Some thermodynamic considerations of phenomenological theory of non-isothermal elastic-plastic deformations

TH. LEHMANN (BOCHUM)

IN THE FRAME work of continuum mechanics non-isothermal elastic-plastic deformations may be described as a process according to the thermodynamics of COLEMAN-NOLL-TRUESDELL. On the other side at least quasi-static homogeneous deformations can be described by state equations. This paper deals with the relations between these two possibilities.

W ramach mechaniki ośrodków ciągłych nieizotermiczne odkształcenia sprężysto-plastyczne opisać można jako proces zgodny z termodynamiką Colemana-Nolla-Truesdella. Z drugiej strony równania stanu opisać mogą co najmniej quas-istatyczny, jednorodny stan odkształcenia. Praca przedstawia rozważania dotyczące związków pomiędzy tymi dwiema możliwościami.

В рамках механики сплошной среды неизотермические упруго-пластические деформации можно описать как процесс согласующийся с термодинамикой Колемана-Нолла-Трус-делла. С другой стороны уравнения состояния могут описать по крайней мере квази-статическое, однородное состояние деформаций. Работа представляет рассуждения, ка-сающиеся соотношений между этими двумя возможностями.

1. Introduction

WITHIN the framework of continuum mechanics the disturbance of a crystal lattice can be described by the deviations of the metric and the (in general independent) lattice connexions from the undisturbed state [1]. The disturbances of the lattice are related to stresses. They are subjected to thermal oscillating motions. Thus only statistical statements can be made about the relation between disturbance of the lattice, stresses and temperature and phenomena connected with the change of state. Provided that the states are homogeneous everywhere, these considerations leads first to a phenomenological theory of non-isothermal, elastic-plastic changes of state of an unlimited single crystal. Then the next steps finally lead through the phenomenological theory of a limited single crystal and further intermediate stages (influence of grain boundaries between two crystals etc.) to a phenomenological theory of non-isothermal, elastic-plastic changes of state of a polycrystalline material. In a similar manner one can proceed with non-crystalline solid bodies of a different structure [1]. It depends on the simplifications made in these considerations as to how many and which of the state variables are finally included in a phenomenological theory derived in this way. Questions connected with these problems will not be discussed here. It must be stated, however, that in principle it must be possible to obtain from the consideration of the state of the crystal lattice, or more generally of the structure of an element of a solid body, a phenomenological theory of non-isothermal, elastic-plastic change of state, though this will need great efforts [2].

It is characteristic of such a theory that it allows the respective state of the body considered as a continuum to be described as a defined function of state variables.

Owing to the difficulties connected with this method, the usual formulations for a phenomenological theory of non-isothermal (and also isothermal) elastic-plastic deformations are in general restricted to a description of the thermomechanical process. It is generally assumed that such a process (with energy supplied only by heat or mechanical work) is uniquely defined, if

(a)   the initial values of all process variables and

(b)   the history of deformation and temperature

are known [3–6]. Hence it is assumed, that only one independent mechanical process variable[1] and one independent (scalar) thermal process variable exist, which, however, can be substituted by other appropriate process variables, as for instance the deformations by the stresses or the temperature by the heat supplied. The history of the dependent process variables can be determined by integrating the system of the differential equations. In general, this integration depends on the path. Thus, in general, the value of the dependent variables at the end of a process depends not only on the initial and final values of the independent process variables, but on the whole course of the process.

Here, however, more has to be said about these two different thermodynamical approaches. An essential controversy[2] can be traced through the whole discussion of thermodynamic aspects of continuum mechanics. It is assumed by some that in the case of processes which proceed through non-equilibrium states it is basically necessary to start from a description of the process [3–6]. Others suppose that also in this case one might assume local equilibrium for the elements of a body and therefore describe the state of the elements in general by state equations (e.g. [8]). The differences become particularly clear by considering the influence of entropy in these two different approaches. In the description of the process [3–6], entropy is a derived quantity and in principle we can proceed without introducing it. In the description by state equations it is on the contrary a necessary state value, which — at least in principle — can be immediately determined.

Restricting ourselves to homogeneous, quasi-statical thermomechanical processes[3], controversial points of view will not here be discussed further. For these processes we also obtain from the general fundamentals of the description of processes that a representation of the processes by state equations must be basically possible. The aim of this paper is to show at least for these homogeneous, quasi-statical non-isothermal, elastic-plastic deformations, what connections exist between the usual description as a process and a possible description by state equations. The considerations are, however, restricted to some examples which can be regarded as first steps to a more general theory. Further, we shall not discusse here whether and how far the considerations can be applied to non-homogeneous, non-quasi-statical processes.

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[1]   In the case of a general continuum, the metric and connexion will be regarded as one tensor variable.

[2]   A good survey of this controversy is given by [7].

[3]   The mere restriction to quasi-statical processes is not sufficient because differences of temperature and therefore non-equilibrium-states can occur in inhomogeneous processes.

2. Preliminary remarks and fundamental equations

Although we shall restrict ourselves here to homogeneous, quasi-statical processes, a method of presentation will be used which formally allows an immediate generalization to non-equilibrium states. Whether such a generalization is physically adequate will not be discussed here.

Further, the considerations here will be restricted to those materials the deformations of which can be completely described by the changes of the metric g_{ik} of a body-fixed coordinate-system ξ^i. Thus the connexion Γ^i_{jk} is regarded as dependent on the metric; it can be simultaneously naturalized with the metric [1]. The fundamental considerations, however, can be applied without essential difficulties also to a general Cosserat-continuum with metric and connexion independent of one another.

The transformation from the undeformed state (metric $\overset{\circ}{g}_{ik}$) to the deformed state can be represented by the tensor([4]):

$$(2.1) \qquad f^i_k = \overset{\circ}{g}^{ir} g_{rk} \quad \text{or} \quad (f^{-1})^i_k = g^{ir} \overset{\circ}{g}_{rk},$$

respectively, since

$$(2.2) \qquad \begin{aligned} g_{ik} &= f^r_i \overset{\circ}{g}_{rk} = \overset{\circ}{g}_{ir} f^r_k, \\ \overset{\circ}{g}_{ik} &= (f^{-1})^r_i g_{rk} = g_{ir}(f^{-1})^r_k. \end{aligned}$$

The strain rate is defined by:

$$(2.3) \qquad d^i_k = \frac{1}{2} g^{ir} \dot{g}_{rk} = -\frac{1}{2} g_{ir} \dot{g}^{rk} = \frac{1}{2}(f^{-1})^i_r (\dot{f})^r_{\cdot k} = -\frac{1}{2}(\dot{f^{-1}})^i_{\cdot r} f^r_k.$$

To decompose the strain rate into an elastic and a plastic part, one can use a non-commutative decomposition of the tensor f^i_k [9]:

$$(2.4) \qquad f^i_k = \underbrace{\overset{\circ}{g}^{ir} \overset{*}{g}_{rm}}_{\substack{f^i_{\cdot m} \\ P}} \underbrace{\overset{*}{g}^{ms} g_{sk}}_{\substack{f^m_k \\ E}}.$$

The metric $\overset{*}{g}_{ik}$ refers here to an in general incompatible intermediate configuration. From (2.4) we obtain an additive decomposition of the strain rate([5]):

$$(2.5) \qquad d^i_k = \underset{E}{d^i_k} + \underset{P}{d^i_k}.$$

For the description of the stress state, we introduce the Kirchhoff stress tensor s^i_k, which is connected with the real Cauchy stress tensor σ^i_k by the relation:

$$(2.6) \qquad s^i_k = \frac{\overset{\circ}{\varrho}}{\varrho} \sigma^i_k,$$

where ϱ denotes the current mass density and $\overset{\circ}{\varrho}$ the mass density in the initial state. The components of the tensor s^i_k and all the other tensors refer to the metric of the deformed body.

([4]) The Einstein summation-convention is used.

([5]) Details are given in [9]. An additive splitting of the strain rate can be obtained also in other ways [10–14]; but some difficulties arise in formulating the constitutive equations.

The rate of work per unit mass is denoted by $\dot{w}$. According to Eq. (2.5), it can be split into an elastic and a plastic part:

$$(2.7) \qquad \varrho\mathring{\dot{w}} = s_k^i d_i^k = \underset{E}{s_k^i d_i^k} + \underset{P}{s_k^i d_i^k} = \underset{E}{s_m^i (f^{-1})_k^m (\mathring{f})_{,i}^k} + \underset{P}{s_k^i d_i^k} = \varrho\underset{E}{\mathring{\dot{w}}} + \varrho\underset{P}{\mathring{\dot{w}}}.$$

The first law of thermodynamics (theorem of conservation of energy) can be written in the form:

$$(2.8) \qquad \varrho\mathring{\dot{u}} = \varrho\mathring{\dot{q}} + s_k^i d_i^k,$$

where u denotes the internal energy per unit mass and q the heat supplied per unit mass. From the second law of thermodynamics we need here only the statement that the rate of dissipation is

$$(2.9) \qquad \underset{D}{\dot{w}} \geqslant 0.$$

The rate of dissipation is connected with the specific entropy change by the relation:

$$(2.10) \qquad \dot{s} = \frac{1}{T}(\dot{q} + \underset{D}{\dot{w}}),$$

where T denotes the absolute temperature ([6]).

The dissipated energy cannot in general be identified with the plastic part of the rate of work — i.e. in general we have

$$(2.11) \qquad \underset{D}{\dot{w}} \neq \underset{P}{\dot{w}}.$$

Expressed here is the fact that a part of the plastic work is transformed into states of micro-stresses, which cause the hardening connected with plastic deformations. Therefore, we have to put [15]

$$(2.12) \qquad \underset{P}{\dot{w}} = \underset{D}{\dot{w}} + \underset{H}{\dot{w}},$$

where the part belonging to the hardening can be

$$(2.13) \qquad \underset{H}{\dot{w}} \lessgtr 0,$$

because in some cases we have to expect the occurrence of a softening (e.g. due to the Bauschinger-effect). Introducing instead of u the free energy (Helmholtz-function)

$$(2.14) \qquad \varphi = u - Ts,$$

and considering (2.8), (2.10) and (2.12), we can also write the first law of thermodynamics in the following form:

$$(2.15) \qquad \dot{\varphi} = -s\dot{T} + \dot{w} - \underset{D}{\dot{w}} = -s\dot{T} + \underset{E}{\dot{w}} + \underset{H}{\dot{w}}.$$

3. Description of non-isothermal, elastic-plastic deformations as a process

In formulating the constitutive equations of non-isothermal, elastic-plastic deformations as description of a process, the following assumptions are usually made:

([6]) The equations for the internal energy (2.8) and entropy (2.10) can be written also as balance-equations. The specific supplied heat q is then replaced by the heat flux q^i in these equations. But it is not necessary to formulate these equations, since they are not needed here.

1. The process is uniquely defined.

2. The stresses s_k^i and the temperature T are regarded as the independent process variables[7].

3. The constitutive law is independent of the time scale.

4. The elastic part of the deformations depends only on the stresses and the temperature.

5. The plastic part of the deformations depends on the stresses s_k^i and the temperature T, and also on other scalar or tensor parameters $(k^2, ...; \alpha_k^i, ...; A_{ks}^{ir}, ...)$, which can be regarded as dependent process variables.

'6. Plastic deformations occur only, if

(a) the stresses and the temperature satisfy a *yield condition* of the form

(3.1) $$F(s_k^i, T; k^2, ..., \alpha_k^i, ..., A_{ks}^{ir}...) = 0$$

and

(b) the increments of the stresses and the temperature satisfy a *loading condition*, which is usually written in the form[8]:

(3.2) $$\frac{\partial F}{\partial s_k^i} s_k^i|_0 = \frac{\partial F}{\partial T} \dot{T} > 0.$$

In most cases it is added that

7. (a) the plastic part of the deformations will depend linearly on the increments of stress and temperature, hence [10, 11, 19]

(3.3) $$\underset{P}{d_k^i} = D_{ks}^{ir} s_r^s|_0 + C_k^i \dot{T}$$

and

(b) [19, 20]

(3.4) $$\underset{P}{d_k^i} = \lambda \frac{\partial F}{\partial s_i^k}.$$

The first assumption is sometimes based on the argument that only in the case of this assumption (3.3) does the constitutive law become independent of time [19]. But this is incorrect. The arguments for the second assumption are often postulates [20, 21] with restrictions more rigorous than the second law of thermodynamics. Therefore both assumptions are not obligatory [10].

From the general assumption stated above [with or without (3.3) and (3.4)], as well as from other special assumptions (concerning the kind of hardening etc.), there results for

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[7] One can also proceed in other ways. In developing the theory this method usually is used. In applications other quantities are often given which then replace the independent variables in the process.

[8] $s_k^i|_0$ denotes the co-rotational time flux. Its definition refers to ZAREMBA [16] and JAUMANN [17]. For a tensor of the second-order — e.g., the stress tensor we have:

$$s_k^i|_0 = (\dot{s})_k^i + d_r^i s_k^r - d_k^r s_r^i.$$

Its meaning appears immediately by introducing the time as 4th coordinate in an adequate manner [18].

loading processes in the plastic range a system of ordinary differential equations of first order:

$$\frac{1}{2} g^{ir}\dot{g}_{rk} = d_k^i = \underset{E}{d_k^i}(s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}) + \underset{P}{d_k^i}(s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}, k^2 ..., \alpha_k^i ..., A_{ks}^{ir} ...),$$

$$k^2 = k^2(s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}, k^2 ..., \alpha_k^i ..., A_{ks}^{ir} ...),$$

(3.5)   ....

$$\alpha_k^i|_0 = \alpha_k^i|_0(s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}, k^2 ..., \alpha_k^i ..., A_{ks}^{ir} ...),$$

....

$$A_{ks}^{ir}|_0 = A_{ks}^{ir}|_0(s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}, k^2 ..., \alpha_k^i ..., A_{ks}^{ir} ...).$$

Of course, its special form depends on the respective assumptions[9]. If it is necessary or appears reasonable, the system of equations can be modified and extended — e.g., by introducing, as further dependent process variables, the whole work w or the plastic part of the work $\underset{P}{w}$ etc.

In the case of purely elastic deformations, the system of equations is reduced to

(3.6) $$\frac{1}{2} g^{ir}\dot{g}_{rk} = \underset{E}{d_k^i}(s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}).$$

This equation can be written in integral form [9]

(3.7) $$\underset{E}{f_k^i} = \underset{E}{f_k^i}(s_k^i, T).$$

4. Description of non-isothermal, elastic-plastic deformations by state equations

Many thermodynamic considerations of non-isothermal, elastic-plastic deformations refer essentially to the general fundamentals which must be observed in describing such phenomena as a thermo-mechanical process, and then discuss in particular which restrictions follow from the second law of thermodynamics (as for instance [12–14, 20–22]). Only a few papers (e.g. [10, 11, 23–26]) attempt to describe completely such processes by state equations. Most of these papers [10, 11, 23–25] introduce (explicitly or implicitly) plastic strains as thermodynamic state variables. But one may conclude from the consideration of the phenomena in the crystal lattice[10] as well as from phenomenological observations[11] that plastic strains in general cannot be regarded as state variables[12]. Furthermore, all these papers [10, 11, 23–26] consider the plastic work as completely dissipated. This, however, is in contradiction to experimental results [15] from which it emerges that one part of plastic work is used for producing states of residual stresses in the lattice, which — phenomenologically considered — cause hardening of the material [14].

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[9] The metric of the deformed body enters the system of equations since all tensor quantities refer to it.

[10] Dislocations, for example, which have completely passed the crystal produce plastic strains but no changes of state.

[11] Most different states of hardening can belong to the same plastic strains (at the same stresses and the same temperature).

[12] This is also emphasized by KESTIN [26].

If it is assumed — as is usual — that in the first approximation the elastic behaviour is not changed by plastic deformations, the free energy φ can be decomposed into two parts. The first $(\underset{E}{\varphi})$ is related to the thermo-elastic phenomena, the second $(\underset{H}{\varphi})$ to the hardening due to plastic deformations. Thus we have:

$$(4.1) \qquad \varphi = \underset{E}{\varphi}(f^i_k, T) + \underset{H}{\varphi}(T, h^2 \ldots, \beta^i_k \ldots, B^{ir}_{ks}).$$

Hence $\underset{E}{\varphi}$ depends only on the elastic strain described by $\underset{E}{f^i_k}$ according to Eqs. (2.4) and the temperature T, whereas $\underset{H}{\varphi}$ is a function of T and the scalar and tensor state variables $(h^2 \ldots, \beta^i_k \ldots, \beta^{ir}_{ks} \ldots)$, which describe the respective state of hardening. For the rate of the energy we have:

$$(4.2) \qquad \dot\varphi = \frac{\partial \underset{E}{\varphi}}{\partial \underset{E}{f^i_k}} (\underset{E}{\dot f})^i_{\cdot k} + \frac{\partial \varphi}{\partial T} \dot T + \frac{\partial \underset{H}{\varphi}}{\partial T} \dot T + \frac{\partial \underset{H}{\varphi}}{\partial h^2} \dot h^2 + \frac{\partial \varphi}{\partial \beta^i_k} \dot\beta^i_k|_0 + \frac{\partial \varphi}{\partial B^{ir}_{ks}} \dot B^{ir}_{ks}|_0$$

from (4.1)(^{13}). Yet, on the other hand,

$$(4.3) \qquad \dot\varphi = -s\dot T + \underset{E}{\dot w} + \underset{H}{\dot w} = -s\dot T + \frac{1}{\overset{\circ}{\varrho}} \underset{E}{s^i_m}(\underset{E}{f^{-1}})^m_k (\underset{E}{\dot f})^k_{\cdot i} + \underset{H}{\dot w}$$

follows from (2.15) and (2.7).

The comparison between (4.1) and (4.2) yields:

$$s = -\left(\frac{\partial \underset{E}{\varphi}}{\partial T} + \frac{\partial \underset{H}{\varphi}}{\partial T}\right) = -\frac{\partial \varphi}{\partial T},$$

$$(4.4) \qquad \frac{1}{\overset{\circ}{\varrho}} \underset{E}{s^i_m}(\underset{E}{f^{-1}})^m_k = \frac{\partial \underset{E}{\varphi}}{\partial \underset{E}{f^i_k}},$$

$$\underset{H}{\dot w} = \frac{\partial \underset{H}{\varphi}}{\partial h^2} \dot h^2 + \frac{\partial \underset{H}{\varphi}}{\partial \beta^i_k} \dot\beta^i_k|_0 + \frac{\partial \underset{H}{\varphi}}{\partial B^{ir}_{ks}} \dot B^{ir}_{ks}|_0.$$

If $\varphi(\underset{E}{f^i_k}, T, k^2, \beta^i_k, B^{ir}_{ks})$ is known, we can determine:

$$s = s(\underset{E}{f^i_k}, T, k^2, \beta^i_k, B^{ir}_{ks}),$$

$$(4.5)$$

$$s^i_k = s^i_k(\underset{E}{f^i_k}, T),$$

and

$$(4.6) \qquad \underset{H}{\dot w} = \underset{H}{\dot w}(T, k^2, \beta^i_k, B^{ir}_{ks}, \dot h^2, \dot\beta^i_k|_0, \dot B^{ir}_{ks}|_0).$$

However, complete description of the thermo-mechanical process according to Sec. 3 cannot yet be derived from this. In addition, we need a further statement from which we can ascertain how the rate of dissipation energy $\underset{D}{\dot w}$ is connected with the state variables and their first time derivatives. This statement can be made in different forms.

(^{13}) Here and in what follows h^2, β^i_k, B^{ir}_{ks} represent all the respective state variables.

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The description of the process will in the next chapters be compared with the description by state equations by means of two examples. From this we shall gain further interesting aspects of non-isothermal, elastic-plastic deformations.

5. Comparison between description of a process and description by state equations

5.1. Isotropic behaviour

In this case, the state of hardening can be described in the first approximation by a scalar state variable (besides the temperature). In order to simplify, we assume here that the hardening is independent of temperature. Then the free energy can be written in the form:

(5.1)
$$\varphi_E(f_k^i, T, h^2) = \varphi_E(f_k^i, T) + \varphi_H(h^2).$$

For the work of hardening it follows that:

(5.2)
$$\dot{w}_H = \frac{d\varphi_H}{dh^2}\dot{h}^2 = \dot{\varphi}_H,$$

$$w_H = \varphi_H(h^2) - \varphi_H(h_0^2).$$

Provided now that there exists a uniquely defined relation depending only on h^2 between the dissipated energy w_D and the work of hardening w_H — i.e.,

(5.3)
$$w_D = c(h^2)\varphi_H(h^2) - c(h_0^2)\varphi_H(h_0^2),$$

then we have

(5.4)
$$\dot{w}_P = \dot{w}_H + \dot{w}_D = \frac{d}{dh^2}\{[1 + c(h^2)]\varphi_H(h^2)\}\dot{h}^2,$$

$$w_P = [1 + c(h^2)]\varphi_H(h^2) - [1 + c(h_0^2)]\varphi_H(h_0^2).$$

The plastic work w_P as well as its parts w_H and w_D can on the assumptions made here, be represented as functions of the state variable h^2.

For description of the process, we obtain first that the yield condition must have the form:

(5.5)
$$F(s_k^i, k^2) = 0,$$

since we have assumed plastic behaviour independent of temperature. k^2 is here a parameter characterizing the hardening([14]). Since, on the other hand, the boundary of elastic behaviour in the stress space can depend only on the state variable h^2

(5.6)
$$k^2 = k^2(h^2) = k^2(w_P)^1$$

holds.

Hence there results from the assumptions made here for the state equations a material behaviour which is known as work hardening in the description of the process. If the

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([14]) Since isotropy is assumed, only the scalar invariants of the stresses enter (5.5).

hardening is isotropic in a more restrictive meaning — i.e., the yield condition does not change its shape during plastic deformations, we have to put[15]

$$(5.7) \qquad F(s_k^i, k^2) = f(s_k^i) - k^2 = 0.$$

Plastic deformations occur if (5.7) holds and the loading condition (3.2) is satisfied at the same time, as can be represented here in the form[16]:

$$(5.8) \qquad \frac{\partial f}{\partial s_k^i} \dot{s}_k^i|_0 > 0.$$

During plastic deformations, we obtain from (5.7):

$$(5.9) \qquad \dot{F} = \frac{\partial f}{\partial s_k^i} \dot{s}_k^i|_0 - \frac{dk^2}{dw}\bigg|_P \dot{w}_P = 0.$$

Taking into consideration that for the plastic strains the assumption (3.4) will be valid, which leads here to

$$(5.10) \qquad d_k^i{}_P = \lambda \frac{\partial f}{\partial s_k^i},$$

we find from (5.7), (6.9) and (5.10) the following system of equations for description of the process:

$$(5.11) \qquad d_k^i = d_k^i{}_E(s_k^i, \dot{s}_k^i|_0, T, \dot{T}, g_{ik}) + \mathring{\varrho}\, \frac{\dfrac{\partial f}{\partial s_s^r} \dot{s}_s^r|_0}{\dfrac{dk^2}{dw}\bigg|_P \dfrac{\partial f}{\partial s_n^m} s_n^m} \frac{\partial f}{\partial s_i^k},$$

$$\dot{w}_P = \frac{\dfrac{\partial f}{\partial s_k^i} \dot{s}_k^i|_0}{\dfrac{dk^2}{dw}\bigg|_P}.$$

The parameter k^2 can be eliminated in this system of equations, since k^2 can be expressed by w_P.

The state equation (5.1) can be found from the description of the process in the following manner:

1. $\varphi_E = \varphi_E(f_k^i, T)$ results from the assumed or derived stress-strain relation

$$d_k^i{}_E = d_k^i{}_E(s_k^i, \dot{s}_k^i|_0, T, \dot{T}, g_{ik})$$

for the elastic part of the strains.

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[15] Also, more general isotropic yield conditions can be formulated.

[16] From the fact that the yield condition is valid only during plastic deformations, one can conclude that in this case we have no state equation. Thus the yield condition belongs to the description of the process whereas Eq. (5.1) is a state equation which is valid for purely elastic as well as plastic deformation.

2. We can use the state variable h^2 — for instance, by letting $\underset{H}{h^2} = w$. Then follows first from (5.2)

$$(5.12) \qquad \underset{H}{\varphi(h^2)} = h^2$$

starting from the initial state $h_0^2 = 0$.

From (5.4) results

$$(5.13) \qquad \underset{P}{w} = [1+c(h^2)]h^2.$$

On the other hand, $\underset{P}{w}$ can be represented from the yield condition (5.7) as a function of $\underset{P}{w}$:

$$(5.14) \qquad \underset{P}{w} = \underset{P}{w}(k^2).$$

From (5.13) and (5.14) we can determine:

$$(5.15) \qquad h^2 = h^2(k^2).$$

In the case of linear hardening with

$$(5.16) \qquad k^2 = k_0^2 + 2B\overset{\circ}{\varrho}\underset{P}{w},$$

we have

$$(5.17) \qquad \underset{P}{w} = \frac{1}{2B\overset{\circ}{\varrho}} (k^2 - k_0^2).$$

By putting $c(h^2) = \text{const} = c$, thus assuming a constant relation between work of hardening and dissipated energy, it finally follows that

$$(5.18) \qquad \underset{H}{\varphi(h^2)} = h^2 = \frac{k^2 - k_0^2}{2B\overset{\circ}{\varrho}(1+c)}.$$

By this, the connection between description of the process and the related state equations is completely given.

5.2. Example for anisotropic behaviour

Provided that the plastic deformations are again independent of temperature, the state equation for the free energy may be assumed approximately in the following form:

$$(5.19) \qquad \varphi = \underset{E}{\varphi}(\underset{E}{f_k^i}, T) + \underset{H}{\varphi}(h^2, \beta_k^i, B_{ks}^{ir}).$$

As a special example, we use the formulation:

$$(5.20) \qquad \varphi = \underset{E}{\varphi}(\underset{E}{f_k^i}, T) + \underbrace{h^2 + B_{ks}^{ir}\beta_i^k\beta_r^s}_{\underset{H}{\varphi}}$$

with

$$B_{ks}^{ir} = B_{sk}^{ri}.$$

For the work of hardening it follows then that

(5.21)
$$\dot{w}_H = \dot{\varphi}_H = \dot{h}^2 + \beta_i^k \beta_r^s B_{ks}^{ir}|_0 + 2B_{ks}^{ir}\beta_i^k\beta_r^s|_0 \quad \text{or}$$

$$w_H = \varphi_H = h^2 + B_{ks}^{ir}\beta_i^k\beta_r^s,$$

respectively, if we put in the initial state $\varphi_H = 0$ (with $h^2 = 0$ and $\beta_k^i = 0$). It can be $\dot{w}_H \gtrless 0$. Again we assume that we have a relation between the dissipated energy and the parts of the work of hardening depending only on state. This assumption, however, must be introduced so that the rate of dissipation never becomes negative. Formulating for instance

(5.22)
$$w_D = \xi_1 h^2 - \xi_2 B_{ks}^{ir}\beta_i^k\beta_r^s$$

(ξ_1, ξ_2 depending on state), we have the condition:

(5.23)
$$\frac{\partial}{\partial t}(\xi_2 B_{ks}^{ir}\beta_i^k\beta_r^s) \leqslant \frac{\partial}{\partial t}(\xi_1 h^2).$$

By appropriate choice of ξ_1 and ξ_2, this condition can be satisfied.

As a special case we consider:

(5.24)
$$\xi_1 = \xi_1(h^2), \quad \xi_2 = 1.$$

Then

(5.25)
$$w_D = \xi_1(h^2)h^2 - B_{ks}^{ir}\beta_i^k\beta_r^s,$$

$$w_P = w_H + w_D = [1 + \xi_1(h^2)]h^2.$$

The plastic work corresponds here only with the change of the scalar state variable h^2, whereas the change of the other state variables is not connected with an external rate of work.

The preceding description by state variables coincides with description of the process which is based on a yield condition of the form:

(5.26)
$$F(s_k^i, k^2, \alpha_k^i, A_{ks}^{ir}) = A_{ks}^{ir}(t_i^k - \alpha_i^k)(t_r^s - \alpha_r^s) - k^2(w_P) = 0,$$

$$A_{ks}^{ir} = A_{sk}^{ri},$$

where

(5.27)
$$t_k^i = s_k^i - \frac{1}{3}s_r^r\delta_k^i$$

denotes the deviatoric part of the stresses.

From the stress-strain relations (3.4), we obtain the equations

(5.28)
$$d_k^i = d_k^i(s_k^i, s_k^i|_0, T, \dot{T}, g_{ik}) + \overset{\circ}{\varrho}\,\frac{\overline{\{A_{sn}^{rm}(t_r^s - \alpha_r^s)(t_m^n - \alpha_m^n)\}}}{\dfrac{dk^2}{dw_P}A_{qi}^{pj}(t_p^q - \alpha_p^q)t_j^i}\,A_{kg}^{if}(t_f^q - \alpha_f^g),$$

$$\dot{w}_P = \frac{1}{\dfrac{dk^2}{dw_P}}\{\overline{A_{ks}^{ir}(t_i^k - \alpha_i^k)(t_r^s - \alpha_r^s)}\}.$$

In order to obtain a complete description of the process this system of equations must be completed by two other equations

(5.29)
$$\alpha_k^i|_0 = \alpha_k^i|_0(t_k^i, t_k^i|_0, \underset{P}{w}, \alpha_k^i, A_{ks}^{ir}),$$
$$A_{ks}^{ir}|_0 = A_{ks}^{ir}|_0(t_k^i, t_k^i|_0, \underset{P}{w}, \alpha_k^i, A_{ks}^{ir}),$$

which must be chosen as suitable. Then the time derivatives of α_k^i and A_{ks}^{ir} can be eliminated in the first two Eqs. (5.28).

From the yield condition (5.26), $\underset{P}{w}(k^2)$ can be determined. From the second equation (5.25) results:

(5.30)
$$[1 + \xi_1(h^2)]h^2 = \underset{P}{w}(k^2)$$

— i.e., the connection between the state variable h^2 in the state equation (5.19) and the parameter k^2 in the yield condition (5.26). It remains undecided for the present which connection exists between the state variables β_k^i, B_{ks}^{ir} on the one hand and the parameters α_k^i, A_{ks}^{ir} of the description of the process on the other. It can be shown that we may put

(5.31)
$$B_{ks}^{ir} = A_{ks}^{ir}$$

and

(5.32)
$$\beta_k^i = \mu\alpha_k^i$$

without contradicting Eq. (5.23), provided that the remaining equations of process are reasonably defined and the proportionality factor μ is suitably chosen. The details cannot be discussed here.

Once more let it be emphasized that the yield condition is not a state equation in the meaning of thermodynamics. This holds even if — as in the two examples discussed above — the parameters of the yield condition are directly connected with the state variables. The yield condition belongs to the equations of the description of the process. It is only valid if plastic deformations occur, whereas state equations hold in general.

6. Some supplementary remarks

The preceding considerations can be extended in many ways. In the theory of irreversible processes the production of entropy is usually interpreted as an inner product of "thermodynamic forces" and the "fluxes" induced by them. The thermodynamic forces have to obey the "Onsager-Casimir-relations" [27].

It remains open as to what are to be regarded as "forces" and what as "fluxes". Since

$$\underset{P}{s_k^i d_i^k} = \underset{P}{\mathring{\varrho}} \dot{w} = \underset{P}{\mathring{\varrho}} (\underset{H}{\dot{w}} + \underset{D}{\dot{w}}),$$

we can write formally, for example

(6.1)
$$\underset{H}{\mathring{\varrho}} \underset{H}{\dot{w}} = \underset{H P}{s_k^i d_i^k},$$
$$\underset{D}{\mathring{\varrho}} \underset{D}{\dot{w}} = \underset{D P}{s_k^i d_i^k},$$

with

$$s_k^i + s_k^i = s_k^i,$$
$$\underset{H}{} \quad \underset{D}{}$$

where d_k^i can be interpreted as the "thermodynamic force" and s_k^i as associated "flux"
$\underset{P}{}$ $\underset{D}{}$
(or vice versa). But a proper physical argument is still lucking. One should, nevertheless, keep this question in view.

The examples discussed in Sec. 5 represent special cases. The considerations can be readily extended also to such problems with plastic deformations depending on temperature. Then we have to formulate $\varphi = \varphi(T, k^2 ...)$. But it is also possible to consider such
$\underset{H}{}$ $\underset{H}{}$
problems with "elastic" deformations no longer independent of plastic deformations. Then an additive decomposition of the free energy into two parts is not possible. We arrive at such problems in studying processes with unloading and reloading [28, 29]. Then we obtain for unloading and reloading different stress-strain relations (see Fig. 1). Negative
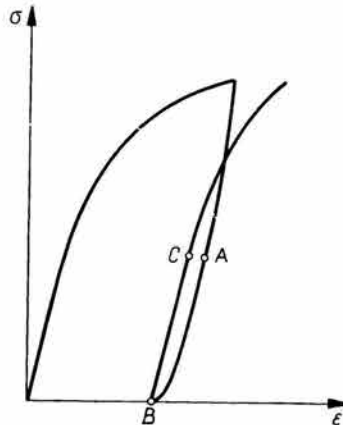

FIG. 1. Unloading and reloading (qualitatively corresponding to experimental results referred to in [28, 29]).

work results for a cycle of stresses (*A-B-C* in Fig. 1). That can be explained taking into consideration that a part of the stored work of hardening is already released during unloading *A-B*. This is a further argument for representing a part of the plastic work as work of hardening.

The stress-strain relations based on the theory of plastic potential lead to certain difficulties in the interpretation of some experimental results. Therefore, it seems to be more advantageous to replace the assumption (3.4) for the plastic strain rate by

(6.2) $$d_k^i = \dot{\lambda}\frac{\partial F}{\partial s_i^k} + \varkappa_{ks}^{ir}s_r^s|_0.$$
$\underset{P}{}$

This approach offers further possibilities to fit theoretical calculations to experimental results [30].

The considerations of this paper can be applied to non-quasi-static, inhomogenous deformation processes if it is accepted that the local state can be described by state equa-

tions also in the case of non-equilibrium states. The basic Eqs. (2.8)–(2.10) given here have then to be replaced by the corresponding balance equations for the flux of free energy and the entropy, with a respective inequality for the production of entropy. A balance equation for the flux of impulse and — if necessary — a balance equation for the flux of moment of momentum, as well as a balance equation for the mass flux have to be added [27]. By this, of course, the problem is considerably extended. The previous basic considerations, however, essentially hold, provided that the principle of local state is accepted.

7. Concluding remarks

The preceding considerations attempt to connect the description of non-isothermal elastic-plastic deformations as a process and the description of such phenomena by state equations. This paper is a first step in this direction. Many questions are still open and need a thorough clarification.

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INSTITUT FÜR MECHANIK
RUHR-UNIVERSITÄT, BOCHUM