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Flows with proportional stretch history

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THE THEORY of motions with constant stretch history is generalized for the case of motions with proportional stretch history, i.e., non-viscometric flows in which all components of the tensor exponent describing stretch histories are proportional to the same smooth function of time. The generalized representation theorem for motions with proportional stretch history is also proved. In the second part of the paper, some applications of the theory are presented for proportional flows in the Maxwell orthogonal rheometer as well as for unsteady simple extensions. Certain possibilities of experimental determination of material functions and general properties of various types of flows are briefly discussed.

Praca uogólnia teorię ruchów ze stałą historią deformacji na przypadek ruchów z proporcjonalną historią deformacji, tj. przepływów niewiskozometrycznych, dla których składowe wykładnika tensorowego, opisującego historię przepływu, są proporcjonalne do tej samej gładkiej funkcji czasu. Udowodniono również uogólnione twierdzenie o reprezentacji dla ruchów z proporcjonalną historią deformacji. W drugiej części pracy przedstawiono zastosowania teorii zarówno do proporcjonalnych przepływów w ortogonalnym reometrze Maxwella, jak i nieustalonych prostych przepływów rozciągających. Krótko przedyskutowano możliwości doświadczalnego określania funkcji materiałowych oraz ogólne własności różnych typów przepływów.

Теория движений с постоянной историей деформирования обобщается в данной работе на случаи течений с пропорциональной историей деформирования, соответствующих невискозиметрическим течениям, в которых составляющие тензорной экспоненты, задающей историю течения, пропорциональны некоторой гладкой функции времени. Доказывается обобщённая теорема о представлении движений с пропорциональной историей деформирования. Во второй части работы изложены приложения данной теории, как к пропорциональным течениям в ортогональном реометре Максвелла, так и к нестационарным простым растягивающим течениям. Вкратце обсуждены возможности опытного определения материальных функций, а также общие свойства разных типов течений.

1. Introduction

It is a well known fact that the behaviour of incompressible simple fluids (cf. [1]) in all steady as well as unsteady viscometric flows is entirely described by three material functions called the viscometric functions. On the other hand, for flows belonging to the class of motions with constant stretch history, as defined by NOLL [2] (cf. also [3, 4, 5, 6, 7]), a knowledge of at least five material functions (two normal stress functions and three shear stress functions) is necessary. These latter flows are of great practical and experimental importance for polymer rheology, since they include, as special cases, not only all steady viscometric flows and "doubly superposed viscometric flows" in the sense of HUILGOL'S definition [4, 5], but also steady extensional flows, pure shearing flows and numerous flows occurring in rheometers recently constructed or proposed (cf. [8, 9].

In the present paper, the theory of motions with constant stretch history is generalized for the case of motions with proportional stretch history (hereafter called briefly

MPSH) — i.e., non-viscometric flows in which the tensor exponent characterizing exponential stretch histories depends on time in such a way that all its components are proportional to the same smooth function of time (Sec. 2). Fundamental constitutive equations for such flows are discussed in greater detail (Sec. 3). We also consider a class of flows in which the corresponding stretch tensors are composed of an arbitrary number of simpler factors, each of them representing a MPSH. These latter flows are called motions with superposable proportional stretch histories, or briefly MSPSH (Sec. 4)(1).

Next, the representation theorem of WANG [3], proved for motions with constant stretch history, is generalized for the case of MPSH. This theorem enables the extra stress tensor to be expressed in terms of the first three Rivlin-Ericksen kinematic tensors (Sec.5). In what follows, we discuss in greater detail the case of proportional Maxwell rheometer flow and its application to possible experimental determination of material functions (Sec. 6). Further considerations concern the case of unsteady simple extensional flows for which previous results obtained for the Maxwell rheometer flow can be used (Sec. 7). Finally, we consider briefly a very particular case of small oscillations of eccentricity in the Maxwell rheometer flow (Sec. 8).

2. Basic relations and definitions

The mechanical behaviour of a material particle is entirely described for the majority of viscoelastic fluids by the history of motion experienced by the particle. In numerous motions, however, which are very important from theoretical and experimental points of view, the histories frequently take particular forms enabling further considerable simplifications. The history of a motion, expressed locally by the history of the deformation gradient at a particle, may be either the same, apart from rigid rotation, at all instants of time or may depend on time in a different particular way. This requirement is met by NOLL's definition of motions with constant stretch history [2] (these motions are equivalent to COLEMAN's substantially stagnant motions [11]), according to which a motion is called a motion with constant stretch history, if and only if, relative to a fixed reference configuration at time 0, the deformation gradient at any time τ is given by

$$\mathbf{F}_0(\tau) = \mathbf{Q}(\tau)\exp(\tau\mathbf{M}), \quad \mathbf{Q}(0) = \mathbf{1}, \tag{2.1}$$

where $\mathbf{Q}(\tau)$ is an orthogonal tensor characterizing the rotation of a particle from the configuration at time 0 up to time τ and $\mathbf{M}$ denotes a constant tensor.

It is possible to generalize the above definition to the case in which the tensor exponent in (2.1) is proportional to a single function of time. Thus we propose the following definition:

DEFINITION. *A motion is called a MPSH (motion with proportional stretch history), if and only if, relative to a fixed reference configuration at time* 0, *the deformation gradient at time τ is given by*

$$\mathbf{F}_0(\tau) = \mathbf{Q}(\tau)\exp\big(\mathbf{M}k(\tau)\big), \quad \mathbf{Q}(0) = \mathbf{1}, \tag{2.2}$$

(1) A similar notion of superposable histories, although understood in a different sense, is used by GODDARD [10] in his attempt to formulate a general class of motions with explicit stress pattern. Unfortunately, the present author is not familiar with details of Goddard's work.

where $\mathbf{Q}(\tau)$ *is an orthogonal tensor,* $\mathbf{M}$ *is a constant tensor and* $k(\tau)$ *is an arbitrary smooth function of time such that* $k(0) = 0$.

Because of the relations

$$\mathbf{F}_t(\tau) = \mathbf{F}(\tau)\mathbf{F}_0^{-1}(t), \quad \mathbf{F}_t(t) = \mathbf{1}, \tag{2.3}$$

we obtain the following expression for the history of the relative deformation gradient:

$$\mathbf{F}(s) \stackrel{\text{df}}{=} \mathbf{F}_t(t-s) = \mathbf{Q}(s)\exp\big(\mathbf{M}g(s)\big)\,\mathbf{Q}^T(t), \quad 0 \leqslant s < \infty, \tag{2.4}$$

where t denotes the actual instant of time, and

$$g(s) = k(t-s) - k(t). \tag{2.5}$$

Thus the history of the relative right Cauchy-Green tensor (cf. [1]) takes the form

$$\begin{aligned} \mathbf{C}(s) &\stackrel{\text{df}}{=} \mathbf{F}^T(s)\mathbf{F}(s) = \mathbf{Q}(t)\exp\big(\mathbf{M}^T g(s)\big)\exp\big(\mathbf{M}g(s)\big)\mathbf{Q}^T(t), \\ \mathbf{C}(s) &= \exp\big(\mathbf{N}^T g(s)\big)\exp\big(\mathbf{N}g(s)\big), \end{aligned} \tag{2.6}$$

where the following notation is used:

$$\mathbf{N}(t) = \mathbf{Q}(t)\mathbf{M}\mathbf{Q}^T(t). \tag{2.7}$$

If we take into account the spatial velocity gradient defined as follows (cf. [1])

$$\mathbf{L}_1(t) = \dot{\mathbf{F}}_0(t)\mathbf{F}_0^{-1}(t) = \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t) + \mathbf{Q}(t)\mathbf{M}\dot{k}(t)\mathbf{Q}^T(t), \tag{2.8}$$

and denote by

$$\mathbf{L}(t) = \mathbf{Q}(t)\mathbf{M}\dot{k}(t)\mathbf{Q}^T(t) = \mathbf{N}(t)\dot{k}(t), \tag{2.9}$$

the tensor, which may be called the rotated parametric tensor (cf. [5, 6, 7]), we also obtain:

$$\mathbf{C}(s) = \exp\left(\mathbf{L}^T \frac{g(s)}{\dot{k}(t)}\right)\exp\left(\mathbf{L}\frac{g(s)}{\dot{k}(t)}\right). \tag{2.10}$$

The relations $(2.6)_3$ and (2.10) may be considered as the equivalent definitions of MPSH, if the tensors $\mathbf{N}$ or $\mathbf{L}$ depend only on actual time t ($s = 0$) and $g(s)$ has the form (2.5).

The rotated parametric tensor $\mathbf{L}$ represents nothing but the velocity gradient at time t, measured in a rotating reference frame, the rotation of which is determined by the time-dependent tensor $\mathbf{Q}(t)$.

In full analogy to the theory of motions with constant stretch history (cf. [5]), we may ask what will happen if $\mathbf{N}$, defined by (2.7), is a constant tensor independent of present time t? The answer results from the following theorem:

THEOREM. *A motion determined by the velocity gradient in the form* $\mathbf{L}_1(t) = \mathbf{N}\dot{k}(t)$, *where* $\mathbf{N}$ *is a constant tensor and* $\dot{k}(t)$ *is an integrable function of time, always belongs to the class of MPSH with rotation tensor* $\mathbf{Q}$ *identically equal to unity.*

P r o o f. The proof is straightforward, since the solution of the differential equation resulting from $(2.8)_1$, namely

$$\frac{d\mathbf{F}_0(\tau)}{d\tau} = \mathbf{N}\dot{k}(\tau)\mathbf{F}_0(\tau), \quad \text{where } \mathbf{N} = \text{const}, \tag{2.11}$$

under the initial condition: $\mathbf{F}_0(0) = \mathbf{1}$, is

$$\mathbf{F}_0(\tau) = \exp\big(\mathbf{N}k(\tau)\big). \tag{2.12}$$

It is seen that the motion under discussion satisfies the definition (2.2) for $\mathbf{Q} \equiv 1$. Q.E.D.

Since for $\mathbf{Q} \equiv \mathbf{1}$, it also results from (2.7)–(2.9) that $\mathbf{N}\dot{k}(t) = \mathbf{L} = \mathbf{L}_1$, and the corresponding velocity fields are homogeneous in space. Thus, we conclude that all spatially homogeneous motions (not necessarily steady) generate proportional stretch histories.

All MPSH realized in three-dimensional Euclidean space can be classified in a manner similar to that employed by NOLL for motions with constant stretch history (cf. [2, 4]). This leads to the following three classes:

$$\text{(2.13)} \qquad \begin{aligned} &\text{(I)} && \mathbf{M}^2 = \mathbf{0};\\ &\text{(II)} && \mathbf{M}^2 = \mathbf{0} \text{ but } \mathbf{M}^3 = \mathbf{0};\\ &\text{(III)} && \mathbf{M} \text{ is not nilpotent, i.e. } \mathbf{M}^n \neq \mathbf{0} \text{ for all } n = 1, 2, 3. \end{aligned}$$

Because of the relations (2.7), (2.9), a similar classification is also valid for the tensors $\mathbf{N}$ and $\mathbf{L}$, respectively.

The first class involves viscometric flows (not necessarily steady) extensively described elsewhere (cf. [1]). The second class contains all unsteady flows which may be considered as certain generalizations of HUILGOL's" doubly superposed viscometric flows" (cf. [4, 5]. The third class of motions represents generalized HUILGOL's "triply superposed viscometric flows" (cf. [6]) as well as numerous types of unsteady non-viscometric flows which are of great experimental and practical importance. This latter class includes certain unsteady pure shearing and extensional flows and many flows which may be realized, in principle, in newly constructed rheometers such as the Maxwell orthogonal rheometer (cf. [12, 9)], the eccentric cylinder rheometer (cf. [8, 9)], etc.

Some possibilities of experimental determination of rheological properties of polymeric fluids, together with some comparisons among different types of unsteady flows belonging to the class of MPSH, are discussed in further sections.

3. Constitutive equations

The general constitutive equation for incompressible simple fluids has the following form (cf. [1]):

$$\text{(3.1)} \qquad \mathbf{T}_E(t) = \mathbf{T}(t) + p\mathbf{1} = \mathop{\mathscr{F}}_{s=0}^{\infty}\big(\mathbf{G}(s)\big), \qquad \mathbf{G}(s) \stackrel{\mathrm{df}}{=} \mathbf{C}(s) - \mathbf{1},$$

where $\mathbf{T}_E$ is the extra-stress tensor at instant t, p — the hydrostatic pressure, and $\mathscr{F}$ denotes the functional mapping of any symmetric tensor in the space of deformation histories into a symmetric extra-stress tensor. The functional $\mathscr{F}$ satisfies the conditions of isotropy, namely:

$$\text{(3.2)} \qquad \mathbf{Q}\mathop{\mathscr{F}}_{s=0}^{\infty}\big(\mathbf{G}(s)\big)\mathbf{Q}^T = \mathop{\mathscr{F}}_{s=0}^{\infty}\big(\mathbf{Q}\mathbf{G}(s)\mathbf{Q}^T\big)$$

for all orthogonal tensors $\mathbf{Q}$ and all histories $\mathbf{G}(s)$ in the domain of $\mathscr{F}$.

Introducing $(2.6)_3$ into (3.1), we obtain:

$$\text{(3.3)} \qquad \mathop{\mathscr{F}}_{s=0}^{\infty}\Big(\exp\big(\mathbf{N}^T g(s)\big)\exp\big(\mathbf{N}g(s)\big) - \mathbf{1}\Big) = \mathop{\mathscr{G}}_{s=0}^{\infty}\big(g(s); \mathbf{N}\big),$$

where $\mathscr{G}$ is a functional of the scalar function $g(s)$ and a function of the tensor $\mathbf{N}$. If we take into account the following properties of tensor exponentials

$$\exp \mathbf{A} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n, \quad (\mathbf{Q}\mathbf{A}\mathbf{Q}^T)^n = \mathbf{Q}\mathbf{A}^n\mathbf{Q}^T, \tag{3.4}$$

we shall easily verify that (cf. [1]):

$$\mathbf{Q} \underset{s=0}{\overset{\infty}{\mathscr{G}}} (g(s); \mathbf{N}) \mathbf{Q}^T = \underset{s=0}{\overset{\infty}{\mathscr{G}}} \left(\alpha g(s); \frac{1}{\alpha} \mathbf{Q}\mathbf{N}\mathbf{Q}^T \right) \tag{3.5}$$

for all orthogonal tensors $\mathbf{Q}$ and all real non-zero α. Thus the extra stress tensor is determined by the functional:

$$T_E(t) = \underset{s=0}{\overset{\infty}{\mathscr{G}}} (g(s); \mathbf{N}). \tag{3.6}$$

Using the rotated parametric tensor $\mathbf{L}$ instead of $\mathbf{N}$ — i.e., introducing into (3.1) the relation (2.10) instead of $(2.6)_3$, we also arrive at

$$\underset{s=0}{\overset{\infty}{\mathscr{F}}} \left(\exp\left(\mathbf{L}^T \frac{g(s)}{\dot{k}(t)} \right) \exp\left(\mathbf{L} \frac{g(s)}{\dot{k}(t)} \right) - \mathbf{1} \right) = \underset{s=0}{\overset{\infty}{\mathscr{H}}} (g(s); \mathbf{L}), \tag{3.7}$$

$$\mathbf{T}_E(t) = \underset{s=0}{\overset{\infty}{\mathscr{H}}} (g(s); \mathbf{L}). \tag{3.8}$$

Explicit representations of the functionals $\mathscr{G}$ or $\mathscr{H}$ can be achieved in the following manner: we shall for the moment assume that the functional $\mathscr{G}$ is a polynomial function of the not necessarily symmetric argument $\mathbf{N}$. Then, according to SPENCER and RIVLIN's results [13, 14], we may write:

$$\mathbf{T}_E = \sum_n \underset{s=0}{\overset{\infty}{\varkappa_n}} (g(s); I_k) (\mathbf{\Pi}_n + \mathbf{\Pi}_n^T), \tag{3.9}$$

where $\mathbf{\Pi}_n$ are certain products formed from the tensors $\mathbf{N}$ and $\mathbf{N}^T$, $\varkappa_n$ — are functionals in $g(s)$ and polynomials in the irreducible integrity basis $I_1, \ldots, I_k$ composed of the tensors $\mathbf{N}$ and $\mathbf{N}^T$. The irreducible integrity basis (cf. [14]) is some finite set of polynomial invariants under the corresponding symmetry group of the initial material — e.g., the set of traces (moments) formed from the tensors $\mathbf{N}$ and $\mathbf{N}^T$, if none of the integrity basis elements is expressible as a polynomial in the others.

According to the arguments of RIVLIN [15] and PIPKIN and WINEMAN [16], we may claim that the representation (3.9) is also valid for the case in which $\mathscr{G}$ is a single-valued function of its arguments rather than a polynomial, since any integrity basis is also a function basis. We should bear in mind, however, that an irreducible integrity basis may not be an irreducible function basis, and some terms occuring in $\varkappa_n$ may be redundant (cf. [15].

Bearing in mind that the extra stress tensor $\mathbf{T}_E$ is necessarily symmetric and that the symmetry group (isotropy group) for the fluid under consideration is the orthogonal group [cf. (3.2)], we obtain, after all possible reductions, the following representation for (3.6):

$$\begin{aligned} \mathbf{T}_E(t) = {} & \alpha_1(\mathbf{N}+\mathbf{N}^T) + \alpha_2 \mathbf{N}\mathbf{N}^T + \alpha_3 \mathbf{N}^T\mathbf{N} + \alpha_4(\mathbf{N}^2 + \mathbf{N}^{T2}) + \alpha_5(\mathbf{N}^T\mathbf{N}^2 + \mathbf{N}^{T2}\mathbf{N}) \\ & + \alpha_6(\mathbf{N}^2\mathbf{N}^T + \mathbf{N}\mathbf{N}^{T2}) + \alpha_7 \mathbf{N}^2\mathbf{N}^{T2} + \alpha_8 \mathbf{N}^{T2}\mathbf{N} + \alpha_9(\mathbf{N}\mathbf{N}^T\mathbf{N}^2 + \mathbf{N}^{T2}\mathbf{N}\mathbf{N}^T) \\ & + \alpha_{10}(\mathbf{N}_T\mathbf{N}\mathbf{N}^{T2} + \mathbf{N}^2\mathbf{N}^T\mathbf{N}) + \alpha_{11}(\mathbf{N}\mathbf{N}^{T2}\mathbf{N}^2 + \mathbf{N}^{T2}\mathbf{N}^2\mathbf{N}^T) + \alpha_{12}(\mathbf{N}^T\mathbf{N}^2\mathbf{N}^{T2} + \mathbf{N}^2\mathbf{N}^{T2}\mathbf{N}), \end{aligned} \tag{3.10}$$

where α_n ($n = 1, \ldots, 12$) are scalar functions of actual time t and scalar functions of the joint invariants (cf. WESOŁOWSKI [17]):

$$I_1 = \mathrm{tr}\mathbf{N} = 0, \quad I_2 = \mathrm{tr}\mathbf{N}^2, \quad I_3 = \mathrm{tr}\mathbf{N}^3,$$
$$I_4 = \mathrm{tr}\mathbf{N}\mathbf{N}^T, \quad I_5 = \mathrm{tr}\mathbf{N}^2\mathbf{N}^T = \mathrm{tr}\mathbf{N}\mathbf{N}^{T2}, \quad I_6 = \mathrm{tr}\mathbf{N}^2\mathbf{N}^{T2}. \tag{3.11}$$

In writing (3.10), we have used the fact that the functionals in $g(s)$

$$\underset{s=0}{\overset{\infty}{\varkappa_n}}\big(g(s); I_k\big) = \alpha_n(t; I_k), \quad n = 1, \ldots, 12, \quad k = 1, \ldots, 6, \tag{3.12}$$

can be treated as the corresponding parametric functions of time t, since, in general, the definition (2.5) involves dependence on t.

4. Superposable proportional stretch histories

It may be useful for certain particular more complex flows to assume that the exponent term in the definition (2.2) is capable to be expressed as a sum of constant tensors $\mathbf{M}_i$ multiplied by arbitrary smooth functions of time $k_i(\tau)$ only. Then,

$$\mathbf{F}_0(\tau) = \mathbf{Q}(\tau)\exp\Big(\sum_{i=1}^{m}\mathbf{M}_i k_i(\tau)\Big), \quad \mathbf{Q}(0) = \mathbf{1}, \tag{4.1}$$

where $\mathbf{Q}$ is again an orthogonal tensor. If all constant tensors $\mathbf{M}_i$ mutually commute, we can also write

$$\mathbf{F}_0(\tau) = \mathbf{Q}(\tau)\prod_{i=1}^{m}\exp\big(\mathbf{M}_i k_i(\tau)\big), \quad \text{if } \mathbf{M}_i\mathbf{M}_j = \mathbf{M}_j\mathbf{M}_i \text{ for } i \neq j. \tag{4.2}$$

The above relation implies the following definition:

DEFINITION. *A motion is called a motion with superposable proportional stretch history* MSPSH, *if and only if, relative to a fixed reference configuration at time* 0, *the deformation gradient at time* τ *is given by* (4.2), *where* $\mathbf{Q}(\tau)$ *is an orthogonal tensor,* $\mathbf{M}_i$ *are mutually commuting constant tensors, and* $k_i(\tau)$ *are arbitrary smooth functions of time, such that* $k_i(0) = 0$.

Motions with superposable proportional stretch histories [MSPSH] can be treated as certain generalizations of MPSH defined in Sec. 2.

Introducing, as previously, the functions

$$g_i(s) = k_i(t-s) - k_i(t), \quad i = 1, \ldots, m, \tag{4.3}$$

we arrive at

$$\mathbf{C}(s) = \exp\Big(\sum_{i=1}^{m}\mathbf{N}_i^T g_i(s)\Big)\exp\Big(\sum_{i=1}^{m}\mathbf{N}_i g_i(s)\Big), \tag{4.4}$$

$$\mathbf{C}(s) = \exp\left(\sum_{i=1}^{m}\mathbf{L}_i^T\frac{g_i(s)}{\dot{k}_i(t)}\right)\exp\left(\sum_{i=1}^{m}\mathbf{L}_i\frac{g_i(s)}{\dot{k}_i(t)}\right), \tag{4.5}$$

where

$$\text{(4.6)} \qquad \mathbf{N}_i(t) = \mathbf{Q}(t)\mathbf{M}_i\mathbf{Q}^T(t), \quad \mathbf{L}_i(t) = \mathbf{Q}(t)\mathbf{M}_i\dot{k}_i(t)\mathbf{Q}^T(t) = \mathbf{N}_i\dot{k}_i(t), \quad i = 1, \ldots, m.$$

A theorem similar to that proved in Sec. 2 can be formulated in the following words:

THEOREM. *A motion determined by the velocity gradient in the form* $\mathbf{L}_1(t) = \sum_{i=1}^{m} \mathbf{L}_{1i}(t) = \sum_{i=1}^{m} \mathbf{N}_i\dot{k}_i(t)$, *where* $\mathbf{N}_i$ *are constant tensors such that* $\mathbf{N}_i\mathbf{N}_j = \mathbf{N}_j\mathbf{N}_i$ *for* $i \neq j$, *and* $\dot{k}_i(t)$ *are integrable functions of time, always belongs to the class of* MSPSH *with rotation tensor* $\mathbf{Q}$ *identically equal to unity.*

P r o o f. The proof is analogous to that in Sec. 2. The solution of the differential equation

$$\text{(4.7)} \qquad \frac{d\mathbf{F}_0(\tau)}{d\tau} = \mathbf{L}_1\mathbf{F}_0(\tau) \equiv \Big(\sum_{i=1}^{m} \mathbf{N}_i\dot{k}_i(\tau)\Big)\mathbf{F}_0(\tau),$$

under the initial condition $\mathbf{F}_0(0) = \mathbf{1}$, is:

$$\text{(4.8)} \qquad \mathbf{F}_0(\tau) = \exp\Big(\sum_{i=1}^{m} \mathbf{N}_i k_i(\tau)\Big) = \prod_{i=1}^{m} \exp\big(\mathbf{N}_i k_i(\tau)\big),$$

since $\mathbf{N}_i$ are commuting tensors. This satisfies the definition (4.2) for $\mathbf{Q} \equiv \mathbf{1}$. Q.E.D. The above motions are also homogeneous in space and, moreover, $\mathbf{L} = \mathbf{L}_1 = \sum_{i=1}^{m} \mathbf{N}_i\dot{k}(t)$.

All MSPSH can be divided into three fundamental classes defined by (2.13). To this end, we can use the total tensor $\mathbf{L} = \sum_{i=1}^{m} \mathbf{L}_i = \sum_{i=1}^{m} \mathbf{N}_i(t)\dot{k}_i(t)$ which may be nilpotent or not. It seems to be much more interesting to determine the necessary and sufficient conditions imposed on viscometric $\mathbf{L}_i$ in order that a given MSPSH may belong to each of the corresponding classes. We shall demonstrate such an approach for the case of motions with doubly superposable proportional stretch histories — i.e., for the case in which $\mathbf{L} = \mathbf{L}' + \mathbf{L}''$.

(I) For viscometric flows we have $\mathbf{L}^2 = 0$, and if $\mathbf{L}'$, $\mathbf{L}''$ also represent viscometric flows — i.e., $\mathbf{L}'^2 = \mathbf{L}''^2 = 0$ — the requirement of commutation implies that the relation

$$\text{(4.9)} \qquad \mathbf{L}'\mathbf{L}'' = \mathbf{L}''\mathbf{L}' = \mathbf{0}$$

is the necessary and sufficient condition for a MSPSH composed of viscometric $\mathbf{L}'$ and $\mathbf{L}''$ to be a viscometric flow. An example of such a flow is an unsteady helical flow, for which the angular velocity as well as the axial velocity are governed by different time-dependent functions $\dot{k}'(\tau)$ and $\dot{k}''(\tau)$. It is easy to check that for helical flows the condition (4.9) is always satisfied.

(II) For generalized "doubly superposed viscometric flows" (cf. Sec. 2) characterized by $\mathbf{L}^2 \neq \mathbf{0}$ but $\mathbf{L}^3 = \mathbf{0}$, we obtain, assuming that they are composed of viscometric flows, the following conditions:

$$\mathbf{L}'\mathbf{L}''+\mathbf{L}''\mathbf{L}' \neq \mathbf{0}, \quad \mathbf{L}''\mathbf{L}'\mathbf{L}''+\mathbf{L}'\mathbf{L}''\mathbf{L}' = \mathbf{0}. \tag{4.10}$$

Since $\mathbf{L}'$ commutes with $\mathbf{L}''$, we obtain the relations:

$$\mathbf{L}'\mathbf{L}'' = \mathbf{L}''\mathbf{L}' \neq \mathbf{0}, \quad \mathbf{L}'\mathbf{L}''(\mathbf{L}'+\mathbf{L}'') = \mathbf{0}. \tag{4.11}$$

(III) For flows characterized by $\mathbf{L}^n \neq 0$, and simultaneously by $\mathbf{L}'^2 = \mathbf{0}$, $\mathbf{L}''^2 = \mathbf{0}$, the only condition which results from the commutation rule is (4.11). This means that, in general, we are not seriously restricted when composing various MSPSH for which the tensors $\mathbf{L}$ are not nilpotent. In this case, only additional properties of tensors such as symmetry, skew-symmetry, etc. may be of importance.

For MSPSH, proceeding similarly as in Sec. 3, we arrive at the constitutive equations as follows:

$$\mathbf{T}_E(t) = \mathop{\mathscr{G}}_{s=0}^{\infty}\big(g_i(s); \mathbf{N}_i\big), \quad i = 1, \dots, m \tag{4.12}$$

and

$$\mathbf{Q} \mathop{\mathscr{G}}_{s=0}^{\infty}\big(g_i(s); \mathbf{N}_i\big)\mathbf{Q}^T = \mathop{\mathscr{G}}_{s=0}^{\infty}\left(\alpha_i g_i(s); \frac{1}{\alpha_i}\mathbf{Q}\mathbf{N}_i\mathbf{Q}^T\right) \tag{4.13}$$

for all orthogonal tensors $\mathbf{Q}$ and all sets of real non-zero α_i.

Similar forms of constitutive equations can be written in terms of the tensors $\mathbf{L}_i$ instead of $\mathbf{N}_i$.

Explicit representations of the functional $\mathscr{G}$ can be achieved, in principle, by means of the method proposed in Sec. 3. It should be noticed, however, that for MSPSH composed of many flows, the number of corresponding material functions may be too large for effective experimental verification and practical computations.

5. Representation theorem. Generalization of Wang's theorems [3]

In 1965, WANG proved the main theorem and the corresponding representation theorem for motions with constant stretch history [3]. His theorems can be generalized for the case of MPSH as defined in Sec. 2.

Let us repeat his preliminary lemma.

LEMMA. *Let* [S] *be a* 3×3 *diagonal matrix and* [W] *a* 3×3 *skew-symmetric matrix*:

$$[\mathbf{S}] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad [\mathbf{W}] = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}; \tag{5.1}$$

then:

(5.2)
i) *if* $a \neq b \neq c$, $[\mathbf{SW}] = [\mathbf{WS}]$ *if, and only if,* $x = y = z = 0$,
ii) *if* $a = b \neq c$, $[\mathbf{SW}] = [\mathbf{WS}]$ *if, and only if,* $y = z = 0$, x is arbitrary
iii) *if* $a = b = c$, $[\mathbf{SW}] = [\mathbf{WS}]$ *for all* x, y, z.

The proof is straightforward by direct multiplication (cf. [3]). If [S] and [W] are component matrices of the tensors $\mathbf{S}$ and $\mathbf{W}$, respectively, relative to the principal orthonormal basis of $\mathbf{S}$, the lemma determines the conditions under which the tensors $\mathbf{S}$ and $\mathbf{W}$ commute — i.e., $\mathbf{SW} = \mathbf{WS}$.

Now, we prove our main theorem.

THEOREM. *In a motion being a* MPSH *the history of the relative Cauchy-Green tensor* $\mathbf{C}(s)$ *is uniquely determined by the first three Rivlin-Ericksen kinematic tensors* $\mathbf{A}_1(t)$, $\mathbf{A}_2(t)$, $\mathbf{A}_3(t)$ *at the fixed actual instant* t *and a given function* $k(\tau)$.

Proof. Bearing in mind (2.9) and (2.10), and taking into account the definition of Rivlin-Ericksen kinematic tensors (cf. [1, 18])

$$\mathbf{A}_n(t) = (-1)^n \frac{d^n\mathbf{C}(s)}{ds^n}\bigg|_{s=0} = \frac{d^n}{d\tau^n}\mathbf{C}_t(\tau)\bigg|_{\tau=t}, \quad n = 1, 2, 3, \ldots, \tag{5.3}$$

we see that the first three of them satisfy the relations:

$$\begin{aligned}
\mathbf{A}_1 &= \mathbf{L}^T+\mathbf{L},\\
\mathbf{A}_2 &= \mathbf{A}_1\frac{\ddot{k}(t)}{\dot{k}(t)}+\mathbf{A}_1\mathbf{L}+\mathbf{L}^T\mathbf{A}_1,\\
\mathbf{A}_3 &= \mathbf{A}_1\frac{\dddot{k}(t)}{\dot{k}(t)}+3(\mathbf{L}\mathbf{A}_1+\mathbf{A}_1\mathbf{L}^T)\frac{\ddot{k}(t)}{\dot{k}(t)}+\mathbf{A}_2\mathbf{L}+\mathbf{L}^T\mathbf{A}_2,
\end{aligned} \tag{5.4}$$

where the function $k(\tau)$ is defined in (2.2) and its relation to $g(s)$ is given by (2.5).

We consider the following three cases:

1) $\mathbf{A}_1$ has three distinct eigenvalues (proper numbers). In this case, $\mathbf{A}_1$ does not commute with any non-zero skew-symmetric tensor (see the lemma). We claim that the tensor $\mathbf{L}$ is uniquely determined by $\mathbf{A}_1$ and $\mathbf{A}_2$. To prove this, suppose that $\mathbf{L}$ is not uniquely determined by $\mathbf{A}_1$ and $\mathbf{A}_2$. Then, taking an $\overline{\mathbf{L}}$, such that

$$\begin{aligned}
\mathbf{A}_1 &= \mathbf{L}^T+\mathbf{L} = \overline{\mathbf{L}}^T+\overline{\mathbf{L}},\\
\mathbf{A}_2 &= \mathbf{A}_1\frac{\ddot{k}(t)}{\dot{k}(t)}+\mathbf{A}_1\mathbf{L}+\mathbf{L}^T\mathbf{A}_1 = \mathbf{A}_1\frac{\ddot{k}(t)}{\dot{k}(t)}+\mathbf{A}_1\overline{\mathbf{L}}+\overline{\mathbf{L}}^T\mathbf{A}_1,
\end{aligned} \tag{5.5}$$

we have

$$(\mathbf{L}-\overline{\mathbf{L}})^T = -(\mathbf{L}-\overline{\mathbf{L}}); \tag{5.6}$$

thus the difference $\mathbf{L}-\overline{\mathbf{L}}$ is skew-symmetric. From $(5.5)_4$ and (5.6), we also see that the skew-symmetric tensor $\mathbf{L}-\overline{\mathbf{L}}$ commutes with $\mathbf{A}_1$, namely

$$(\mathbf{L}-\overline{\mathbf{L}})\mathbf{A}_1 = \mathbf{A}_1(\mathbf{L}-\overline{\mathbf{L}}). \tag{5.7}$$

Since $\mathbf{A}_1$ does not commute with any non-zero skew-symmetric tensor, we have $\mathbf{L}-\overline{\mathbf{L}} = \mathbf{0}$, and thus $\mathbf{L}$ is uniquely determined by $\mathbf{A}_1$ and $\mathbf{A}_2$.

2) $\mathbf{A}_1$ has only two distinct eigenvalues — i.e., the matrix of $\mathbf{A}_1$ can be written in the form:

$$[\mathbf{A}_1] = \begin{bmatrix} a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & b\end{bmatrix}, \quad a \neq b. \tag{5.8}$$

Now, we have the following two subcases:

a) In the same orthonormal basis, in which $\mathbf{A}_1$ is determined by (5.8), $\mathbf{A}_2$ has the form:

$$[\mathbf{A}_2] = \begin{bmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{bmatrix}. \tag{5.9}$$

Then we claim that

$$u = a\frac{\ddot{k}(t)}{\dot{k}(t)} + a^2, \quad v = b\frac{\ddot{k}(t)}{\dot{k}(t)} + b^2, \tag{5.10}$$

and the tensor $\mathbf{C}(s)$ is uniquely determined by $\mathbf{A}_1$ and $\mathbf{A}_2$. Although the tensor $\mathbf{L}$ is not uniquely determined by $\mathbf{A}_1$ and $\mathbf{A}_2$, the corresponding class of $\mathbf{L}$'s determines $\mathbf{C}(s)$ uniquely. Any solution $\mathbf{L}$ of the system $(5.5)_{1,3}$ has the component form (in the basis of $[\mathbf{A}_1]$):

$$[\mathbf{L}] = \begin{bmatrix} \frac{a}{2} & x & 0 \\ -x & \frac{a}{2} & 0 \\ 0 & 0 & \frac{b}{2} \end{bmatrix}, \tag{5.11}$$

where x is arbitrary. Such a class of $\mathbf{L}$'s determines only one function $\mathbf{C}(s)$ in the form:

$$[\mathbf{C}(s)] = \exp\left(\frac{g(s)}{\dot{k}(t)}\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}\right). \tag{5.12}$$

To prove this, let us observe that, on the basis of $(5.5)_1$, $\mathbf{L}$ must be of the form:

$$[\mathbf{L}] = \begin{bmatrix} \frac{a}{2} & x & y \\ -x & \frac{a}{2} & z \\ -y & -z & \frac{b}{2} \end{bmatrix}. \tag{5.13}$$

Thus by direct computation from $(5.5)_3$:

$$[\mathbf{A}_2] = \left[\mathbf{A}_1\frac{\ddot{k}(t)}{\dot{k}(t)} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1\right] = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}\frac{\ddot{k}(t)}{\dot{k}(b)} + \begin{bmatrix} a^2 & 0 & (a-b)y \\ \cdot & a^2 & (a-b)z \\ \cdot & \cdot & b^2 \end{bmatrix}. \tag{5.14}$$

If $a \neq b$, we can see from (5.9) and (5.14) that (5.10) and (5.11) are satisfied. Furthermore, substituting (5.11) into (2.10), we arrive at (5.12), since from the lemma

$$[(\mathbf{L}+\mathbf{L}^T)(\mathbf{L}-\mathbf{L}^T)] = [(\mathbf{L}-\mathbf{L}^T)(\mathbf{L}+\mathbf{L}^T)] \tag{5.15}$$

— i.e. the symmetric part of $\mathbf{L}$ commutes with the skew-symmetric part. Thus $\mathbf{A}_1$ and $\mathbf{A}_2$ determine $\mathbf{C}(s)$ uniquely.

b) The component matrix of $\mathbf{A}_2$ is not of the form (5.9). Then we claim that the tensor $\mathbf{L}$ is uniquely determined by $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$; which means that the solution $\mathbf{L}$ of the system (5.4) is unique.

To prove this, we suppose that $\bar{\mathbf{L}}$ is another solution of the system. Then, from $(5.4)_{1,2}$, $\mathbf{L}-\bar{\mathbf{L}}$ is a skew-symmetric tensor and $\mathbf{L}-\bar{\mathbf{L}}$ commutes with $\mathbf{A}_1$. Therefore, from the lemma, we see that $\mathbf{L}-\bar{\mathbf{L}}$ must have the form:

$$[\mathbf{L}-\bar{\mathbf{L}}] = \begin{bmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.16}$$

Since from $(5.4)_3$, $\mathbf{L}-\bar{\mathbf{L}}$ commutes with $\mathbf{A}_2$, and by assumption $[\mathbf{A}_2]$ is not of the form (5.9), it results from direct multiplication that $x = 0$. Thus $\mathbf{L} = \bar{\mathbf{L}}$.

3) $\mathbf{A}_1$ has the same three eigenvalues — i.e., $\mathbf{A}_1 = a\mathbf{1}$. According to the lemma, $\mathbf{A}_1$ commutes with any skew-symmetric tensor. From (2.10):

$$\mathbf{C}(s) = \exp\left(\frac{g(s)}{\dot{k}(t)}\mathbf{A}_1\right) = \exp\left(\frac{g(s)}{\dot{k}(t)}a\mathbf{1}\right). \tag{5.17}$$

Although the tensor $\mathbf{L}$ is not unique, any solution $\mathbf{L}$ of the Eq. $(5.5)_1$ determines the same tensor $\mathbf{C}(s)$ given by (5.17).

Thus the generalized theorem has been proved.

It should be added, however, that if the tensors $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are quite arbitrary, it may happen that the systems (5.4) or (5.5) do not have a solution for $\mathbf{L}$, since in MPSH the Rivlin-Ericksen kinematic tensors must satisfy compatibility conditions similar to those derived by WANG for motions with constant stretch history (cf. [3] p. 335).

If the material under consideration is an incompressible simple fluid, the following equations are direct consequences of the main theorem:

$$\mathbf{T}_E(t) = \underset{s=0}{\overset{\infty}{\mathscr{H}}}\big(g(s); \mathbf{A}_1, \mathbf{A}_2\big), \quad \operatorname{tr}\mathbf{T}_E = 0, \tag{5.18}$$

for the cases 1) and 2a), and

$$\mathbf{T}_E(t) = \underset{s=0}{\overset{\infty}{\mathscr{K}}}\big(g(s); \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\big), \quad \operatorname{tr}\mathbf{T}_E = 0, \tag{5.19}$$

for the case 2b). It must be borne in mind, however, that our main theorem, similarly to the Wang theorem, is valid for any simple material in MPSH, not necessarily being an incompressible fluid.

Since the two cases 1) and 2a) cover the majority of interesting flows, such as Poiseuille-torsional flow, Maxwell orthogonal rheometer flow (cf. Sec. 6), and other flows occurring in new rheometers (cf. [8, 9]), we present a more explicit representation for the constitutive equation (5.18). Proceeding as in Sec. 3, and using RIVLIN's fundamental result [18], we have:

$$\begin{aligned}\mathbf{T}_E = {} & \gamma_1\mathbf{A}_1+\gamma_2\mathbf{A}_1^2+\gamma_3\mathbf{A}_2+\gamma_4\mathbf{A}_2^2+\gamma_5(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1)+\gamma_6(\mathbf{A}_1^2\mathbf{A}+\mathbf{A}_2\mathbf{A}_1^2) \\ & +\gamma_7(\mathbf{A}_1\mathbf{A}_2^2+\mathbf{A}_2^2\mathbf{A}_1)+\gamma_8(\mathbf{A}_1^2\mathbf{A}_2^2+\mathbf{A}_2^2\mathbf{A}_1^2),\end{aligned} \tag{5.20}$$

where γ_i ($i = 1, \ldots, 8$) are either the functionals in $g(s)$ or the functions of actual time t, and the functions of the following integrity basis (function basis):

$$\begin{gathered} I_1 = \mathrm{tr}\mathbf{A}_1 = 0, \quad I_2 = \mathrm{tr}\mathbf{A}_1^2, \quad I_3 = \mathrm{tr}\mathbf{A}_1^3, \quad I_4 = \mathrm{tr}\mathbf{A}_2, \\ I_5 = \mathrm{tr}\mathbf{A}_2^2, \quad I_6 = \mathrm{tr}\mathbf{A}_2^3, \quad I_7 = \mathrm{tr}\mathbf{A}_1\mathbf{A}_2, \quad I_8 = \mathrm{tr}\mathbf{A}_1^2\mathbf{A}_2, \\ I_9 = \mathrm{tr}\mathbf{A}_1\mathbf{A}_2^2, \quad I_{10} = \mathrm{tr}\mathbf{A}_1^2\mathbf{A}_2^2. \end{gathered} \tag{5.21}$$

All functions γ_i are not entirely independent, since some redundancies always exist in the expansion (5.20), and it is frequently possible to choose six linearly independent combinations of terms. If we use the normalization condition: $\mathrm{tr}\mathbf{T}_E = 0$, the number of independent terms will decrease by one. An example of such an approach is demonstrated in Sec. 6 for the case of Maxwell rheometer flow.

6. Proportional Maxwell rheometer flow

In 1965, MAXWELL and CHARTOFF [12] presented experiments with the rheometer consisting of two flat disks fixed at distance b one from the other, and rotating with the same angular velocity $\tilde{\omega}$ about parallel axes displaced at distance a. Special devices are provided to measure the forces X, Y, Z acting on the upper (or lower) disk in three directions mutually at right angles (Fig. 1). The theory and experiment for the Maxwell ortho-

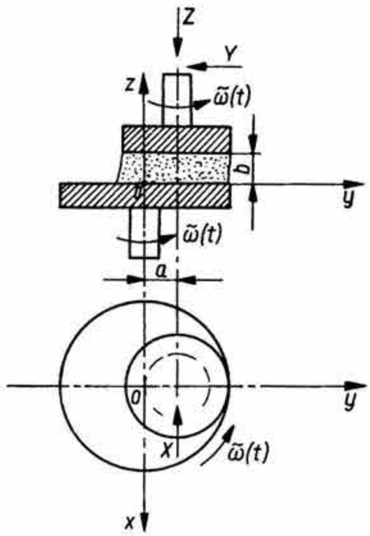

FIG. 1.

gonal rheometer are to be found elsewhere (cf. [12, 9, 19, 20]). In what follows we are, primarily, interested in possibilities of experimental determination of the material functions for MPSH.

A kinematic analysis of flow realized in the Maxwell orthogonal rheometer leads to the velocity field in the form (cf. [9, 19]):

$$v^1 = -\tilde{\omega}(t)y + \tilde{\omega}(t)\psi z, \quad v^2 = \tilde{\omega}(t)x, \quad v^3 = 0, \quad \tilde{\omega}(t) = \omega\dot{k}(t) \tag{6.1}$$

where $\psi = a/b$, and v^i denote Cartesian components of the velocity field (cf. Fig. 1).

Since for our homogeneous flow

$$(6.2)\qquad [\mathbf{L}] = [\mathbf{N}]\dot{k}(t) = \begin{bmatrix} 0 & -\omega & \omega\psi \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{k}(t), \qquad [\mathbf{A}_1] = \begin{bmatrix} 0 & 0 & \omega\psi \\ 0 & 0 & 0 \\ \omega\psi & 0 & 0 \end{bmatrix} \dot{k}(t),$$

according to (2.2) and $(2.6)_2$, we have

$$(6.3)\qquad \mathbf{F}_0(\tau) = \exp(\mathbf{N}k(\tau)), \qquad \mathbf{C}(s) = \exp(\mathbf{N}^T g(s))\exp(\mathbf{N}g(s)).$$

Taking use of the constitutive Eq. (3.10), we arrive at the following expressions for physical components of extra-stresses:

$$(6.4)\qquad \begin{aligned}
T_E^{(11)} &= (\alpha_2+\alpha_3-2\alpha_4)\omega^2+\alpha_2\omega^2\psi^2+(\alpha_7+\alpha_8-2\alpha_9-2\alpha_{10})\omega^4-2\alpha_9\omega^4\psi^2,\\
T_E^{(22)} &= (\alpha_2+\alpha_3-2\alpha_4)\omega^2+(\alpha_7+\alpha_8-2\alpha_9-2\alpha_{10})\omega^4+\alpha_7\omega^4\psi^2-2\alpha_{10}\omega^4\psi^2,\\
T_E^{(33)} &= \alpha_3\omega^2\psi^2+\alpha_8\omega^4\psi^2,\\
T_E^{(12)} &= \alpha_6\omega^3\psi^2+(\alpha_{12}-\alpha_{11})\omega^5\psi^2,\\
T_E^{(13)} &= \alpha_1\omega\psi+(\alpha_{11}+\alpha_{12})\omega^5\psi+\alpha_{11}\omega^5\psi^3,\\
T_E^{(23)} &= (\alpha_4-\alpha_3)\omega^2\psi+(\alpha_9-\alpha_8)\omega^4\psi+\alpha_{10}\omega^4\psi+\alpha_{10}\omega^4\psi^3,
\end{aligned}$$

with the additional relation:

$$(6.5)\qquad \begin{aligned}\operatorname{tr}\mathbf{T}_E = 2(\alpha_2+\alpha_3-2\alpha_4)\omega^2+(\alpha_2+\alpha_3)\omega^2\psi^2+2(\alpha_7+\alpha_8-2\alpha_9-2\alpha_{10})\omega^4\\
+(\alpha_7+\alpha_8-2\alpha_9-2\alpha_{10})\omega^4\psi^2 = 0.\end{aligned}$$

Since the invariants (3.11) can be expressed as certain combinations of the terms ω^2, $\omega^2\psi^2$, the functions α_i ($i = 1, \ldots, 12$) can be considered as depending on time t and the above arguments. The expressions (6.4), (6.5) are too complex for further calculations or experimental verification, although certain reductions of terms are, in principle, possible. Thus we propose an alternative approach using the constitutive Eqs. (5.20), (5.21).

To this end, let us introduce the following quantities (cf. (5.4)):

$$(6.6)\qquad \overline{\mathbf{A}}_1 = \mathbf{A}_1 = \mathbf{L}^T+\mathbf{L}, \qquad \overline{\mathbf{A}}_2 = \mathbf{A}_2-\mathbf{A}_1\frac{\ddot{k}(t)}{\dot{k}(t)} = \mathbf{A}_1\mathbf{L}+\mathbf{L}^T\mathbf{A}_1.$$

For these quantities the Eqs. (5.20) are still valid but with different material functions β_i (instead of γ_i) depending on actual time t as well as on the invariants (5.21) calculated for the tensors $\overline{\mathbf{A}}_1$ and $\overline{\mathbf{A}}_2$.

A reduction of terms appearing in equations of the type (5.20) is accomplished by making use of HUILGOL's procedure [6] based on the notion of a non-singular linear operator. In that way, two terms of higher order can be reduced, since three eigenvalues of the tensor $\overline{\mathbf{A}}_1$ are distinct. Thus we arrive at:

$$(6.7)\qquad \mathbf{T}_E(t) = \beta_1\overline{\mathbf{A}}_1+\beta_2\overline{\mathbf{A}}_1^2+\beta_3\overline{\mathbf{A}}_2+\beta_4\overline{\mathbf{A}}_2^2+\beta_5(\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2+\overline{\mathbf{A}}_2\overline{\mathbf{A}}_1)+\beta_6(\overline{\mathbf{A}}_1^2\overline{\mathbf{A}}_2^2+\overline{\mathbf{A}}_2^2\overline{\mathbf{A}}_1^2).$$

The following relations justify the above reduction:

$$(6.8)\qquad \begin{aligned}
\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2^2+\overline{\mathbf{A}}_2^2\overline{\mathbf{A}}_1 &= \omega^4\psi^2\overline{\mathbf{A}}_1+2\omega^2\psi^2(\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2+\overline{\mathbf{A}}_2\overline{\mathbf{A}}_1),\\
\overline{\mathbf{A}}_1^2\overline{\mathbf{A}}_2^2+\overline{\mathbf{A}}_2^2\overline{\mathbf{A}}_1^2 &= -\omega^4\psi^2\overline{\mathbf{A}}_2+\omega^2(1+2\psi^2)(\overline{\mathbf{A}}_2^2\overline{\mathbf{A}}_2+\overline{\mathbf{A}}_2\overline{\mathbf{A}}_1^2).
\end{aligned}$$

Using (6.7), we obtain finally

$$\begin{aligned}
&T_E^{(11)} = \beta_2\omega^2\psi^2, \quad T_E^{(22)} = \beta_4\omega^4\psi^2, \quad T_E^{(12)} = -\beta_5\omega^3\psi^2,\\
&T_E^{(13)} = (\beta_1+2\beta_5\omega^2\psi^2)\omega\psi,\\
&T_E^{(33)} = (\beta_2+2\beta_3)\omega^2\psi^2+\beta_4(\omega^4\psi^2+4\omega^4\psi^4)+\beta_6(2\omega^6\psi^4+8\omega^6\psi^6),\\
&T_E^{(23)} = -(\beta_3+2\beta_4\omega^2\psi^2+2\beta_6\omega^4\psi^4)\omega^2\psi,
\end{aligned} \tag{6.9}$$

$$2(\beta_2+\beta_3)\omega^2\psi^2+2\beta_4(\omega^4\psi^2+2\omega^4\psi^4)+2\beta_6(\omega^6\psi^4+4\omega^6\psi^6) = 0, \tag{6.10}$$

where β_i $(i = 1, \ldots, 6)$ are functions of actual time t, and the following terms: $\omega^2\psi^2$, $\omega^4\psi^2$.

Certain relations between the material functions appearing in (3.10) and those in (6.7) can also be established under the assumption of polynomial forms of α_i and β_i. Disregarding, for example, in both equations the terms of order $O(\omega^4)$(which is reasonable for moderately slow flows), we obtain:

$$\begin{aligned}
&\beta_1 = \alpha_1+2\alpha_6\omega^2\psi^2, \quad \beta_5 = -\alpha_6, \quad \beta_2 = \alpha_2,\\
&\beta_3 = \frac{1}{2}(\alpha_3-\alpha_2), \quad \beta_2+\beta_3 = 0, \quad \alpha_3 = -\alpha_2.
\end{aligned} \tag{6.11}$$

Since the only direct measurements possible in the Maxwell rheometer are those concerning the forces X, Y and Z, which is equivalent to measurements of the extra-stresses $T_E^{(23)}$, $T_E^{(13)}$ and $T_E^{(33)}$, respectively, it is seen from (6.9) that not all the functions β_i $(i = 1, \ldots, 6)$ can be determined from experiments. If we assume, however, that certain further simplifications are admissible in (6.9), an experimental determination of certain β_i or some of their combinations will very likely be possible. To demonstrate this, let us observe (cf. [12, 20]) that the parameter ψ is usually very small and, in existing rheometers, can be changed in a somewhat narrow range $(0 \leqslant \psi \ll 1)$. On the other hand, the parameter ω, characterizing the variable angular velocity $\tilde{\omega}(t) = \omega\dot{k}(t)$, may essentially depend on the range of angular velocities used in experiments. Its magnitude is usually restricted by inertia effects, viscosities of investigated fluids, etc. Thus, under the assumption of moderately slow flows and polynomial forms of β_i, we may disregard higher order terms in ψ and ω by comparison with lower order terms.

For example, rejecting in (6.9), (6.10) all terms of orders $O(\omega^5)$ and $O(\psi^4)$, respectively, we arrive at the following expressions for extra-stresses connected directly with the forces X, Y, Z:

$$\begin{aligned}
&(Z) \quad T_E^{(33)}(t) \approx \beta_3(t)\omega^2\psi^2,\\
&(X) \quad T_E^{(23)}(t) \approx -\big(\beta_3(t)+2\beta_4(t)\omega^2\psi^2\big)\omega^2\psi,\\
&(Y) \quad T_E^{(13)}(t) \approx \big(\beta_1(t)+2\beta_5(t)\omega^2\psi^2\big)\omega\psi,
\end{aligned} \tag{6.12}$$

where, in view of the assumed orders, β_3, β_4 and β_5 are the functions of t only. Moreover, β_1 may be considered as a linear function of $\omega^2\psi^2$.

The above formulae could be utilized in experiments as follows. First, for any chosen values of the parameters ω and ψ, we measure $T_E^{(33)}$ as a function of time t. This procedure gives the function $\beta_3(t)$ and may be averaged, taking into account any other values of ω

and ψ. Next, on the basis of the experimentally determined function $T_E^{(23)}(t)$, we can calculate the function $\beta_4(t)$. Then the function $\beta_2(t)$ is given by [cf. (6.10)]

$$\beta_2(t) \approx -\big(\beta_4(t)\omega^2+\beta_3(t)\big). \tag{6.13}$$

In a similar manner, the measurement of $T_E^{(13)}(t)$ implies the forms of $\beta_1(t)$ and $\beta_5(t)$, if we assume that $\beta_1(t)$ is entirely responsible for time-dependence of zero shear-rate viscosity. Some information on the form of β_1 as a function of $\omega^2\psi^2$ can be extracted from a steady-state experiment for small shear-rates $\omega\psi$ (cf. [12, 20]). Thereby, repeating measurements for various values of the parameters ω and ψ, we could, in principle, construct families of the corresponding curves for all required functions β_i.

Such an approach is not restricted from the theoretical point of view as regards the form of $\tilde{\omega}(t) = \omega\dot{k}(t)$, which may be an oscillatory function, a monotonically increasing function, etc. Any unforeseen difficulties may rather be connected with experimental realization and measurement of input and output parameters.

It is not our intention in the present paper to outline any definite and effective programme of experiments. We only wish to demonstrate certain possibilities resulting from the theory of MPSH, according to which any material functions determined, even approximately, in certain types of experiments may be useful as providing the material behaviour in other important flows.

7. Unsteady simple extensional flow

In this section, we present an attempt to apply the results obtained for the Maxwell rheometer flow to the case of unsteady simple extensional flow. Extensional flows of various nature, apart from their great importance for polymer processing, represent the third class of MPSH (cf. Sec. 2) and can be investigated within the theory outlined in this paper.

For a proportional simple extension, we have

$$v^1 = -\frac{1}{2}q\dot{k}(t)x, \quad v^2 = -\frac{1}{2}q\dot{k}(t)y, \quad v^3 = q\dot{k}(t)z, \tag{7.1}$$

where q is a constant. This velocity field gives:

$$[\mathbf{L}] = [\mathbf{N}]\dot{k}(t) = \begin{bmatrix} -\frac{q}{2} & 0 & 0 \\ 0 & -\frac{q}{2} & 0 \\ 0 & 0 & q \end{bmatrix}\dot{k}(t), \quad [\mathbf{A}_1] = [\overline{\mathbf{A}}_1] = \begin{bmatrix} -q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & 2q \end{bmatrix}\dot{k}(t). \tag{7.2}$$

Thus, according to (2.2), $(2.6)_2$ and (5.12), we obtain:

$$\mathbf{F}_0(\tau) = \exp\big((\mathbf{N}k(\tau)\big), \quad \mathbf{C}(s) = \exp\left(\frac{g(s)}{\dot{k}(t)}\mathbf{A}_1\right). \tag{7.3}$$

Since for the above type of flow $[\mathbf{A}_2] = [\overline{\mathbf{A}}_1^2] = [\mathbf{A}_1^2]$, we can use the constitutive equation (6.7) in the form:

$$\mathbf{T}_E = \beta_1\mathbf{A}_1+\beta_2\mathbf{A}_1^2+\beta_3\mathbf{A}_1^2+\beta_4\mathbf{A}_1^4+2\beta_5\mathbf{A}_1^3+2\beta_6\mathbf{A}_1^6, \tag{7.4}$$

where β_i ($i = 1, \ldots, 6$) are functions of t and polynomials in the following invariants: $\mathrm{tr}\,\mathbf{A}_1^2 = 6q^2\dot{k}^2(t)$, $\mathrm{tr}\,\mathbf{A}_1^3 = 6q^3\dot{k}^3(t)$. After using the Cayley-Hamilton theorem (cf. e.g. [1]), we can also write:

$$\mathbf{T}_E = \{\beta_1+2\beta_4 q^3+6\beta_5 q^2+24\beta_6 q^5\}\mathbf{A}_1+\{\beta_2+\beta_3+3\beta_4 q^2+18\beta_6 q^4\}\mathbf{A}_1^2. \tag{7.5}$$

Taking into account the boundary condition $T^{(11)} = T^{(22)} = 0$, we obtain finally:

$$T^{(33)} = T_E^{(33)}-T_E^{(11)} = \{3\beta_1+18\beta_5 q^2+6\beta_4 q^3+72\beta_6 q^5\}q + \{3\beta_2+3\beta_3+9\beta_4 q^2+54\beta_6 q^4\}q^2, \tag{7.6}$$

where β_i are functions of t and q^2, q^3.

It follows from (7.6) that up to the terms of order $O(q^4)$, the stress $T^{(33)}$ is determined by the same functions of time: β_1, β_2, β_3, β_4 and β_5 which appeared in (6.12) for moderately slow Maxwell rheometer flows. If we assume, as in Sec. 6, that $\beta_1(t)$ is responsible for the time-dependence of viscosity at zero shear-rates, $3\beta_1(t)$ will represent the time-dependence of Trouton's viscosity at zero extension rates.

Introducing the notion of extensional (tensile) time-dependent viscosity (cf. [7]), we arrive at the following approximate formula:

$$\eta^*(t, q) = \frac{T^{(33)}}{q} \approx 3\beta_1(t)+18\beta_5(t)q^2+6\beta_4(t)q^3 + \{3\beta_2(t)+3\beta_3(t)+9\beta_4(t)q^2\}q+O(q^4), \tag{7.7}$$

where β_1 may depend on q^2 at most linearly. Thus the behaviour of a fluid in unsteady simple extensional flows can be determined, at least approximately, on the basis of measurements made for the case of moderately slow flows in the Maxwell orthogonal rheometer. The time-dependence involved in the derivation of (7.7) is not restricted in any sense, apart from the requirement that the flow under consideration really belongs to the class of MPSH.

By way of illustration, let us consider three cases in which time-dependence of the deformation gradient is known a priori:

1. The case of constant strain rate. For this steady flow, the deformation gradient has the form $(7.3)_1$ with

$$k(\tau) = \tau, \quad k(0) = 0. \tag{7.8}$$

Since the flow is motion with constant stretch history, all β_i in (7.6) or (7.7) may be considered as material constants independent of time.

2. The case of constant velocity. For this unsteady flow, the deformation gradient has the form $(7.3)_1$ with

$$k(\tau) = \frac{1}{q}\ln(1+q\tau), \quad k(0) = 0, \quad q = v_0/l_0, \tag{7.9}$$

where v_0 is the velocity of extension, and l_0 denotes the initial length of a specimen. All β_i in (7.6) or (7.7) are, of course, functions of time t and may be determined, in principle, from the Maxwell rheometer flow, at the same function $k(\tau)$.

3. The case of oscillatory extension. For this unsteady flow, the deformation gradient has the form $(7.3)_1$ with

$$k(\tau) = \sin\nu\tau, \quad k(0) = 0, \quad \nu = \mathrm{const}, \tag{7.10}$$

where ν is the angular frequency of oscillations. All β_i in (7.6) or (7.7) depend on time t as well as frequency ν and, since the dependence on t can be expressed in harmonic form (cf. Sec. 8), the quantities finally measured are functions of ν only.

It should be emphasized to conclude this section that, apart from the Maxwell orthogonal rheometer, any other rheometers realizing flows belonging to the third class of MPSH or MSPSH, might be more practical and even more precise for determination of extensional flow characteristics. This question is open to further investigation.

8. The case of special oscillatory flow

Let us assume that in the Maxwell rheometer flow (cf. Fig. 1), two disks rotate with the same constant angular velocity ω, while the eccentricity expressed by the parameter $\tilde{\psi}(\tau) = \psi \sin \nu\tau$ oscillates with a constant angular frequency ν. This leads to

$$[\mathbf{F}_0(\tau)] = \exp[\mathbf{N}(\tau)] = \exp\begin{bmatrix} 0 & -\omega & \omega\psi\sin\nu\tau \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.1}$$

and

$$\mathbf{C}(s) = \exp\big(\mathbf{N}^T(t-s)-\mathbf{N}^T(t)\big)\exp\big((\mathbf{N}(t-s)-\mathbf{N}(t)\big). \tag{8.2}$$

Since the flow defined above does not belong to the class of MPSH (not all components of $\mathbf{N}(\tau)$ are proportional to the same function of time), we have to use a different approach.

If we assume that the amplitude of oscillations ψ is small, we can use the constitutive equations corresponding to the case of what is called finite linear viscoelasticity (cf. [1]). This means that the functional defined in (3.1) can be written in the form of the linear integral operator:

$$\mathbf{T}_E(t) = \int_0^\infty m(s)\mathbf{G}(s)ds, \quad \mathbf{G}(s) = \mathbf{C}(s)-\mathbf{1}. \tag{8.3}$$

Expanding two exponential terms in (8.2) into the following series:

$$\mathbf{C}(s) = \sum_{n=0}^\infty \frac{1}{n!}\big(\mathbf{N}^T(t-s)-\mathbf{N}^T(t)\big)^n \sum_{m=0}^\infty \frac{1}{m!}\big(\mathbf{N}(t-s)-\mathbf{N}(t)\big)^m, \tag{8.4}$$

multiplying and disregarding higher order terms in ψ by comparison with ψ^2, we obtain:

$$\begin{aligned} G_{11} &= G_{22} = G_{12} = G_{23} = 0, \\ G_{13} &= \omega\big(\tilde{\psi}(t-s)-\tilde{\psi}(t)\big), \\ G_{33} &= \omega^2\big(\tilde{\psi}(t-s)-\tilde{\psi}(t)\big)^2. \end{aligned} \tag{8.5}$$

These expressions can also be written in the form:

$$\begin{aligned} G_{13} &= \omega\psi\{\sin\nu t(\cos\nu s-1)-\cos\nu t\sin\nu s\}, \\ G_{33} &= \omega^2\psi^2(1-\cos\nu s)\{1+\cos2\nu t\cos\nu s+\sin2\nu t\sin\nu s\}. \end{aligned} \tag{8.6}$$

Introducing (8.6) into (8.3), we obtain finally:

$$\begin{aligned} T_E^{(13)}(t) &= \omega\psi\{A\sin\nu t + D\cos\nu t\}, \\ T_E^{(33)}(t) &= -\omega^2\psi^2\{A + B\cos 2\nu t + C\sin 2\nu t\}, \end{aligned} \tag{8.7}$$

where

$$\begin{aligned} A(\nu) &= -\int_0^\infty m(s)\,(1-\cos\nu s)\,ds, \\ B(\nu) &= -\int_0^\infty m(s)\,(1-\cos\nu s)\cos\nu s\,ds, \\ C(\nu) &= -\int_0^\infty m(s)\,(1-\cos\nu s)\sin\nu s\,ds, \\ D(\nu) &= -\int_0^\infty m(s)\sin\nu s\,ds, \end{aligned} \tag{8.8}$$

are characteristic material functions of ν.

The above results are very similar to those obtained for oscillatory shear flows (cf. [21]). The normal extra-stress $T_E^{(33)}$ oscillates with double the frequency of the shear stress $T_E^{(13)}$. This latter stress oscillates with the same frequency as the eccentricity parameter $\tilde{\psi}(t)$, but is shifted in phase. By contrast with LODGE's result (cf. [21]), the same material function $A(\nu)$ is responsible for the in-phase component of $T^{(13)}$ and the constant component of $T_E^{(33)}$.

The functions $A(\nu)$ and $D(\nu)$ may be interpreted as the dynamic modulus $G'(\nu)$ and the dynamic viscosity multiplied by the frequency $\nu\eta'(\nu)$, respectively.

The results obtained in this section may be useful in an experimental investigation of simple oscillatory extensional flows, for which

$$[\mathbf{F}_0(\tau)] = \exp[\mathbf{N}(\tau)] = \exp\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} q\sin\nu\tau. \tag{8.9}$$

Oscillations of this form lead, assuming that the terms up to q^2 are retained, to the following result:

$$T^{(33)} = T_E^{(33)} - T_E^{(11)} \approx 3q(A\sin\nu t + D\cos\nu t) - \frac{3}{2}q^2(A + B\cos 2\nu t + C\sin 2\nu t), \tag{8.10}$$

where the functions A, B, C, D are defined by (8.8).

Thus, for small amplitudes, the material functions measured in the special Maxwell rheometer flow can be used to determine the behaviour of fluids in oscillatory simple extensional flows.

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