

## The rigidity of gases

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MAXWELL'S kinetic theory of gases assumes that gases possesses an elasticity of shape measured by a modulus of rigidity. In the present paper expressions are derived in accordance with his theory for pure shear. It is shown that for the elastic strains the Hencky measure must be used. This is applied to the problem of simple shear. In this case there appear in addition to the first-order tangential stresses also second-order cross-stresses. They can be demonstrated in a centripetal pump designed by REINER.

Kinetyczna teoria gazów Maxwella przyjmuje, że gaz ma sztywność postaciową mierzoną modulem sztywności. W niniejszej pracy wyprowadzono wyrażenia na naprężenia dla przypadku czystego ścinania. Wykazano, że należy korzystać z miary sprężystej deformacji Hencky'ego. Miara ta została zastosowana do problemu prostego ścinania. W tym przypadku obok naprężeń stycznich pierwszego rzędu występują poprzeczne naprężenia rzędu drugiego. Naprężenia te można zaobserwować w pompie centropetalnej, zaprojektowanej przez REINERA.

Кинетическая теория газов Максвелла принимает, что газ обладает упругостью формы, измеряемой модулем жесткости. В настоящей работе выведены выражения на напряжения для чистого сдвига согласно его теории. Показано, что следует пользоваться мерой Хенки для упругих деформаций. Эта мера применена к проблеме простого сдвига. В этом случае наряду с тангенциальными напряжениями первого порядка появляются поперечные напряжения второго порядка. Их можно наглядно увидеть в центропетальном насосе спроектированном Рейнером.

### 1. Introduction

THE PRESENT investigation deals, as indicated in the title, with "gases" and their property named "rigidity".

To define these two terms, we may say that "a gas is a volume of matter which cannot maintain a free surface", where a free surface is one on which no stresses act.

"Rigidity" is the elastic response to a *change of form* of an elastic material. Elasticity is the property of a material to have the deformation, caused by certain forces, entirely or partly reversed, when these forces are removed. Such recovered deformation, which may be called *strain*, will consist in either a *change of volume* or a *change of shape*, the *distortion*, or a combination of both. The voluminal deformation is elastically opposed by an effort measured by the "bulk modulus", the elastic distortion is measured by the "modulus of rigidity".

These terms have originally been conceived to describe the elasticity of *solid materials*, following Hooke's law. This is formulated in the classical theory of elasticity as

$$(1.1) \quad m_l s = \kappa \varepsilon_v (\delta_{lm}) + 2\mu (m_l \varepsilon^\circ),$$

where  $m_l s$  is the stress tensor,  $m_l \varepsilon$  the strain tensor,  $m_l \varepsilon^\circ$  its deviator,  $\kappa$  the bulk modulus,  $\mu$  the modulus of rigidity,  $v$  as an index indicates volume and  $\delta$  is Kronecker's delta. The indices  $l$  and  $m$  are tensor indices, the sign  $^\circ$  over the symbol denotes the deviator.

Through the deformation the coordinates  ${}_i x$  of the particles of the initial state are transformed to the final coordinates  $x_i$ , where an index on the left refers to the initial state, one on the right refers to the final state. The displacement vector is defined by

$$(1.2) \quad {}_i u = x_i - {}_i x$$

and the strain by

$$(1.3) \quad {}_m \varepsilon = \frac{1}{2} ({}_{m,i} u + {}_{i,m} u).$$

In "classical" elasticity it is assumed the  ${}_i u$  is "infinitesimal" and so are the gradients  ${}_{m,i} u$ , where the comma indicates the derivative.

Of fluids it is usually supposed that they exhibit a viscous resistance to rate of change of shape, following Newton's law, but have an elasticity of volume only. Their constitutive equation is formulated in the classical theory of hydrodynamic viscosity as

$$(1.4) \quad s_{im} = -p \delta_{im} + 2\eta \dot{f}_{im},$$

where  $p$  is the isotropic pressure,  $\eta$  the coefficient of viscosity and  $\dot{f}_{im}$  the deviator of the flow tensor. The flow tensor is

$$(1.5) \quad f_{im} = \frac{1}{2} (v_{i,m} + v_{m,i})$$

with  $v$  the velocity, while its deviator is

$$(1.6) \quad \dot{f}_{im} = \dot{f}_{im} - \frac{f_{\alpha\alpha}}{3} \delta_{im}.$$

If we write  $v_i = \dot{u}_i$ , we have  $f_{im} = \dot{\varepsilon}_{im}$ . This implies an imperfection of the classical approach, assuming that "flow" is defined by an infinitesimal deformation.

We shall presently see that this view is too narrow and does not apply to "elastic liquids".

It was long known that there are elastic liquids which exhibit an elasticity of shape, a property called "elastico-viscosity" (SCHWEDOFF, 1889. See also HATSCHEK, 1928, p. 224). Their theory has only recently been developed mainly for polymers. However, the rheological equation of an elastico-viscous liquid was postulated by Maxwell already in 1866. He wrote it in a form which relates the complete stress to a complete strain and its time derivatives. It was first pointed out by REIGER (1919) that the isotropic part of both viscous liquids and elastic solids must be dealt with separately, as under isotropic pressure the liquid may be assumed to be simply elastic similar to a solid. For elastico-viscosity we therefore have to look for a relation between the elastic deviator

$$(1.7) \quad \dot{\varepsilon}_{im} = \frac{\dot{s}_{im}}{2\mu}$$

and the viscous deviator

$$(1.8) \quad \dot{f}_{im} = \frac{\dot{s}_{im}}{2\eta}.$$

Maxwell assumed that the rate of deformation  $\dot{d}_{im}$  of an elastico-viscous material is equal to the co-axial sum of the elastic and the viscous rates of deformation. For the

elastic rate of deformation he introduced  $\dot{\varepsilon}$  and for the viscous rate of deformation  $\dot{f}$  and wrote (in different form and with different notation)

$$(1.9) \quad \dot{d} = \frac{\dot{s}}{2\mu} + \frac{s}{2\eta}$$

which means that he assumed that the measure of flow is equal to the time derivative of the measure of strain, or

$$(1.10) \quad \dot{f}_{lm} = \frac{d}{dt} \dot{\varepsilon}_{lm}.$$

We shall now examine under what conditions such an equality can exist.

## 2. The Hencky-measure of strain

In accordance with (1.10), flow is expressed as the rate of relative deformation. This is only possible if we consider finite deformations and drop the classical assumption that  $\varepsilon$  is infinitesimal. To consider infinitesimal deformation of a liquid means excluding real flow. When there is flow, the deformation is finite.

Therefore consider dilatational flow. Consider a cylindrical tube of increasing length  $l$ , surrounding a straight streamline of a viscous liquid. Let the origin of the coordinate system permanently coincide with one end of the tube. Let the other end move with the velocity  $v_1$ , and let the flow be homogeneous, so that  $v_x = (v_1/l)x$ , where  $v_1 = dl/dt$ . In accordance with the arguments of classical hydrodynamics, the longitudinal rate of deformation or "flow" which causes and determines the viscous resistance is  $f_1 = v_1/l$ . Introducing the above expression for  $v$ , we find  $f_1 = (dl/dt)/l = (d/dt) \log(l/l_0)$ , where  $l_0$  is the "original" length of the tube at some arbitrary time  $t = 0$ . If the flow is not homogeneous, we must consider an *element* of length, say  $dx$ , the two ends of which move with the velocities  $v_x$  and  $v_{x+dx} = v_x + (dv_x/dx)dx$ , respectively, so that the relative velocity (our previous  $v_1$ ) is  $(dv_x/dx)dx$  and  $f_{xx} = dv_x/dx = v_{x,x}$ . We then find as before  $f_{xx} = d[\log(\delta x_1/\delta x_0)]/dt$ , where  $x_0$  and  $x_1$  are the values of  $x$  for  $t = 0$  and  $t = t$ , respectively. But  $\delta x_1 = \delta x_0 + u_x(x_0 + \delta x_0, t) - u_x(x_0, t)$ , where  $u_x(x_0, t)$  is the relative displacement as a function of the initial coordinates and time. Neglecting terms of second order and higher order ( $u_{x,x}$  cannot vanish identically) against the first order term, there results  $\delta x_1 = \delta x_0 + u_{x,x}(x_0, 0)\delta x_0$  and therefore  $f_{xx} = (d/dt)[\log(1 + u_{x,x})] = (d/dt) \times \log(1 + I_{xx})$ . Only when the displacement gradient is infinitesimal, can we, by developing

$$(2.1) \quad d_{xx} = \log(1 + \varepsilon_{xx}) = \varepsilon_{xx} - \varepsilon_{xx}^2/2 + \dots$$

use the Cauchy deformation, Eq. (1.3). When the displacement gradient is finite, as it will be in every case of real flow, whether viscous or plastic, we must measure the deformation in the logarithmic measure introduced systematically by Hencky in which

$$(2.2) \quad e^H = \log(l/l_0)$$

replaces the Cauchy measure

$$(2.3) \quad e^C = l/l_0 - 1.$$

We find then

$$(2.4) \quad f_l = \frac{d}{dt} (e^H)$$

and the designation "rate of deformation" for  $f_{lm}$  is justified.

The assertion that only the Hencky-measure is suitable for the description of a finite deformation due to viscous flow is important enough to warrant support by an additional argument. Consider the elongation of a bar of "original" length  $l_0$  in stages. If we elongate the bar first by a small increment  $\Delta l$ , the extension will be  $\Delta l/l_0$ . Now elongate the extended bar of length  $l_0 + \Delta l$  by another increment  $\Delta l$ . The question then arises how to express its extension. In accordance with the Cauchy measure, if the second elongation follows the first immediately, so that both can be considered as forming one single operation upon the original length  $l_0$ , the second extension will again be  $\Delta l/l_0$  and the total  $2\Delta l/l_0$ .

But, if the second extension was carried out after some time as an entirely new operation, the original length  $l_0$  not being known, the second extension would be  $\Delta l/(l_0 + \Delta l)$  and the total  $\Delta l[1/l_0 + 1/(l_0 + \Delta l)]$  which is different from  $2\Delta l/l_0$ .

Now, let us consider the progressive elongation in the light of the Hencky measure. We have  $l_1 = l_0 + \Delta l$ ,  $l_2 = l_1 + \Delta l = l_0 + 2\Delta l$  and the final extension is

$$(2.5) \quad e^H = \ln(l_2/l_0).$$

The expression on the right side can, however, be developed as follows:

$$(2.6) \quad \ln(l_2/l_0) = \ln\left(\frac{l_2}{l_1} \cdot \frac{l_1}{l_0}\right) = \ln(l_2/l_1) + \ln(l_1/l_0).$$

Therefore, if a bar of length  $l_0$  is extended by  $\Delta l$  in a first operation and by another  $\Delta l$  in a second operation, the result is the same as if it were extended in one single operation by  $2\Delta l$ , provided extensions are defined as by HENCKY, but not so if defined as by CAUCHY. Mathematically, this is expressed by the statement that Hencky-extensions form a group, while Cauchy extensions do not form a group. In describing flow, we, therefore, must use the Hencky measure.

### 3. The relaxation-time

Maxwell wrote Eqs. (1.9) in the form

$$(3.1)^{(1)} \quad s = 2\mu\dot{d} - \frac{\dot{s}}{T},$$

where

$$(3.2) \quad T = \eta/\mu$$

is the "relaxation time", a term which can be understood from the example presently given.

Let the body, the rheological equation of which is Eq. (1.9), be stressed until a certain stress  $s_0$  is reached. Start the experiment at that time  $t = 0$  by fixing the deformation. We then have from (1.9) putting  $\dot{d} = 0$

(<sup>1</sup>) For convenience, we omit the subscripts  $l, m$  and the superscript  $\circ$ .

$$(3.3) \quad \frac{\dot{s}}{s} = -\frac{\mu}{\eta}.$$

By integration this gives

$$(3.4) \quad [\ln s]_0 = -\frac{\mu}{\eta} t$$

or

$$(3.5) \quad s = s_0 e^{-(\mu/\eta)t}.$$

We see that the initial stress  $s_0$  is gradually reduced until it vanishes at  $t = \infty$ . The stress accordingly relaxes in the measure of

$$(3.6) \quad -\left. \frac{s}{\dot{s}} \right|_{d=0} = \frac{\eta}{\mu} = T.$$

For an inelastic material  $\mu \rightarrow \infty$  and  $T \rightarrow 0$ , the relaxation is instantaneous. Alternatively, when  $\eta \rightarrow \infty$ ,  $T \rightarrow \infty$  which is the case of an elastic solid, the stress does not relax.

#### 4. Maxwell's kinetic theory of a gas

It is generally stated that a gas is a kind of fluid, which possesses an elasticity of volume, but no elasticity of shape or rigidity. The elasticity of volume is expressed by the law of Boyle-Mariotte as

$$(4.1) \quad p = R\varrho,$$

where  $p$  is the isotropic pressure in a steady state,  $\varrho$  is the density and  $R$  is a constant when the temperature is constant. It is, however, maintained that a gas has no elasticity of shape or "rigidity".

MAXWELL, in his quoted paper (1866), assumes such a rigidity on theoretical grounds. We shall presently treat with Maxwell's work. But this work has been entirely ignored. It is not mentioned in any volume of the "Encyclopedia of Physics" dealing with gas, or in some other book known to me. To make sure that it is ignored everywhere, I inquired Professor GOLDSTEIN (Harvard University) who wrote me, "I cannot remember at all having seen a modern exposition about the shear elasticity of an ideal gas". In any case, it was assumed that, if such a rigidity existed, it could not be observed. This was formulated by JEFFREYS (1952) as follows: "In any experiment that tests the behaviour over large intervals, the flow will be much greater than the elastic deformation and the insertion of an elasticity term in the stress-strain relation will affect nothing observable". JEFFREYS does not consider the case when there is elasticity in a direction where there is no displacement and no flow. Such elasticity has been called "cross-elasticity". We shall come back to it in § 7.

Maxwell's reasoning is based on a kinetic theory of gases which pictures the gas as a crowd of molecules, each moving on its own independent path.

Suppose the molecules to be confined in a rectangular vessel with the dimensions  $a$ ,  $b$ ,  $c$ , so that the volume of the vessel is

$$(4.2) \quad V = a \cdot b \cdot c.$$

Let the sides of the vessel be perfectly elastic. Let the molecules have no action on one another, so that they never strike one another, or cause each other to deviate from their rectilinear path.

We now consider one molecule of mass moving in the direction  $a$  with velocity  $u$ . By impact with one of the sides normal to  $a$ , when the side is at rest, it transfers the momentum  $mu - (-mu) = 2mu$ . It returns to the same side after a lapse of time  $2a/u$ . The impact upon the side per unit time or the pressure force is equal to the rate of change of momentum or equal to the momentum  $2mu$  divided by the time  $2a/u$ , or

$$(4.3) \quad bcp_{aa} = \frac{2mu}{2a/u} = \frac{mu^2}{a},$$

where  $p_{aa}$  is the stress and  $bc$  the area on which it acts.

From (4.3), considering (4.2), we have from one molecule

$$(4.4) \quad p_{aa} = \frac{mu^2}{abc} = mu^2/V.$$

From  $N$  molecules with  $\bar{u}$  the mean value of velocities and  $n$  the number density of the molecules, or

$$(4.5) \quad n = \frac{N}{V},$$

we find

$$(4.6) \quad p_{aa} = \frac{Nm\bar{u}^2}{V} = nm\bar{u}^2.$$

In the steady state, the motions of the molecules have no preference for particular directions in space and  $m\bar{u}^2 = m\bar{v}^2 = m\bar{w}^2$ , where  $v$  and  $w$  are the velocities in the directions  $b$  and  $c$ . We then have for the pressure in the steady state

$$(4.7) \quad p_{aa} = p_{bb} = p_{cc} = p = nm \frac{1}{3} \frac{\Sigma m(\bar{u}^2 + \bar{v}^2 + \bar{w}^2)}{\Sigma m}.$$

Considering that  $nm$  is the density  $\rho$ , this is the kinetic expression for the Boyle-Mariotte law, Eq. (4.1).

Now let us move one of the sides normal to  $a$ , a distance  $\delta a$  with the velocity  $U$  which is smaller than  $u$ . Due to the "perfect elasticity" of the side, the velocity of the molecule is reversed by collision. Relative to the moving side, the velocity of the molecule is  $u - U$  before collision, and  $-(u - U) = -u + U$  after the collision. Relative to a fixed system, the velocity of the molecule is therefore  $u - U + U = u$  before collision and  $(-u + U) + U = 2U - u$  after collision. By one collision, the velocity  $u$  with which the molecule hits the wall is reduced by  $u - (2U - u) = -2U$ . This process goes on for the interval  $T$ , where

$$(4.8) \quad TU = \delta a.$$

As said before, the molecule, hitting the side once, returns to it after the time  $2a/u$ . Therefore, the molecule hits the side during the time  $T$ ,  $Tu/2a$  times. Introducing  $T$  from (4.8),

this is equal to  $(\delta_a/U)(u/2a)$ . By one collision the velocity is reduced by  $-2U$  to  $u-2U$ . By  $(\delta_a/U)(u/2a)$  collisions it is reduced to

$$(4.9) \quad u-2U \frac{\delta_a}{U2a} u = u \left(1 - \frac{\delta_a}{a}\right).$$

The pressure is now in analogy with (4.3)

$$(4.10) \quad bc\bar{p}_{aa} = \frac{mu^2 \left(1 - \frac{\delta a}{a}\right)^2}{a + \delta a}.$$

To the first order this is

$$(4.11) \quad bc\bar{p}_{aa} = \frac{mu^2}{a} \left(1 - 3 \frac{\delta a}{a}\right).$$

However, the area  $bc$  is changed by movements  $\delta b$  and  $\delta c$  in the direction  $b$  and  $c$  from  $bc$  to  $\bar{bc} = b \left(1 + \frac{\delta b}{b}\right) c \left(1 + \frac{\delta c}{c}\right)$  or approximately to  $bc \left(1 + \frac{\delta b}{b} + \frac{\delta c}{c}\right)$ . Therefore (4.11) becomes

$$(4.12) \quad \bar{p}_{aa} = \frac{mu^2}{a} \left(1 - 3 \frac{\delta a}{a}\right) / \bar{bc}.$$

But from

$$(4.13) \quad \bar{bc} = bc \left(1 + \frac{\delta b}{b} + \frac{\delta c}{c}\right),$$

we have to the first order

$$(4.14) \quad 1/\bar{bc} = \frac{1}{bc} \left(1 - \frac{\delta b}{b} - \frac{\delta c}{c}\right)$$

and therefore to the same order

$$(4.15) \quad \bar{p}_{aa} = \frac{mu^2}{abc} \left(1 - 3 \frac{\delta a}{a} - \frac{\delta b}{b} - \frac{\delta c}{c}\right).$$

From (4.4), (4.6) and (4.7), we have

$$(4.16) \quad \frac{mu^2}{abc} = nm\bar{u}^2 = p_0,$$

where  $p_0$  is the pressure if there is no change of volume.

Finally, we can write

$$(4.17) \quad \begin{aligned} \bar{p}_{aa} &= p_0 \left(1 - 3 \frac{\delta a}{a} - \frac{\delta b}{b} - \frac{\delta c}{c}\right), \\ \bar{p}_{bb} &= p_0 \left(1 - \frac{\delta a}{a} - 3 \frac{\delta b}{b} - \frac{\delta c}{c}\right), \\ \bar{p}_{cc} &= p_0 \left(1 - \frac{\delta a}{a} - \frac{\delta b}{b} - 3 \frac{\delta c}{c}\right). \end{aligned}$$

So far did Maxwell develop his theory.

Note that

$$(4.18) \quad \frac{\delta a}{a} + \frac{\delta b}{b} + \frac{\delta c}{c} = \frac{\delta V}{V} = \varepsilon_v$$

is equal to the cubical dilatation  $\varepsilon_v$  if the extensions are infinitesimal, as assumed ("very small quantities").

From (4.18), the mean pressure  $\bar{p}_m$  is

$$(4.19) \quad \bar{p}_m = (\bar{p}_{aa} + \bar{p}_{bb} + \bar{p}_{cc})/3 = p_0 \left(1 - \frac{5}{3} \varepsilon_v\right)$$

and  $\bar{p}_m$  is equal to  $p_0$  only if there is no change of volume and  $\varepsilon_v = 0$ .

## 5. Derivation of the rigidity

We now introduce the infinitesimal extensions

$$(5.1) \quad \varepsilon_a = \frac{\delta a}{a}, \quad \varepsilon_b = \frac{\delta b}{b}, \quad \varepsilon_c = \frac{\delta c}{c}, \quad \varepsilon_v = \varepsilon_a + \varepsilon_b + \varepsilon_c.$$

Consider the case

$$(5.2) \quad \varepsilon_c = 0, \quad \varepsilon_b = -\varepsilon_a, \quad \varepsilon_v = 0.$$

Accordingly, from (4.17),

$$(5.3) \quad \bar{p}_{aa} = p_0(1 - 2\varepsilon_a), \quad \bar{p}_{bb} = p_0(1 + 2\varepsilon_a), \quad \bar{p}_{cc} = p_0,$$

while from (4.19),

$$(5.4) \quad \bar{p}_m = p_0 = p.$$

This is a case of pure shear in which  $a, b, c$ , are the principal directions and the elastic stresses are, according to Hooke's law,

$$(5.5) \quad s_i = p_0 + 2\mu\varepsilon_i, \quad s_j = p_0 - 2\mu\varepsilon_i, \quad s_k = p_0.$$

Putting  $\bar{p}_{bb}$  for  $s_i$ ,  $\bar{p}_{aa}$  for  $s_j$ , and  $\bar{p}_{cc}$  for  $s_k$ , we find the rather surprising result

$$(5.6) \quad \mu = p_0.$$

This has been expressly stated by MAXWELL in the words "The coefficient (modulus) of <Rigidity> is therefore =  $p$ . This rigidity, however, cannot be directly observed, because the molecules continually deflect each other from their rectilinear courses, and so equalise the pressure in all directions". This equalization is connected with the time of relaxation. We can now calculate the time of relaxation of a gas such as air by putting in (3.6)  $p$  for  $\mu$ . We find that  $T$  is of the order of  $10^{-10}$  s. In the statement of Jeffreys quoted above, he considers as "large intervals" those compared with  $10^{-12}$  s. which he calculates as the time of relaxation of water.

## 6. The strain tensor

In Sec. 2, we have defined the Hencky measure of strain (Eq. 2.2) from which we can calculate the principal components of the Hencky strain tensor.



Introducing the stretch

$$(6.1) \quad \lambda = l/l_0,$$

we have in the Cauchy measure

$$(6.2) \quad e^C = \frac{\Delta l}{l_0} = \frac{l-l_0}{l_0} = \frac{l}{l_0} - 1 = \lambda - 1,$$

and in the Hencky measure

$$(6.3) \quad e^H = \int_0^1 \ln \lambda = (\lambda - 1) - \frac{1}{2}(\lambda - 1)^2 + \dots$$

If the stretch is infinitesimal, this is reduced to

$$(6.4) \quad \varepsilon^H = \lambda - 1$$

which is the same as the classical measure of elasticity defined by (6.2). This gives us the principal components of infinitesimal pure strain as written in (5.1).

Pure strain is defined by the absence of rotation while general strain presupposes a rotation. In §5, we considered the case of pure shear. If we want to know conditions in, say, simple shear, we must use the tensor of finite strain and then specialize it for the infinitesimal case.

Tensors of finite strain were derived by MURNAGHAM (1937) only for two cases named after GREEN and ALMANSI. Let a body given by initial coordinates  ${}_i a$  be deformed, when the coordinates become  $x_i$ , and let the coordinate gradients be

$$(6.5) \quad \lambda_{im} = \frac{\partial x_i}{\partial {}_m a}, \quad {}_m \lambda = \frac{\partial {}_i a}{\partial x_m},$$

then the deformation tensors for the two measures are for finite deformations  ${}_m D$  and  $D_{im}$

$$(6.6) \quad 2{}_m D^G = \lambda_{\alpha i} \lambda_{\alpha m} - \delta_{im}, \quad 2D_{im}^A = \delta_{im} - {}_i \lambda_{\alpha m} \lambda_{\alpha i}$$

As can be seen from the manner of notation, the Green-tensor refers to the initial state and the Almansi tensor to the final state. HANIN and REINER (1956) expressed the final Hencky-tensor as a function of the Almansi tensor as follows:

$$(6.7) \quad 2D_{im}^H = F_0 \delta_{im} + F_1 (2D_{im}^A) + F_2 (4D_{i\alpha}^A D_{\alpha m}^A),$$

where the scalars  $F_0, F_1, F_2$  are functions of the invariants of  $2D_{im}^A$ . They found the following expressions for  $F_0, F_1, F_2$ , where  $x$  stands for  $2D_{im}^A$ ;

$$(6.8) \quad \begin{aligned} F_0 &= \text{III}_x \int_0^1 \frac{\sigma^2 d\sigma}{1 - \text{I}_x \sigma + \text{II}_x \sigma^2 - \text{III}_x \sigma^3}, \\ F_1 &= \int_0^1 \frac{(1 - \text{I}_x \sigma) d\sigma}{1 - \text{I}_x \sigma + \text{II}_x \sigma^2 - \text{III}_x \sigma^3}, \\ F_2 &= \int_0^1 \frac{\sigma d\sigma}{1 - \text{I}_x \sigma + \text{II}_x \sigma^2 - \text{III}_x \sigma^3}. \end{aligned}$$

## 7. Simple shear

For the case of simple shear, we write

$$(7.1) \quad a = x - \Gamma y, \quad b = y, \quad c = z,$$

where  $\Gamma$  is the displacement gradient.

The coordinate gradients are from (6.5)

$$(7.2) \quad {}_{mI}\lambda = \begin{bmatrix} 1 & -\Gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and, therefore, from (6.6)<sub>2</sub>,

$$(7.3) \quad 2D_{Im}^A = \Gamma \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\Gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We find from (7.3)

$$(7.4) \quad I_x = -\Gamma^2, \quad II_x = -\Gamma^2, \quad III_x = 0.$$

Furthermore,

$$(7.5) \quad 4D_{Ia}^A D_{am}^A = \Gamma^2 \begin{bmatrix} 1 & -\Gamma & 0 \\ -\Gamma & 1 + \Gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From (6.8)<sub>1</sub>,

$$(7.6) \quad F_0 = 0.$$

Therefore, from (6.7),

$$(7.7) \quad 2D_{Im}^H = \Gamma \begin{bmatrix} F_2 \Gamma & F_1 - F_2 \Gamma^2 & 0 \\ F_1 - F_2 \Gamma^2 & -F_1 \Gamma + F_2 \Gamma(1 + \Gamma^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For  $F_1$  and  $F_2$ , we find from (6.8)

$$(7.8) \quad F_1 = \frac{\Gamma^2 + 2}{\Gamma^2 C} \ln \left( \frac{C+1}{C-1} \right), \quad F_2 = \frac{1}{\Gamma^2 C} \ln \left( \frac{C+1}{C-1} \right), \quad C = \sqrt{1+4/\Gamma^2}.$$

Finally,

$$(7.9) \quad D_{Im}^H = \frac{1}{C\Gamma} \ln \left( \frac{C+1}{C-1} \right) \begin{bmatrix} \Gamma/2 & 1 & 0 \\ 1 & -\Gamma/2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which yields

$$(7.10) \quad I_H = 0, \quad II_H = -\frac{1}{4} \left[ \ln \left( \frac{C+1}{C-1} \right) \right]^2, \quad III_H = 0.$$

We require the principal strains

$$(7.11) \quad D_{Im}^H = \begin{bmatrix} D_i^H & 0 & 0 \\ 0 & D_j^H & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

We find these from the equation

$$(7.12) \quad -\lambda^3 + \text{II}\lambda^2 - \text{III}\lambda + \text{III} = 0.$$

Considering (7.10), this is simplified to

$$(7.13) \quad \lambda^3 + \text{III}\lambda = 0$$

from which

$$(7.14) \quad \lambda_{1,2} = \pm \sqrt{-\text{III}} = \pm \frac{1}{2} \ln \left( \frac{C+1}{C-1} \right), \quad \lambda_3 = 0.$$

In the case of infinitesimal strain, we calculate the linear approximation of  $\frac{1}{2} \ln \frac{C+1}{C-1}$  as follows. We have for  $C > 1$

$$(7.15) \quad \frac{1}{2} \ln \left| \frac{C+1}{C-1} \right| = \frac{1}{C} + \frac{1}{3C^3} + \dots$$

Keeping the first term only and neglecting  $\Gamma^2$  in comparison with 4, we have

$$(7.16) \quad \lambda_{1,2} = \pm \frac{1}{C} = \frac{\Gamma}{\sqrt{4+\Gamma^2}} \sim \frac{\Gamma}{2}$$

and

$$(7.17) \quad d_{im}^H = \Gamma \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Comparison of this expression with (5.2) shows that we can write for (5.5)

$$(7.18) \quad \bar{s}_{im} = p_0 \left( \delta_{im} + \Gamma \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right).$$

and  $D_{im}^H$  of (7.9) becomes for infinitesimal  $\Gamma$

$$(7.19) \quad d_{im}^H = \frac{\Gamma}{2} \begin{bmatrix} \Gamma/2 & 1 & 0 \\ 1 & -\Gamma/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$(7.20) \quad s_{im} = p_0 \left( \delta_{im} + \Gamma \begin{bmatrix} \Gamma/2 & 1 & 0 \\ 1 & -\Gamma/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right).$$

We, therefore, see that if we calculate the stress in infinitesimal simple shear using the Hencky measure of strain there will not only be shearing reactions  $\mu\Gamma$  as in the classical theory, but also "cross stresses" of the second order, namely a tension  $\mu\Gamma^2/2$  in the  $x$  direction and a pressure  $-\mu\Gamma^2/2$  in the  $y$  direction. These effects can be used in order to check experimentally Maxwell's theory of the rigidity of gases.

## 8. The centripetal-pump effect

Simple shear with the determination of  $s_{11}$  and  $s_{22}$  is difficult to carry out. However, REINER has used an instrument consisting of two concentric circular discs parallel at some distance  $d$ . Of these two discs, one, the "stator" is at rest. The other, the "rotor" revolves at some velocity  $\Omega$ . The distance  $d$  can be changed through movement of one or the other disc in a normal direction ( $y$ ). A gas between both discs is sheared, the streamlines (direction  $x = \theta$ ) being circles. If the stress component in this direction ( $s_{\theta\theta}$ ) is a tension, "strangulation" stresses in the  $r$  direction are set in. They are pressures ( $s_{rr}$ ) operating towards the centre and will, therefore, cause a centripetal pumping action. A pressure in the normal ( $y$ ) direction pushes the plates apart. This would be in accordance with (7.20).

Such an apparatus was designed by POPPER and REINER (1956) and demonstrated at the 9th Congress of Applied Mechanics (1958). The stator had an opening connected to a manometer. The distance  $d$  could be reduced from 0.75 mm to 0.015 mm. The rotor could be given speeds of up to 10 000 r.p.m. When the rotor was given a speed of about 7 000 r.p.m. and the gap  $d$  was the largest, the monometer registered a vacuum of about 1 cm water. When the gap was reduced to about 0.02 mm, the vacuum turned into a pressure which reached values higher than 1/2 atm. This instrument was later developed in a more efficient form. Cross-stresses were determined in the flow of different gases (1965) and it was found that the effect was the same with mono-, di- and tri- atomic gases, decreasing with decreasing ambient pressure. It can, therefore, be said that the "Reiner effect" (as this centripetal pumping effect is now generally called) proves Maxwell's kinetic theory of the rigidity of gases.

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## References

1. M. HANIN and M. REINER, *ZAMP*, **7**, 5, 377-393, 1956.
2. E. HATSCHKE, *The viscosity of liquids*, London 1928.
3. H. HENCKY, *Ann. der Phys.*, **2**, 5, 617-630, 1929.
4. H. JEFFREYS, *The earth*, Cambridge University Press, 1929.
5. J. C. MAXWELL, *On the dynamical theory of gases*, *Phil. Trans.*, **157**, 1866, or *Scientific Papers*, **2**, p. 26.
6. F. D. MURNAGHAM, *Amer. J. Math.*, **59**, 238-260, 1937.
7. R. REIGER, *Ber. d. deutschen Phys. Ges.*, **132**, 421-434, 1919.
8. M. REINER (in collaboration with A. FOUX), *Cross-stresses in the flow of different gases*, In *Topics in Applied Mechanics*, Amsterdam 1965.
9. M. REINER, (in collaboration with B. POPPER), *Br. J. Appl. Phys.*, **7**, 452-453, 1956; **8**, 493-494, 1957.
10. T. SCHWEDOFF, *J. de Phys.*, **9**, 2, 34, 1890.

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