty}^{\tau}\big(d, \gamma_B(d, \sigma)\big), \quad A, B = 1, 2, \ldots, K, \tag{5.2}$$

where $\mathbf{P}^A$ are given constitutive functionals. Equations (5.1), (5.2) and the "geometric" equations given in Sec. 4

$$\gamma_a(d, \tau) = \varphi_A\big(d, q^a(d, \tau), \Delta_A q^a(d, \tau)\big), \quad A = 1, 2, \ldots, K, \tag{5.3}$$

constitute an alternative form of the basic equations of the mechanics of discretized bodies. In the special case of an elastic discretized body, we obtain $\varepsilon = \varepsilon\big(d, \gamma_A(d, \tau)\big)$ and $p^A(d, \tau) = \partial\varepsilon\big(d, \gamma_B(d, \tau)\big)/\partial\gamma_A(d, \tau)$. If we have to deal with an elastic-plastic body, then we have $\tilde{\dot{\gamma}}_A = \lambda\partial\Phi/\partial p^A$, where $\Phi = \Phi\big(d, p^A(d, \tau)\big)$ is the elastic potential.

In many special problems, it is convenient to assign to each $d \in D$ a separate configuration space V_d^n and a dual space V_d^{*n}. Let there be given on $(D_d, \mathscr{E}_d)$ an allowable difference structure; now, we have to assume additionally that there is given a connexion between spaces V_d^n, $V_{f_A d}^n$ (if we are to write the constitutive equations) and between spaces V_d^{*n}, $V_{f_{-A}d}^{*n}$ (if we are to write the equations of motion). Such connexions are determined, for any fixed A, by the non-singular $n\times n$ matrices $\big(\delta_\beta^\alpha + G_{A\beta}^{\ \alpha}(d)\big)$, $\big(\delta_\beta^\alpha + \overset{*}{G}{}_{\alpha A}^{\beta}(d)\big)$, $\alpha, \beta = 1, 2, \ldots, n$, where

$$q_{(A)}^\alpha(d, \tau) = \big(\delta_\beta^\alpha + G_{A\beta}^{\ \alpha}(d)\big) q^\beta(f_A d, \tau), \qquad T_{\alpha(A)}(d, \tau) = \big(\delta_\alpha^\beta + \overset{*}{G}{}_{\alpha A}^{\beta}(d)\big) T_\beta(f_{-A} d, \tau), \tag{5.4}$$

are components of the vectors after parallel transport from $V_{f_A d}^n$ to V_d^n and from $V_{f_{-A}d}^{*n}$ to V_d^{*n}, respectively. Defining the following "absolute" differences

$$\begin{aligned} \delta_A q^\alpha(d, \tau) &\overset{\mathrm{df}}{=} q_{(A)}^\alpha(d, \tau) - q^\alpha(d, \tau) = \Delta_A q^\alpha(d, \tau) + G_{A\beta}^{\ \alpha}(d) q^\beta(f_A d, \tau), \quad d \in D_A, \\ \bar{\delta}_A T_\alpha(d, \tau) &\overset{\mathrm{df}}{=} T_{\alpha(A)}(d, \tau) - T_\alpha(d, \tau) = \bar{\Delta}_A T_\alpha(d, \tau) + \overset{*}{G}{}_{\alpha A}^{\beta}(d) T_\beta(f_{-A} d, \tau), \quad d \in D_{-A}, \end{aligned} \tag{5.5}$$

we have to replace the differences $\bar{\Delta}_A q^a(d, \tau)$, $\bar{\Delta}_A T_a{}^A(d, \tau)$ by the absolute differences $\delta_A q^\alpha(d, \tau)$, $\delta_A T_\alpha{}^A(d, \tau)$, respectively. Thus we arrive at the following equations of motion:

$$\bar{\delta}_A T_\alpha{}^A + t_\alpha + f_\alpha = r_\alpha, \quad d \in D, \tag{5.6}$$

and the constitutive equations:

$$\begin{aligned} T_\alpha{}^A(d, \tau) &= \overset{\tau}{\underset{\sigma=-\infty}{\mathbf{S}_\alpha^A}} \big(d, q^\beta(d, \sigma), \delta_A q^\beta(d, \sigma)\big), \\ t_\alpha(d, \tau) &= \overset{\tau}{\underset{\sigma=-\infty}{\mathbf{S}_\alpha}} \big(d, q^\beta(d, \sigma), \delta_A q^\beta(d, \sigma)\big). \end{aligned} \tag{5.7}$$

In the special case of a discretized elastic body, we obtain from (5.7)

$$T_\alpha{}^A(d, \tau) = \frac{\partial\varepsilon(d, \ldots)}{\partial\delta_A q^\alpha(d, \tau)}, \qquad t_\alpha(d, \tau) = -\frac{\partial\varepsilon(d, \ldots)}{\partial q^\alpha(d, \tau)}, \tag{5.8}$$

where $\varepsilon(d, \ldots) = \varepsilon\big(d, q^\beta(d, \tau), \delta_A q^\beta(d, \tau)\big)$ is the elastic potential. For an elastic-plastic discretized body, we can write:

$$\delta_A \dot{\tilde{q}}^\alpha(d, \tau) = \lambda\frac{\partial\Phi(d, \ldots)}{\partial T_\alpha{}^A(d, \tau)}, \qquad \dot{\tilde{q}}^\alpha(d, \tau) = -\lambda\frac{\partial\Phi(d, \ldots)}{\partial t_\alpha(d, \tau)}, \tag{5.9}$$

where $\Phi\big(d, T_a{}^A(d, \tau), t_a(d, \tau)\big)$ is the plastic potential. We use here the indices α, β instead of a, b, to underline the fact that the separate vector spaces V_d^n, V_d^{*n} are introduced for each $d \in D$. The problem of connexion in the bundle V_d^n, $d \in D$ of the vector spaces is discussed in [5] and applied in [4].

6. Final remarks

The main feature of the mechanics of discretized bodies is the simple and general form of the basic Eqs. (2.5), (3.1), their independence of the process of discretization (Eqs. (2.5), (3.1) are the same for all discretized bodies), and their formal resemblance to the known equations of continuum mechanics. The basic equations given in the present paper are only the starting point for further considerations which concern the linear and small deformation theory [8], the theory of variated states with vibration and stability problems [9], and the solution of various special problems.

We can also observe that the equations of mechanics of discretized bodies in quasi-static as well as in simple dynamical problems can be directly used as the basic equations of the finite element method, if we are to obtain numerical solutions of these problems. In more complicated dynamical or time-dependent problems, we can arrive at the system of algebraic equations of the finite element method after performing discretization with respect to the time coordinate in the equations given in the paper.

To conclude the paper, let us compare the mechanics of discretized bodies with continuum mechanics. It is easy to see that all problems which can be discussed and solved within the mechanics of continuous media are not of interest as problems of the mechanics of discretized bodies. On the other hand, it must be born in mind that it is only simple boundary value problems of continuum mechanics that we are able to solve and to discuss; in particular, the region occupied by the continuous body and the external forces have to be given by simple analytical expressions. In the boundary value problems of continuum mechanics in which there are many singularities and discontinuities, also the finite difference approach to boundary value problems leads to serious numerical difficulties. Such difficulties do not exist in the mechanics of discretized bodies, where we do not deal with any boundary conditions and where the non-continuous and singular character of external agents (concentrated forces) does not introduce any additional difficulties in solving the problem. Thus we can state that the theory of discretized bodies can be applied for somewhat complicated technical problems such as non-smooth shells, shells and plates made of different materials or loaded by many concentrated forces, systems composed of many plates and rods, etc. All such problems, which are too complex to be solved by use of the continuum approach, will be disccussed in separate papers.

Appendix

Introduction to discrete element calculus

In finite difference calculus, we deal with real valued functions defined on finite or countable sets, provided that all points of such sets belong to a given lattice $\mathscr{L}$ in the n-th dimensional affine space (E, V^N)(3). Such lattices are introduced in order to replace the

(3) The affine space (E, V^N) is a space of points $\mathbf{x} \in E$ and vectors $\mathbf{t} \in V^N$, endowed with the mapping $E \times V^N \to E$ (which is denoted by the symbol $+$, — i.e., $\mathbf{x}+\mathbf{t} = \mathbf{x}' \in E$) which satisfy additional conditions, [3]. The subset $\mathscr{L} \subset E$ is said to be the lattice in (E, V^N), if it is a set of all points $\mathbf{x} = \mathbf{x}_0 + \alpha_1 \mathbf{t}_1 + \ldots + \alpha_N \mathbf{t}_N$, where $(\mathbf{x}_0; \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_N)$ is a given basis in (E, V^N) and $\alpha_1, \alpha_2, \ldots, \alpha_N$ are arbitrary integers.

differential equations by finite difference equations. The finite differences of an arbitrary function $\varphi: L \to R$, $L \subset \mathscr{L}$ are defined as the differences $\Delta_K\varphi(\mathbf{x}) \overset{\text{df}}{=} \varphi(\mathbf{x}+\mathbf{t}_K)-\varphi(\mathbf{x})$, $\bar{\Delta}_K\varphi(\mathbf{x}) \overset{\text{df}}{=} \varphi(\mathbf{x})-\varphi(\mathbf{x}-\mathbf{t}_K)$, $K = 1, 2, \ldots, N$, assuming that $\mathbf{x} \in L, \mathbf{x}+\mathbf{t}_K \in L, \mathbf{x}-\mathbf{t}_K \in L$ [2]. In the mechanics of discretized bodies or in the finite element method, we also deal with real valued functions defined on finite or countable sets, but points of these sets do not belong to any lattice in (E, V^N), provided that we do not take into account certain special situations [1]. In all problems of mechanics of discretized bodies, as well as in the finite element method, there exists a certain covering of the given finite or countable set. Elements of such coverings will be called discrete elements. In all such problems, finite difference calculus has to be replaced by a more general approach called discrete element calculus.

Let us denote by D the finite or countable set of points $d \in D$, where $\overline{\overline{D}} > 1$. Let $\mathscr{E}$ be a covering of D with finite subsets $E \in \mathscr{E}$; we assume that $\bigwedge_{E\in\mathscr{E}} (\overline{\overline{E}} > 1)$, $\bigwedge_{E',E''\in\mathscr{E}} (E' \cap E'' \subset E')$, and that each $d \in D$ is an element of the finite number of subsets $E \in \mathscr{E}$. In the pair $(D, \mathscr{E})$, elements of $\mathscr{E}$ will be called discrete elements, and the points of D are said to be the particles.

DEFINITION 1. Each arrangement of all particles belonging to a given discrete element $E \in \mathscr{E}$ in the form of a sequence will be called the coordinate system at E or the local coordinate system in $(D, \mathscr{E})$.

Each local coordinate system is a mapping $f: E \to (d, f_{\mathrm{I}}d, f_{\mathrm{II}}d, \ldots, f_m d)$, where $d \in E$ and $f_\Lambda d \in E$ for $\Lambda = I, II, \ldots, m$, $m = m(E) = \overline{\overline{E}}-1$, the sense of the symbol $f_\Lambda d$ will be explained in what follows. If at the discrete element E a coordinate system is prescribed, then for an arbitrary real-valued function $f: S \to R$, $E \subset S \subset D$, there exist m numbers $\Delta_\Lambda\varphi(d) = \varphi(f_\Lambda d)-\varphi(d)$. The sequence of m numbers $\Delta_\Lambda\varphi(d)$ is said to be the discrete gradient of φ at E; each discrete gradient of the given function $\varphi: S \to R$ can be defined only in a certain local coordinate system in (D, E).

Let there be given at E two coordinate systems: $f: E \to (d, f_{\mathrm{I}}d, \ldots, f_m d)$, $f': E \to (d', f_{\mathrm{I}'} d', \ldots, f_{m'} d')$. The permutation $(d, f_{\mathrm{I}}d, \ldots, f_m d) \to (d' f_{\mathrm{I}'} d', \ldots, f_{m'} d')$ given by $f' \circ f^{-1}$ will be called the local transformation of the coordinate system at E; all such transformations form a group with $(m+1)$ elements. To write any local transformation in explicit form, let us denote by $U = U(E)$ the set of all subsets of E and let us define the mapping $\{0, 1\} \times U \to U$ by assuming that $1u = u$, $0u = \phi$ for each $u \in U$. An arbitrary transformation of the coordinate system at E can now be put in the matrix form:

$$\begin{bmatrix} d' \\ f_{\Lambda'}d' \end{bmatrix} = \begin{bmatrix} 'a & 'a^{\Lambda} \\ a_{\Lambda'} & A_{\Lambda'}{}^{\Lambda} \end{bmatrix} \begin{bmatrix} d \\ f_\Lambda d \end{bmatrix}, \tag{1}$$

where the symbol + stands foı the symbol $\bigcup$ and where

$$a' = \begin{cases} 1 \text{ when } d' = d, \\ 0 \text{ whən } d' \neq d, \end{cases} \quad 'a^{\Lambda} = \begin{cases} 1 \text{ when } d' = f_\Lambda d, \\ 0 \text{ when } d' \neq {}_{f\Lambda} d. \end{cases} \quad a_\Lambda = \begin{cases} 1 \text{ when } f_{\Lambda'} d' = d, \\ 0 \text{ when } f_{\Lambda'} d' \neq d. \end{cases} \tag{2}$$

$$A_{\Lambda'}^{\Lambda} = \begin{cases} 1 \text{ when } f_{\Lambda'} d' = f_\Lambda d, \\ 9 \text{ when } f_{\Lambda'} d' \neq f_\Lambda d. \end{cases}$$

The $(m+1)\times(m+1)$ matrix in Eq. (1) is said to be the local transformation matrix; it is an orthogonal matrix, the elements of which are either 1 or 0. An arbitrary ro wand an arbitrary column of such matrix has one element equal to 1 and m elements equal to 0; if an element in the k-th row and the l-th column of the matrix is equal to one, then the k-th element in the sequence $(d', f_I d', \ldots, f_{m'} d')$ is identical with the l-th element in the sequence $(d, f_I d, \ldots, f_m d)$. It can be seen that the set of all local transformation matrices forms the group which will be denoted by 0^{m-1}; this is a finite group with $(m+1)$ elements.

DEFINITION 2. At the discrete element E, there is defined an object with s components, if to every coordinate system at E a sequence of s numbers has been assigned. The object at E is called geometric if its components in the "new" local coordinate system can be expressed by its components in the "old" local coordinate system and by local transformation.

The Definition 2 has the same form as the known Wundheiler definition [6]. We shall also introduce the concept of the concomitant of the given object; the object at E, with components Ω_A, $A = 1, 2, \ldots, S$, is called the concomitant of the object at E with components ω_a, $a = 1, 2, \ldots, s$, if there exists a function $\Omega_A = \Psi_A(\omega_a)$ which does not depend on the choice of the coordinate system at E.

Now, we shall give some examples of geometrical objects which are used in the mechanics of discretized bodies. Let us assign to each coordinate system at E the sequence of $m+1$ numbers $I, I_\Lambda^A, \Lambda = \text{I}, \text{II}, \ldots, m$, each of them equal to 1. From

$$\begin{bmatrix} I' \\ I_{\Lambda'} \end{bmatrix} = \begin{bmatrix} 'a & 'a^\Lambda \\ a_{\Lambda'} & A_{\Lambda'}{}^\Lambda \end{bmatrix} \begin{bmatrix} I \\ I_\Lambda \end{bmatrix},$$

we obtain $I' = I, I_{\Lambda'} = I_\Lambda$ for each $\Lambda = \Lambda'$. It follows that $I, I_I, \ldots, I_m$ are components of a geometrical object. Let $\varphi: E \to R$ be an arbitrary function and let us define, in an arbitrary coordinate system at E, the sequence of $m+1$ numbers $\varphi \overset{\text{df}}{=} \varphi(d), \varphi_\Lambda \overset{\text{df}}{=} \Delta_\Lambda \varphi(d)$. Simple calculations show that:

$$\begin{bmatrix} \varphi' \\ \varphi_{\Lambda'} \end{bmatrix} = \begin{bmatrix} 1 & 'a_\Lambda \\ 0 & B_{\Lambda'}{}^\Lambda \end{bmatrix} \begin{bmatrix} \varphi \\ \varphi_\Lambda \end{bmatrix}, \tag{3}$$

where

$$B_{\Lambda'}{}^\Lambda = A_{\Lambda'}{}^\Lambda - I_{\Lambda'} {}'a^\Lambda, \quad B_{\Lambda'}{}^\Lambda = \begin{cases} 1 \text{ when } f_{\Lambda'} d' = f_\Lambda d, \\ -1 \text{ when } d' = f_\Lambda d, \\ 0 \text{ in other cases}. \end{cases}$$

We can prove that the set of $(m+1) \times (m+1)$ matrices in (3) forms a group which will be denoted by U^{m-1} and is a subgroup of the group of unimodular matrices. By virtue of

$$\begin{bmatrix} 1 & ''a^\Lambda \\ 0 & B_{\Lambda''}{}^\Lambda \end{bmatrix} = \begin{bmatrix} 1 & ''a^{\Lambda'} \\ 0 & B_{\Lambda'}{}^{\Lambda'} \end{bmatrix} \begin{bmatrix} 1 & 'a^\Lambda \\ 0 & B_{\Lambda'}{}^\Lambda \end{bmatrix},$$

we arrive at $''a^\Lambda = 'a^\Lambda + ''a^{\Lambda'} B_{\Lambda'}{}^\Lambda$, $B_{\Lambda''}{}^\Lambda = B_{\Lambda''}{}^{\Lambda'} B_{\Lambda'}{}^\Lambda$. Putting $''a^\Lambda = 0$, $B_{\Lambda''}{}^\Lambda = \delta^\Lambda_{\Lambda''}$ and $''a^{\Lambda'} = a^{\Lambda'}$, $B^{\Lambda'}_{\Lambda''} = B_\Lambda{}^{\Lambda'}$, we obtain:

$$'a^\Lambda = -a^{\Lambda'} B_{\Lambda'}{}^\Lambda, \quad B_\Lambda{}^{\Lambda'} B_{\Lambda'}{}^\Phi = \delta^\Phi_\Lambda. \tag{4}$$

The set of all matrices $(B_{\Lambda'}{}^\Lambda)$ forms a group which is denoted by $\overline{U}_m$. Assuming that in (3) the condition $'a^\Lambda = 0$ always holds, we arrive at a subgroup of U^{m+1} which is also a subgroup of 0^{m+1}. This is a group of all $(m+1)\times(m+1)$ matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & Q_{\Lambda'}{}^{\Lambda} \end{bmatrix}, \tag{5}$$

where $(Q_{\Lambda'}{}^{\Lambda}) \in 0^m$, and it will be denoted by 0_0^{m+1}. All these groups are encountered in the mechanics of discretized bodies. We conclude that the object with $m+1$ components φ, φ_Λ is geometrical. It will be called the covector with respect to the group U^{m+1}. The object with m components φ^Λ (a discrete gradient) is also geometrical and is called the covector with respect to the group $\overline{U}^m$. The object with $m+1$ components ψ, ψ^Λ and with the transformation formula

$$\begin{bmatrix} \psi' \\ \psi^{\Lambda'} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a^{\Lambda'} & B_{\Lambda}{}^{\Lambda'} \end{bmatrix} \begin{bmatrix} \psi \\ \psi^{\Lambda} \end{bmatrix} \tag{6}$$

will be called a vector with respect to the group U^{m+1}. By virtue of

$$(\varphi', \varphi^{\Lambda'}) \begin{pmatrix} \psi' \\ \psi^{\Lambda'} \end{pmatrix} = (\varphi, \varphi_\Lambda) \begin{pmatrix} 1 & 0 \\ 'a^{\Lambda} & B^{\Lambda'\Lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a^{\Lambda'} & B_{\Phi}{}^{\Lambda'} \end{pmatrix} \begin{pmatrix} \psi \\ \psi^{\Phi} \end{pmatrix}$$
$$= (\varphi, \varphi_\Lambda) \begin{pmatrix} 1 & 0 \\ 'a^{\Lambda} + B_{\Lambda'}{}^{\Lambda} a^{\Lambda'} & B_{\Lambda'}{}^{\Lambda} B_{\Phi}{}^{\Lambda'} \end{pmatrix} \begin{pmatrix} \psi \\ \psi^{\Phi} \end{pmatrix},$$

and using (4), we conclude that

$$(\varphi', \varphi_{\Lambda'}) \begin{pmatrix} \psi' \\ \psi^{\Lambda'} \end{pmatrix} = (\varphi, \varphi_\Lambda) \begin{pmatrix} \psi \\ \psi^{\Lambda} \end{pmatrix}. \tag{7}$$

It follows that the form (7) is an invariant under the local transformation group. All the foregoing considerations are also valid if we denote $\overline{\overline{E}} = s+1$, and if we introduce a local coordinate system $f: E \to (d, f_{\Lambda_1}d, \ldots, f_{\Lambda_s}d)$, where $(\Lambda_1, \Lambda_2, \ldots, \Lambda_s)$ is a subsequence of the sequence $(\mathrm{I}, \mathrm{II}, \ldots, m)$, $s < m$. In this case, the indices Λ, Φ take the values Λ_1, $\Lambda_2, \ldots, \Lambda_s$.

In the theory of discretized bodies and in the finite element approach, the local coordinates cannot be prescribed at each $E \in \mathscr{E}$ independently, but have to be induced by what is called the allowable difference structure [5]. Let us denote by $\mathscr{E}_d$ the set of all discrete elements containing the particle d, $\mathscr{E}_d \subset \mathscr{E}$, and let us define the positive integer m, putting $1+m = \max(\overline{\overline{E}}, \overline{\overline{\mathscr{E}}}_d)$, $E \in \mathscr{E}$, $d \in D$.

DEFINITION 3. The allowable difference structure on $(D, \mathscr{E})$ is a sequence of m one-one mappings $f_\Lambda: D_\Lambda \to D_{-\Lambda}$, $D_\Lambda, D_{-\Lambda} \subset D$, $\Lambda = \mathrm{I}, \mathrm{II}, \ldots, m$, satisfying the conditions:

1. $\bigwedge_{\Lambda} \bigwedge_{d \in D_\Lambda} (f_\Lambda d \neq d)$,
2. $\bigwedge_{\Lambda} \bigwedge_{\Phi} \bigwedge_{d \in D_\Lambda \cap D_\Phi} [(f_\Lambda d = f_\Phi d) \Rightarrow (\Lambda = \Phi)]$,
3. $\mathscr{E} = (E_d)_{d \in D_*}$,

where the symbol $f_\Lambda d$ stands for $f_\Lambda(d)$, $D^* = \bigcup_{\Lambda=\mathrm{I}}^{m} D_\Lambda$, and $E_d \subset D$ is a subset given for each $d \in D_*$, containing the particle d and all particles $f_\Lambda d$.

It may happen that on $(D, \mathscr{E})$ the allowable difference structure does not exist. In this case, we can introduce the allowable difference structure on each $(D_d, \mathscr{E}_d)$, $d \in D$, where $\mathscr{E}_d$ is a set of all discrete elements containing the particle d, and D_d is a subset of D which is covered by $\mathscr{E}_d$. We have to replace in Definition 3 the symbols D, $\mathscr{E}$, D_Λ, $D_{-\Lambda}$, m by D_d, $\mathscr{E}_d$, D_d^Λ, $D_d^{-\Lambda}$, m_d, respectively, where $m_d+1 = \max(\overline{\overline{E}}, \overline{\overline{\mathscr{E}_d}})$, $E \in \mathscr{E}_d$. Such an approach may be applied in all problems of mechanics of discretized bodies or when we are to formulate the theoretical basis for the finite element method. In what follows, we assume that the difference structure is given, and we use the symbols f_1 for one-one mappings which determine the allowable difference structure either on $(D, \mathscr{E})$ or on $(D_d, \mathscr{E}_d)$. Moreover, we denote $f_{-\Lambda} \overset{\text{df}}{=} f_\Lambda^{-1}$ and use the symbols $f_{-\Lambda}d$ instead of $f_{-\Lambda}(d)$.

Let us define the subsets $R_d \subset \{\mathrm{I}, \mathrm{II}, \ldots, m\}$, $L_d \subset \{\mathrm{I}, \mathrm{II}, \ldots, m\}$, assuming that $(\Lambda \in R_d) \Leftrightarrow (d \in D_\Lambda)$, $(\Lambda \in L_d) \Leftrightarrow (d \in G)_{-\Lambda}$. We see that $\overline{\overline{E}}_d = 1+\overline{\overline{R}}_d > 1$ for each $d \in D_*$. For an arbitrary function $\varphi: D_d \to R$, we can define uniquely $\overline{\overline{R}}_d$ numbers $\Delta_\Lambda\varphi(d) \overset{\text{df}}{=} \varphi(f_\Lambda d)-\varphi(d)$, $\Lambda \in R_d$, and $\overline{\overline{L}}_d$ numbers $\overline{\Delta}_\Lambda\varphi(d) \overset{\text{df}}{=} \varphi(d)-\varphi(f_{-\Lambda}d)$, $\Lambda \in L_d$. They are called the right and left side differences of the function $\varphi: D_d \to R$ at the particle d.

Now, let the difference structure be given on $(D, \mathscr{E})$, and let S be an arbitrary subset of D, $S \subseteq D$. We define for each Λ the subsets $S_\Lambda \subset S$ and $S_{-\Lambda} \subset S$, assuming that $(d \in S_\Lambda) \Leftrightarrow (f_\Lambda d \in S)$ and $(d \in S_{-\Lambda}) \Leftrightarrow (f_{-\Lambda}d \in S)$; some of the subsets S_Λ, $S_{-\Lambda}$ or even all of them, may be empty. Let $\varphi: S \to R$ be an arbitrary function. Assigning to each $d \in S_\Lambda$ the real number $\Delta_\Lambda\varphi(d) = \varphi(f_\Lambda d)-\varphi(d)$, we define the function $\Delta_\Lambda\varphi: S_\Lambda \to R$. Assigning to each $d \in S_{-\Lambda}$ the real number $\overline{\Delta}_\Lambda\varphi(d) = \varphi(d)-\varphi(f_{\Lambda-}d)$, we define the function $\overline{\Delta}_\Lambda\varphi: S_{-\Lambda} \to R$. The $2m$ functions $\Delta_\Lambda\varphi: S_\Lambda \to R$, $\overline{\Delta}_\Lambda\varphi: S_{\Lambda-} \to R$ (some of them may be defined on empty sets) are said to be the first differences of the function $\varphi: S \to R$. Because all first differences of an arbitrary real valued function φ are themselves real-valued functions, then we are able to define the second and higher differences of this function. For example, the second differences of the function $\varphi: S \to R$ are given by $\Delta_\Lambda\Delta_\Phi\varphi \overset{\text{df}}{=} \Delta_\Phi(\Delta_\Phi\varphi): S_{\Phi,\Lambda} \to R$, being defined as the first differences of the functions $\Delta_\Phi\varphi: S_\Phi \to R$. It is easy to prove that the relations $\Delta_\Lambda\Delta_\Phi\varphi(d) = \Delta_\Phi\Delta_\Lambda\varphi(d)$, $d \in S_{\Lambda,\Phi} \cap S_{\Phi,\Lambda}$, hold, if and only if, $f_\Lambda f_\Phi d = f_\Phi f_\Lambda d$.

If on $(D_d, \mathscr{E}_d)$ an allowable difference structure is given, then $\mathscr{E}_d = (E_{d'})_{d' \in D_d^*}$, where $D_d^* = \bigcup_\Lambda D_d^\Lambda$. Let a sequence of $1+m_d$ numbers $\psi = \psi(d')$, $\psi^\Lambda = \psi^\Lambda(d')$ be given at each $d' \in D_d^*$, such that $\psi^\Lambda(d) \overset{\text{df}}{=} 0$ when $d \sim\in D_\Lambda$, $\psi(d') \overset{\text{df}}{=} 0$ when $d' \sim\in D_d^*$, $\psi^\Lambda(f_{-\Lambda}d) \overset{\text{df}}{=} 0$ when $d \sim\in D_{-\Lambda}$. If $1+\overline{\overline{E}}_{d'}$ numbers $\psi(d')$, $\psi^\Lambda(d')$ are components of the vector (with respect to the group $U^{m_\alpha+1}$) at $E_{d'}$, $d' \in D_d^*$, then it is possible to prove that the expression

$$\overline{\Delta}_\Lambda\psi^\Lambda(d)+\psi(d),$$

where the summation convention for $\Lambda = \mathrm{I}, \ldots, m_d$ holds, is an invariant under all transformations of the allowable difference structure on $(D_d, \mathscr{E}_d)$. The proofs of the theorems given here will be published in the separate paper.

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