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# Optimisation theory of elastic-rigid bodies under repeated variable deformation

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THE OPTIMISATION problems of limit displacements on a prescribed surface and of body parameters in the volume are formulated for elastic-rigid bodies. The mathematical models of dual pairs of mathematical programming problems in functional spaces are derived by way of extremum energy theorems. Mathematical models of problems concerning non-repeated deformation derived for deformations and stress velocities are shown to be a particular type of problems of repeated variable deformations derived for residual deformations and velocities of residual stresses.

Zdefiniowano problemy optymalizacji wartości przemieszczeń granicznych na powierzchni ciała lub optymalizacji rozkładu charakterystyk materiału wewnątrz ciała przy zastosowaniu modelu sprężysto-sztywnego. Modele matematyczne tych problemów zostały zbudowane w oparciu o twierdzenia energetyczne i mają postać dualnych zadań programowania matematycznego w przestrzeni funkcyjnej. Wykazano, że modele matematyczne dla odkształcania jednorazowego (w terminach odkształceń i przyrostów naprężeń) stanowią przypadek szczególny problemów odkształcania cyklicznego (w terminach odkształceń rezydualnych i przyrostów rezydualnych naprężeń).

Даны формулировки задач оптимизации величин предельных перемещений на поверхности тела или оптимизации распределения характеристик материала внутри тела, описываемого упруго-жесткой моделью. Математические модели рассмотренных задач построены на основе энергетических теорем и имеют форму двойственных задач математического программирования в функциональном пространстве. Показано, что математическое модели для однократных деформаций (сформулированные в терминах деформаций и приращений напряжений) являются частными случаями циклического деформирования (в терминах остаточных деформаций и приращений остаточных напряжений).

LET US consider the problem of repeated variable deformation of an elastic-rigid body (Fig. 1), referred to as perfectly locking in [1, 5]. This problem is similar to that of repeated variable loading (the shake-down problem) of elastic-plastic bodies.

Repeated variable deformation is understood to be a set displacements on a certain part of the body surface, every displacement or some subsets of displacements varying independently within the prescribed limits. In the case of non-repeated deformation, locking occurs when certain limit values of displacements are obtained.

Henceforth this state is referred to as simple locking. On reaching this state, the body deformation remains constant, while the increase of loading and stresses may be unlimited. Under repeated variable deformation, there may occur cycles of deformations, the repetition of which may cause locking at displacement values lower than in the case of non-repeated deformation. Henceforword, this type of locking is referred to as cyclic locking.

The purpose of the present paper is to formulate and derive the mathematical models of the general design problem (optimisation being included) of an elastic-rigid body subjected to the repeated variable deformation reffered to above. Note that all design parameters of the body will be determined in and not before the locking state.

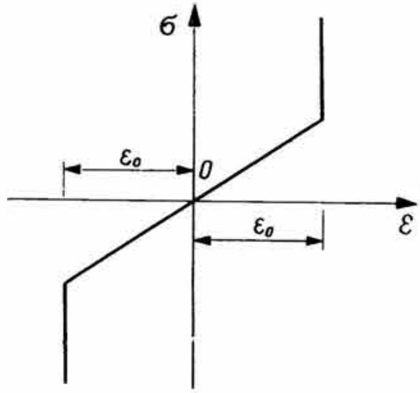


Fig. 1.

Thus the deformation programme as a whole need not be considered: it suffices to determine the limit state of the body. To this end, let us formulate the general extremum energy theorems determining which parameters of the given problems may be varied freely. The mathematical models corresponding to these theorems will be derived as dual pairs of mathematical programming problems in functional spaces [3].

## 1. Notion and definitions

Let us consider an elastic-rigid body of volume $V$ with respect to the Cartesian coordinates $\mathbf{x}$ $(x_1, x_2, x_3)$. The regular surface $S$ of the body is divided into two parts: $S_u$, where the displacements are zero, and $S_p$, where the limit displacements are either prescribed completely by their value and direction or by some scalar multiplier determining only their value. Small deformations are dealt with, volume forces being disregarded. Vector-matrix notation is used to denote fields and operators.

| | |
|---|---|
| $\mathbf{u} \equiv (u_1, u_2, u_3)^T$ | vector field of displacements, |
| $\mathbf{u}^+$ and $\mathbf{u}^-$ | vector fields of displacements, determining the upper and lower bounds of limit displacements on $S_p$, |
| $\mathbf{w} \equiv (w_1, w_2, w_3)^T$ | vector field of residual displacements, |
| $\boldsymbol{\epsilon} \equiv (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})^T$ | vector field of deformations, |
| $\boldsymbol{\epsilon}^+$ and $\boldsymbol{\epsilon}^-$ | vector fields of "elastic" extremum deformations, |
| $\boldsymbol{\xi} \equiv (\xi_{11}, \xi_{22}, \xi_{33}, \xi_{12}, \xi_{23}, \xi_{31})^T$ | vector field of residual deformations, |
| $\boldsymbol{\sigma} \equiv (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})^T$ | vector field of stress velocities, |
| $\boldsymbol{\sigma}^+$ and $\boldsymbol{\sigma}^-$ | vector fields of stress velocities, corresponding to $\mathbf{u}^+$ and $\mathbf{u}^-$, |
| $\boldsymbol{\rho} \equiv (\varrho_{11}, \varrho_{22}, \varrho_{33}, \varrho_{12}, \varrho_{23}, \varrho_{31})^T$ | vector field of velocities of residual stresses, |
| $\mathbf{P} \equiv (P_1, P_2, P_3)^T$ | vector fields of external load velocities, |
| $\mathbf{P}^+$ and $\mathbf{P}^-$ | vector fields of external load velocities corresponding to the upper and lower bounds of limit displacements $\mathbf{u}^+$ and $\mathbf{u}^-$, |

| | |
|---|---|
| $\mathbf{R} \equiv (R_1, R_2, R_3)^T$ | vector field of velocities of residual reactions on $S_u$, |
| $\lambda^+$ and $\lambda^-$ | scalar fields of proportionality multipliers, |
| $t$ | time, |
| $\tau$ | duration of cycle, |
| $\nabla$ | differential operator of static equilibrium, |
| $N$ | algebraic operator of equilibrium equations on surface $S$, |
| $\mathbf{e} \equiv (e_1, e_2, e_3)^T$ | unit vector field, determining the displacement direction on $S_p$, |
| $K$ | scalar field of the locking constant, determined in $V$, |
| $\Lambda$ | scalar field of weight multipliers of the optimality criterion of the body, prescribed in $V$, |
| $\mathbf{T}^+$ and $\mathbf{T}^-$ | vector fields of weight multipliers of the optimality criterion of limit displacements, prescribed in $S_p$. |

Hereinafter, we shall introduce the intensity of certain values — i.e., scalar fields rather than vector fields. In the present paper, they are given in standard form. If the variables in some expressions are the functions both of the coordinates and time, they will be given with both indices in parenthesis — e.g., $\mathbf{u}(\mathbf{x}, t)$.

Now let us deal with some basic definitions.

The field of deformations is considered to be kinematically admissible if it satisfies the condition of kinematic compatibility — i.e., for actual deformations it is subject to:

$$-\nabla^T \mathbf{u} + \boldsymbol{\epsilon} = 0, \tag{1.1}$$

and for residual deformations:

$$-\nabla^T \mathbf{w} + \boldsymbol{\xi} = 0. \tag{1.2}$$

The field of residual deformations is admissible if together with extremum "elastic" deformations it does not exceed the deformations corresponding to the locking condition — i.e.,

$$\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) \leqslant K, \quad \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) \leqslant K, \tag{1.3}$$

where the locking function $\varphi$ is a concave homogeneous function.

The field of residual deformations is kinematically admissible if it satisfies the conditions (1.2) and (1.3).

The relationship between the displacements on $S_p$ and the deformations in the body volume may, if its behaviour is elastic, be expressed as follows:

$$\boldsymbol{\epsilon}(\mathbf{x}, t) = \int_{S_p} \alpha(\mathbf{x}) \mathbf{u}(\mathbf{x}, t)\, ds, \tag{1.4}$$

where $\alpha(\mathbf{x})$ is an influence operator of the elastic solution.

Extremum values of elastic deformations will have the form:

$$\boldsymbol{\epsilon}^+(\mathbf{x}) = \max_{0 \leqslant t \leqslant \tau} \boldsymbol{\epsilon}(\mathbf{x}, t), \quad \boldsymbol{\epsilon}^-(\mathbf{x}) = \min_{0 \leqslant t \leqslant \tau} \boldsymbol{\epsilon}(\mathbf{x}, t).$$

Corresponding to the definition of the upper and lower bounds of displacements, we obtain:

$$\boldsymbol{\epsilon}^+ = \int_{S_p} (\alpha^+ \mathbf{u}^+ - \alpha^- \mathbf{u}^-)\, ds, \quad \boldsymbol{\epsilon}^- = \int_{S_p} (\alpha^- \mathbf{u}^+ - \alpha^+ \mathbf{u}^-)\, ds, \tag{1.5}$$

where $\alpha^+$ and $\alpha^-$ are known as extremum operators of the elastic solution. In (1.5), displacements corresponding to the lower negative bound are assumed to be positive — i.e., $\mathbf{u}^- \geqslant 0$.

Let us take the expressions of some energy values.

The displacement power in a cycle on the surface $S_p$ has the form:

$$W = \int_{\tau} \int_{S_p} u(\mathbf{x}, t)\mathbf{e}^T(\mathbf{x}) \cdot \mathbf{P}(\mathbf{x}, t)\, dt\, ds, \tag{1.6}$$

where both the displacement intensity and the external load velocity are the function of the coordinates and time. It is taken that $\mathbf{u} = u\mathbf{e}$. Thus, the displacement power may be determined by consideration of the deformation process related to time. Nevertheless, the introduction of the extremum values of the displacements and external load velocities which are independent of time makes it possible to avoid such a consideration in the limit analysis, when the extremum of expression (1.6) is to be determined. Thus, we have:

$$W = \int_{S_p} u^+\mathbf{e}^T \cdot \mathbf{P}^+\, ds + \int_{S_p} u^-\mathbf{e}^T \cdot \mathbf{P}^-\, ds, \tag{1.7}$$

where the lower bound of the vector field $\mathbf{P}^-$ is assumed to be positive, as well as $\mathbf{u}^-$.

Taking into account the fact that the residual deformations at the end of a cycle become the same as the initial ones at the beginning of it, the plastic dissipation per cycle is as follows:

$$\mathscr{D} = \int_V \boldsymbol{\sigma}^{+T} \cdot \boldsymbol{\epsilon}^+\, dv + \int_V \boldsymbol{\sigma}^{-T} \cdot \boldsymbol{\epsilon}^-\, dv = \int_V \lambda^+ K dv + \int_V \lambda^- K dv = \int_V (\lambda^+ + \lambda^-) K dv. \tag{1.8}$$

Now we shall deal with some general features of the presentation of the variable repeated deformation problems. As in the case of non-repeated deformation, we have two types of optimisation problems. The displacement design problem is as follows: for a given body ($K$ being prescribed) it is necessary to determine on the surface $S_p$ the value of the limit displacements (their direction being prescribed), at which cyclic locking occurs and the displacements satisfy a certain optimality criterion. The body design problem is formulated as follows: for a prescribed repeated variable deformation on $S_p$ it is necessary to determine the body parameters ($K$ is to be found) satisfying a certain optimality criterion, on condition that locking is cyclic. In both problems, $V$ is assumed to be constant.

Thus, first the optimality criteria of these two different types of problems are to be determined. The optimality criterion of the displacement design problem is:

$$\max \Big\{ \int_{S_p} u^+\mathbf{e}^T \cdot \mathbf{T}^+\, ds + \int_{S_p} u^-\mathbf{e}^T \cdot \mathbf{T}^-\, ds \Big\}, \tag{1.9}$$

and that of the body design problem

$$\min \int_V \Lambda K dv. \tag{1.10}$$

A more detailed discussion of the optimality criteria is given in [4].

## 2. Mathematical models of problems

First let us formulate the general energy theorems of an elastic-rigid body in order to derive the mathematical models of the above problems. The theorems refferred to will be as before [2].

### 2.1. Displacement design problem

The problem if based on the kinematic theorem of limit displacements:

Of all kinematically admissible fields of residual deformations at cyclic locking the actual one corresponds to the maximum value of the cycle of the external displacements.

The mathematical model corresponding to this theorem is:

$$\max\Big\{\int_{S_p} u^+\mathbf{e}^T\cdot\mathbf{P}^+\,ds+\int_{S_p} u^-\mathbf{e}^T\cdot\mathbf{P}^-\,ds\Big\}, \tag{2.1}$$

subject to:

$$\left.\begin{aligned}\varphi(\boldsymbol{\epsilon}^+ +\boldsymbol{\xi}) &\leqslant K\\ \varphi(-\boldsymbol{\epsilon}^- -\boldsymbol{\xi}) &\leqslant K\\ -\nabla^T\mathbf{w}+\boldsymbol{\xi} &= 0\end{aligned}\right\}\quad \text{in } V, \tag{2.2}$$

$$\mathbf{w}=0 \quad \text{on } S_u,$$
$$u^+\geqslant 0,\ u^-\geqslant 0 \quad \text{on } S_p.$$

For the derivation of the optimisation problem, use is made of the above theorem and its mathematical model. The maximisation of (2.1) is possible at any fixed values of $P^+$ and $P^-$, including $P^+\geqslant T^+$ and $P^-\geqslant T^-$, since they are not introduced in the constraints (2.2). Hence, the kinematic formulation of the mathematical model of the problem is:

$$\max\Big\{\int_{S_p} u^+\mathbf{e}^T\cdot\mathbf{T}^+\,ds+\int_{S_p} u^-\mathbf{e}^T\cdot\mathbf{T}^-\,ds\Big\}, \tag{2.3}$$

subject to:

$$\left.\begin{aligned}\varphi(\boldsymbol{\epsilon}^+ +\boldsymbol{\xi}) &\leqslant K\\ \varphi(-\boldsymbol{\epsilon}^- -\boldsymbol{\xi}) &\leqslant K\\ -\nabla^T\mathbf{w}+\boldsymbol{\xi} &= 0\end{aligned}\right\}\quad \text{in } V, \tag{2.4}$$

$$\mathbf{w}=0 \quad \text{on } S_u,$$
$$u^+\geqslant 0,\ u^-\geqslant 0 \quad \text{on } S_p.$$

The static formulation of the problem may be obtained formally on the ground of the theory of functional analogues of convex programming problems [3].

Let us derive the Lagrange functional for problems (2.3)–(2.4)

$$\begin{aligned}(2.5)\quad &\mathscr{F}(u^+,u^-,\boldsymbol{\epsilon}^+,\boldsymbol{\epsilon}^-,\boldsymbol{\xi},\mathbf{w},\lambda^+,\lambda^-,\boldsymbol{\rho},\mathbf{R},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)\\ &=\int_{S_p} u^+\mathbf{e}^T\cdot\mathbf{T}^+\,ds+\int_{S_p} u^-\mathbf{e}^T\cdot\mathbf{T}^-\,ds-\int_V \lambda^+\varphi(\boldsymbol{\epsilon}^+ +\boldsymbol{\xi})\,dv+\int_V \lambda^+K\,dv\\ &\quad-\int_V \lambda^-\varphi(-\boldsymbol{\epsilon}^- -\boldsymbol{\xi})\,dv+\int_V \lambda^-K\,dv-\int_V \boldsymbol{\rho}^T\cdot\nabla^T\mathbf{w}\,dv+\int_V \boldsymbol{\rho}^T\cdot\boldsymbol{\xi}\,dv\\ &\quad+\int_{S_u}\mathbf{R}^T\cdot\mathbf{w}\,ds-\int_{S_p}\boldsymbol{\nu}_1^T\cdot u^+\mathbf{e}\,ds+\int_{S_p}\boldsymbol{\nu}_2^T\cdot u^-\mathbf{e}\,ds.\end{aligned}$$

Setting the functional variations on $\mathbf{u}^+$, $\mathbf{u}^-$, $\boldsymbol{\xi}$, $\mathbf{w}$, equal to zero yields, the constraints of the problem dual to problems (2.3)–(2.4) are:

$$\left.\begin{aligned} -\lambda^+\frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}-\lambda^-\frac{\partial\varphi(-\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}+\boldsymbol{\rho}=0\\ -\nabla\boldsymbol{\rho}=0\\ \lambda^+\geqslant 0,\quad \lambda^-\geqslant 0\end{aligned}\right\}\quad\text{in } V; \tag{2.6}$$

$$\mathbf{R}-N\boldsymbol{\rho}=0\quad \text{on } S_u,$$

$$\left.\begin{aligned} -N\boldsymbol{\rho}=0\\ \mathbf{T}^+-\int\limits_V\lambda^+\frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\mathbf{u}^+}dv-\int\limits_V\lambda^-\frac{\partial\varphi(-\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial u^+}dv+\boldsymbol{\nu}_1=0\\ \mathbf{T}^--\int\limits_V\lambda^+\frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\mathbf{u}^-}dv-\int\limits_V\lambda^-\frac{\partial\varphi(-\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial\mathbf{u}^-}dv+\boldsymbol{\nu}_2=0\\ \boldsymbol{\nu}_1\geqslant 0,\quad \boldsymbol{\nu}_2\geqslant 0\end{aligned}\right\}\quad\text{on } S_p.$$

The variations with respect to $\mathbf{w}$ are based on:

$$\int\limits_V\boldsymbol{\rho}^T\cdot\nabla^T\mathbf{w}\,dv=\int\limits_V\mathbf{w}^T\cdot\nabla\boldsymbol{\rho}\,dv+\int\limits_{S_u}\mathbf{w}^T\cdot N\boldsymbol{\rho}\,ds+\int\limits_{S_p}\mathbf{w}^T\cdot N\boldsymbol{\rho}\,ds.$$

Substituting the condition (2.6) in (2.5) and bearing in mind that the residual deformations at the end of a cycle obtain the initial value at the beginning of it — i.e., $\int\limits_V\boldsymbol{\xi}^T\cdot\boldsymbol{\rho}\,dv=$ $=0$ — and taking into account the following relations:

$$\begin{aligned} \frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\mathbf{u}^+}&=\alpha^{+T}\frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\boldsymbol{\xi}},\\ \frac{\partial\varphi(\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial\mathbf{u}^-}&=\alpha^{-T}\frac{\partial\varphi(\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial\boldsymbol{\xi}},\\ \frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\mathbf{u}^-}&=-\alpha^{-T}\frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\boldsymbol{\xi}},\\ \frac{\partial\varphi(-\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial\mathbf{u}^-}&=-\alpha^{+T}\frac{\partial\varphi(-\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\end{aligned}\tag{2.7}$$

and

$$\begin{aligned} \left[\lambda^+\frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T\cdot\int\limits_{S_p}[\alpha^+\mathbf{u}^+-\alpha^-\mathbf{u}^-]ds&=\left[\lambda^+\frac{\partial\varphi(\boldsymbol{\epsilon}^++\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T\cdot\boldsymbol{\epsilon}^+,\\ \left[\lambda^-\frac{\partial\varphi(-\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T\cdot\int\limits_{S_p}[\alpha^-\mathbf{u}^+-\alpha^-\mathbf{u}^-]ds&=\left[\lambda^-\frac{\partial\varphi(-\boldsymbol{\epsilon}^--\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T\cdot\boldsymbol{\epsilon}^-,\end{aligned}\tag{2.8}$$

we obtain the static formulation of the mathematical model of the displacement design problem:

$$(2.9)\quad \min\left\{\int_V \left[\lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right]^T \cdot \boldsymbol{\epsilon}^+ dv + \int_V \left[\lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right]^T \cdot \boldsymbol{\epsilon}^- dv - \int_V \lambda^+ [\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) - K] dv - \int_V \lambda^- [\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) - K] dv\right\},$$

subject to:

$$\left.\begin{aligned} -\lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} - \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} + \boldsymbol{\rho} &= 0 \\ -\nabla \boldsymbol{\rho} &= 0 \\ \lambda^+ \geqslant 0, \quad \lambda^- &\geqslant 0 \end{aligned}\right\} \quad \text{in } V;$$

$$\mathbf{R} - N\boldsymbol{\rho} = 0 \quad \text{on } S_u;$$

$$(2.10)\quad \left.\begin{aligned} -N\boldsymbol{\rho} &= 0 \\ \int_V \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \mathbf{u}^+} dv + \int_V \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \mathbf{u}^+} dv &\geqslant \mathbf{T}^+ \\ \int_V \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \mathbf{u}^-} dv + \int_V \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \mathbf{u}^-} dv &\geqslant \mathbf{T}^- \end{aligned}\right\} \quad \text{on } S_p.$$

Now let us explain the physical sense of the problem.

The first condition in (2.10) expresses the relation between the velocities of residual stresses and the deformations. If we denote:

$$(2.11)\quad \boldsymbol{\sigma}^+ = \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}}, \quad \boldsymbol{\sigma}^- = \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \boldsymbol{\xi}},$$

then

$$(2.12)\quad \boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^- = \boldsymbol{\rho},$$

which leads to the concept of the velocity of residual stresses in a cycle.

The condition $-\nabla\boldsymbol{\rho} = 0$ (differential equilibrium equations) together with the conditions $\mathbf{R} - N\boldsymbol{\rho} = 0$ on $S_u$ and $-N\boldsymbol{\rho} = 0$ on $S_p$ determine the statically admissible field of residual stresses.

Corresponding to the relations (2.7), the conditions inequalities on $S_p$ may be presented as follows:

$$(2.13)\quad \begin{aligned} &\int_V \alpha^{+T} \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} dv + \int_V \alpha^{-T} \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} dv \geqslant \mathbf{T}^+. \\ &-\int_V \alpha^{-T} \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} dv - \int_V \alpha^{+T} \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} dv \geqslant \mathbf{T}^-, \end{aligned}$$

or, by way of (2.11)

$$(2.14)\quad \begin{aligned} \mathbf{T}^{+} &\leqslant \int_V (\alpha^{+T}\boldsymbol{\sigma}^{+}+\alpha^{-T}\boldsymbol{\sigma}^{-})dv \equiv \mathbf{P}^{+},\\ \mathbf{T}^{-} &\leqslant \int_V (-\alpha^{-T}\boldsymbol{\sigma}^{+}-\alpha^{+T}\boldsymbol{\sigma}^{-})dv \equiv \mathbf{P}^{-}. \end{aligned}$$

This implies that $\mathbf{P}^{+} \geqslant \mathbf{T}^{+}$ and $\mathbf{P}^{-} \geqslant \mathbf{T}^{-}$ — i.e., the vector field of the weigt multipliers of the optimality criterion defines the constraints on the external load velocity. Hence, its origin is "force".

Thus, all conditions (2.10) taken together determine the statically admissible field of the velocities of residual stresses.

The physical sense of the cost function (2.9) is obvious — it is the plastic dissipation, since

$$\int_V \left[\lambda^{+}\frac{\partial\varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T\cdot\boldsymbol{\epsilon}^{+}dv+\int_V \left[\lambda^{-}\frac{\partial\varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T\cdot\boldsymbol{\epsilon}^{-}dv = \int_V \boldsymbol{\sigma}^{+T}\cdot\boldsymbol{\epsilon}^{+}dv+\int_V \boldsymbol{\sigma}^{-}\cdot\boldsymbol{\epsilon}^{-}dv,$$

and the third and the fourth terms for optimal solution are zero.

The mathematical model (2.9)–(2.10) corresponds, therefore, to the following extremum theorem:

Of all statically admissible fields of velocities of residual stresses the actual one in the state of cyclic locking corresponds to the minimum value of plastic dissipation in a cycle.

Hereinafter this extremum principle will be referred to as a static theorem of limit displacement. The mathematical model (2.9)–(2.10) corresponding to this theorem is dual to the problem (2.3)–(2.4), which in turn corresponds to the kinematic theorem of limit displacements. The kinematic and static theorems therefore are dual.

The first theorem of duality of the mathematical programming theory for the optimal solutions of both problems denoted by asterisks implies that the cost functions of both problems are equal — i.e.,

$$(2.15)\quad \int_{S_p} \mathbf{u}^{+*}\mathbf{e}^T\cdot\mathbf{T}^{+}ds+\int_{S_p} \mathbf{u}^{-*}\mathbf{e}^T\cdot\mathbf{T}^{-}ds = \int_V \left[\lambda^{+*}\frac{\partial\varphi(\boldsymbol{\epsilon}^{+*}+\boldsymbol{\xi}^*)}{\partial\boldsymbol{\xi}}\right]^T\cdot\boldsymbol{\epsilon}^{+*}dv+\int_V \left[\lambda^{-*}\frac{\partial\varphi(-\boldsymbol{\epsilon}^{-*}-\boldsymbol{\xi}^*)}{\partial\boldsymbol{\xi}}\right]^T\cdot\boldsymbol{\epsilon}^{-*}dv.$$

From a physical point of view, this means that the external displacements and the plastic dissipation in a cycle are equal. This conclusion will be referred to as the first theorem of duality of elastic-rigid bodies.

According to the second theorem of duality of mathematical programming, we have:

$$(2.16)\quad \lambda^{+*}[\varphi(\boldsymbol{\epsilon}^{+*}+\boldsymbol{\xi}^*)-K]=0,\quad \lambda^{-*}[\varphi(-\boldsymbol{\epsilon}^{-*}-\boldsymbol{\xi}^*)-K]=0.$$

These relations will be referred to as the second theorem of duality of elastic-rigid bodies. The consequence of this theorem taken together with expression (2.11) gives the associated locking rule for cyclic deformation:

$$
\begin{aligned}
&\boldsymbol{\sigma}^{+} = \lambda^{+}\frac{\partial\varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}; \quad && \lambda^{+} > 0 \quad \text{if} \quad \varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi}) = K, \\
& && \lambda^{+} = 0 \quad \text{if} \quad \varphi(\boldsymbol{\epsilon}^{+}-\boldsymbol{\xi}) < K. \\
&\boldsymbol{\sigma}^{-} = \lambda^{-}\frac{\partial\varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi})}{\partial\boldsymbol{\xi}} \quad && \lambda^{-} > 0 \quad \text{if} \quad \varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi}) = K, \\
& && \lambda^{-} = 0 \quad \text{if} \quad \varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi}) < K.
\end{aligned}
\tag{2.17}
$$

The stress velocity of cyclic locking is seen to be determined only to within non-negative multipliers $\lambda^{+}$ and $\lambda^{-}$.

Finally, by the mathematical programming theory, according to which the solution or each dual problem is equivalent to the solution of all conditions of both problems taken together, and completed by (2.16), we may express all constitutive conditions of a perfectly elastic-rigid body at repeated-variable deformation as follows

$$
\left.
\begin{aligned}
&\varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi}) \leqslant K, \\
&\varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi}) \leqslant K, \\
&-\nabla^{T}\mathbf{w}+\boldsymbol{\xi} = 0, \\
&\nabla\boldsymbol{\rho} = 0, \\
&\boldsymbol{\rho} = \lambda^{+}\frac{\partial\varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi})}{\partial\boldsymbol{\xi}} + \lambda^{-}\frac{\partial\varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}, \\
&\boldsymbol{\sigma}^{+} = \lambda^{+}\frac{\partial\varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}; \quad \lambda^{+} > 0 \ \text{if} \ \varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi}) = K, \quad \lambda = 0 \ \text{if} \ \varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi}) < K; \\
&\boldsymbol{\sigma}^{-} = \lambda^{-}\frac{\partial\varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}; \quad \lambda^{+} > 0 \ \text{if} \ \varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi}) = K, \quad \lambda = 0 \ \text{if} \ \varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi}) < K; \\
&\boldsymbol{\epsilon}^{+} = \int_{S_p}(\alpha^{+}\mathbf{u}^{+}-\alpha^{-}\mathbf{u}^{-})\,ds, \\
&\boldsymbol{\epsilon}^{-} = \int_{S_p}(\alpha^{-}\mathbf{u}^{+}-\alpha^{+}\mathbf{u}^{-})\,ds, \\
&\mathbf{u}^{+} \geqslant 0, \quad \mathbf{u}^{-} \geqslant 0;
\end{aligned}
\right\} \text{in } V,
\tag{2.18}
$$

$$
\mathbf{w} = 0, \quad -N\boldsymbol{\rho}+\mathbf{R} = 0 \quad \text{on} \quad S_u,
$$

$$
\left.
\begin{aligned}
&\int_V \lambda^{+}\frac{\partial\varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi})}{\partial\mathbf{u}^{+}}dv + \int_V \lambda^{-}\frac{\partial\varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi})}{\partial\mathbf{u}^{+}}dv \geqslant \mathbf{T}^{+} \\
&\int_V \lambda^{+}\frac{\partial\varphi(\boldsymbol{\epsilon}^{+}+\boldsymbol{\xi})}{\partial\mathbf{u}^{-}}dv + \int_V \lambda^{-}\frac{\partial\varphi(-\boldsymbol{\epsilon}^{-}-\boldsymbol{\xi})}{\partial\mathbf{u}^{-}}dv \geqslant \mathbf{T}^{-} \\
&-N\boldsymbol{\rho} = 0
\end{aligned}
\right\} \quad \text{on } S_p.
$$

It is obvious, that the solution of every separate dual problem is simpler than the solution of the generalised Lagrange problem, expressed by conditions (2.18). Every dual problem is convex due to the convexity of the locking function, and to solve it mathematical programming technique may be used.

Now let us show that the mathematical model (2.3)–(2.4) and its dual are also the mathematical models for non-repeated deformation problems. To this end, some formal transformations of the variables will be carried out. At non-repeated deformation, $\mathbf{u}^+ = \mathbf{u}^- = \mathbf{u}$, $\mathbf{T}^+ + \mathbf{T}^- = \mathbf{T}$, $\alpha^+ + \alpha^- = \alpha$, $\boldsymbol{\epsilon}^+ = -\boldsymbol{\epsilon}^- = \boldsymbol{\epsilon}^\circ$, $\boldsymbol{\sigma}^+ = -\boldsymbol{\sigma}^- = \boldsymbol{\sigma}^\circ$, where $\boldsymbol{\epsilon}^\circ$ and $\boldsymbol{\sigma}^\circ$ are appropriate elastic deformations and stress velocities. Expressing the actual deformations $\boldsymbol{\epsilon}$ the displacements $\mathbf{u}$ and the stress velocities $\boldsymbol{\sigma}$ as the sum of elastic and residual parts: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^\circ + \boldsymbol{\xi}$, $\mathbf{u} = \mathbf{u}^\circ + \mathbf{w}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\circ + \boldsymbol{\rho}$, and taking into consideration that the elastic deformations and displacements are kinematically compatible while the velocities of elastic stresses are statically compatible, we obtain the following dual pair of problems:

(2.19a)
$$\max \int_{S_p} \mathbf{u}^T \cdot \mathbf{T}\, ds,$$

(2.20a)
$$\left.\begin{aligned} \varphi(\boldsymbol{\epsilon}) &\leqslant K \\ -\nabla^T \mathbf{u} + \boldsymbol{\epsilon} &= 0 \end{aligned}\right\} \quad \text{in } V, \qquad \mathbf{u} = 0 \quad \text{on } S_u,$$

subject to:

(2.19b)
$$\min \left\{ \left[ \lambda \frac{\partial \varphi(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}} \right]^T \cdot \boldsymbol{\epsilon}\, dv - \int_V \lambda [\varphi(\boldsymbol{\epsilon}) - K) dv \right\},$$

subject to:

(2.20b)
$$\left.\begin{aligned} \boldsymbol{\sigma} - \lambda \frac{\partial \varphi(\boldsymbol{\epsilon})}{\partial (\boldsymbol{\epsilon})} &= 0 \\ \lambda &\geqslant 0 \\ -\nabla \boldsymbol{\sigma} &= 0 \end{aligned}\right\} \quad \text{in } V,$$
$$\mathbf{R} - N\boldsymbol{\sigma} = 0 \quad \text{on } S_u, \qquad N\boldsymbol{\sigma} \geqslant \mathbf{T} \quad \text{on } S_p.$$

Thus, the general theorems of plastic-rigid bodies, expressed by way of deformations and stress velocities, may be considered as particular cases of the general theorems of elastic-rigid bodies, expressed by way of residual deformations and residual stress velocities.

### 2.2. Body design problem

The purpose of the present problem is to determine the composition of the body — i.e., to find the distribution of the locking constant, satisfying the optimality criterion of the body (1.10). Thus, the upper and lower bounds of displacements on the body surface $S_p$ are assumed to be prescribed. Let us determine the scalar field $K$ according to the locking conditions. The solution of the present problem may depend on two different primal conditions:

1) the locking constant of the material does not depend on the elastic constants of the material;

2) the locking constant depends on the elastic parameters of the body.

In the first case, the extremum elastic deformations $\boldsymbol{\epsilon}^+$ and $\boldsymbol{\epsilon}^-$ do not depend on $K$ but are assumed to be constant, when varying the locking constant with respect to the variable $K$. Such problems are usually referred to as minimum cost problems [4].

The second case includes the problems where the influence operators of the elastic solution $\alpha^+$ and $\alpha^-$ depend on $K$. In this case, one of the possible ways of solving such problems is by iterating the solutions of some problems of the first type. Here, the first case will be dealt with. To derive the mathematical model of the problem we shall apply the following extremum principle, called the kinematic theorem of cyclic locking.

Of all kinematically admissible fields of residual deformations at cyclic locking the actual one corresponds to the minimum value of plastic dissipation in a cycle. This theorem has the following mathematical model:

$$\min \int_V (\lambda^+ + \lambda^-) K dv, \tag{2.21}$$

subject to:

$$\left.\begin{aligned} -\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) + K \geqslant 0 \\ -\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) + K \geqslant 0 \\ -\nabla^T \mathbf{w} + \boldsymbol{\xi} = 0 \\ K \geqslant 0 \end{aligned}\right\} \quad \text{in } V, \tag{2.22}$$

$$\mathbf{w} = 0 \quad \text{on } S_u.$$

The application of this problem and its mathematical model for the derivation of the optimisation problem is analogous to the displacement design problem.

Since the values $\lambda^+$ and $\lambda^-$ are not included in the conditions of the problem, then the minimisation of (2.2) is possible at any fixed values of them, including $\lambda^+ + \lambda^- \leqslant \Lambda$.

Thus, we obtain the following mathematical model of the body design problem in the kinematic formulation:

$$\min \int_V \Lambda K dv, \tag{2.23}$$

subject to:

$$\left.\begin{aligned} -\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) + K \geqslant 0 \\ -\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) + K \geqslant 0 \\ -\nabla^T \mathbf{w} + \boldsymbol{\xi} = 0 \\ K \geqslant 0 \end{aligned}\right\} \quad \text{in } V, \tag{2.24}$$

$$\mathbf{w} = 0 \quad \text{on } S_u.$$

Now let us turn to the dual problem.

The Lagrange functional for problems (2.23)–(2.24) has the form:

$$(2.25)\quad \mathscr{F}(K, \boldsymbol{\xi}, \mathbf{w}, \lambda^+, \lambda^-, \boldsymbol{\rho}, \mu, \mathbf{R}) = \int_V \Lambda K dv + \int_V \lambda^+ \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) dv - \int_V \lambda^+ K dv + \int_V \lambda^- \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) dv - \int_V -\lambda^- K dv + \int_V \boldsymbol{\rho}^T \cdot \nabla^T \mathbf{w} dv - \int_V \boldsymbol{\rho}^T \cdot \boldsymbol{\xi} dv - \int_V \mu K dv - \int_V \mathbf{R}^T \cdot \mathbf{w} ds.$$

Its variations with respect to $K$, $\boldsymbol{\xi}$ and $\mathbf{w}$, when set equal to zero, yield the conditions of the problem, dual to problem (2.23)–(2.24):

$$(2.26)\quad \left.\begin{aligned} \lambda^+ \frac{\partial \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} + \lambda^- \frac{\partial \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} - \boldsymbol{\rho} &= 0 \\ \nabla \boldsymbol{\rho} &= 0 \\ \Lambda - (\lambda^+ + \lambda^-) - \mu &= 0 \\ \lambda^+ \geqslant 0,\ \lambda^- \geqslant 0,\ \mu &\geqslant 0 \end{aligned}\right\} \text{ in } V,$$

$$N\boldsymbol{\rho} - \mathbf{R} = 0 \quad \text{on } S_u,$$
$$N\boldsymbol{\rho} = 0 \quad \text{on } S_p.$$

Substituting (2.26) in (2.25), and a taking into account $\int_V \boldsymbol{\rho}^T \boldsymbol{\xi} dv = 0$, we obtain the static formulation of the mathematical model of the body design problem for an elastic-rigid body:

$$(2.27)\quad \max \left\{ \int_V \lambda^+ \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) dv + \int_V \lambda^- \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) dv, \right.$$

subject to

$$(2.28)\quad \left.\begin{aligned} \lambda^+ \frac{\partial \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} + \lambda^- \frac{\partial \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} - \boldsymbol{\rho} &= 0 \\ \nabla \boldsymbol{\rho} &= 0 \\ \Lambda \geqslant \lambda^+ + \lambda^-, \quad \lambda^+ \geqslant 0, \quad \lambda^- &\geqslant 0 \end{aligned}\right\} \text{ in } V,$$

$$N\boldsymbol{\rho} - R = 0 \quad \text{on } S_u,$$
$$N\boldsymbol{\rho} = 0 \quad \text{on } S_p.$$

Let us explain the physical sense of the problem (2.27)–(2.28). The inequality $\Lambda \geqslant \lambda^+ + \lambda^-$ expresses the concept of admissibility for the field of velocity of residual stresses. It is a form of normalisation and allows for derivation of the field of stress velocities to within a constant multiplier. Thus, all conditions (2.28) taken together express the statically admissible field of residual stresses.

The cost function expresses power per cycle. This statement is proved by the following relations, which hold for a constant multiplier:

$$(2.29)\quad \int_V \lambda^+ \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) dv + \int_V \lambda^- \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) dv = \int_V \left[ \lambda^+ \frac{\partial \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \right]^T \cdot \boldsymbol{\epsilon}^+ dv + \int_V \left[ \lambda^- \frac{\partial \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \right]^T \cdot \boldsymbol{\epsilon}^- dv = \int_V \int_{S_p} \left[ \lambda^+ \frac{\partial \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \right]^T \cdot [\alpha^+ \mathbf{u}^+ - \alpha^- \mathbf{u}^-] dv\, ds$$

$$+\int_V \int_{S_p} \left[\lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T \cdot [\alpha^- \mathbf{u}^+ - \alpha^+ \mathbf{u}^-] dv\, ds = \int_V \int_{S_p} \left[\alpha^{+T} \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right.$$

$$\left. + \alpha^{-T} \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T \cdot \mathbf{u}^+ dv\, ds - \int_V \int_{S_p} \left[\alpha^{-T} \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right.$$

$$\left. + \alpha^{+T} \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right]^T \cdot \mathbf{u}^- dv\, ds = \int_{S_p} \int_V [\alpha^{+T} \boldsymbol{\sigma}^+ + \alpha^{-T} \boldsymbol{\sigma}^-]^T \cdot \mathbf{u}^+ ds\, dv$$

$$- \int_{S_p} \int_V [\alpha^{-T} \boldsymbol{\sigma}^+ + \alpha^{+T} \boldsymbol{\sigma}^-]^T \cdot \mathbf{u}^- ds\, dv = \int_{S_p} \mathbf{P}^{+T} \cdot \mathbf{u}^+ ds + \int_{S_p} \mathbf{P}^{-T} \cdot \mathbf{u}^- ds.$$

Mathematical model (2.27)–(2.28), therefore, corresponds to the following extremum theorem:

Of all statically admissible fields of residual stresses at cyclic locking the actual one corresponds to the maximum value of the cycle of the external power.

Hereinafter, this theorem is referred to as the static theorem of cyclic locking. It is dual to the kinematic theorem.

The first theorem of duality will be:

$$\int_V \Lambda K^* dv = \int_V \lambda^{+*} \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}^*) dv + \int_V \lambda^{-*} \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}^*) dv. \tag{2.30}$$

This implies that the plastic dissipation and the external displacements in a cycle for statically and kinematically admissible fields of residual deformations and residual stress velocities are equal.

The second theorem of duality yields the associated cyclic locking rule, having the form of (2.17).

The generalised Lagrange problem consists of all the cyclic constraints and conditions:

$$\left.\begin{array}{c}
-\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) + K \geqslant 0 \\
-\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) + K \geqslant 0 \\
-\nabla^T \mathbf{w} + \boldsymbol{\xi} = 0 \\
\nabla \boldsymbol{\rho} = 0 \\
\boldsymbol{\rho} = \lambda^+ \dfrac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial\boldsymbol{\xi}} + \lambda^- \dfrac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial\boldsymbol{\xi}} \\
\boldsymbol{\sigma}^+ = \lambda^+ \dfrac{\partial\varphi(\boldsymbol{\epsilon}^+ - \boldsymbol{\xi})}{\partial\boldsymbol{\xi}}; \quad \begin{array}{lll} \lambda^+ > 0 & \text{if} & \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) = K \\ \lambda^+ = 0 & \text{if} & \varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi}) < K \end{array} \\
\boldsymbol{\sigma}^- = \lambda^- \dfrac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial\boldsymbol{\xi}}; \quad \begin{array}{lll} \lambda^- > 0 & \text{if} & \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) = K \\ \lambda^- = 0 & \text{if} & \varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi}) < K \end{array} \\
\Lambda \geqslant \lambda^+ + \lambda^-, \quad \lambda^+ \geqslant 0, \quad \lambda^- \geqslant 0, \\
\boldsymbol{\epsilon}^+ = \displaystyle\int_{S_p} (\alpha^+ \mathbf{u}^+ - \alpha^- \mathbf{u}^-) ds \\
\boldsymbol{\epsilon}^- = \displaystyle\int_{S_p} (\alpha^- \mathbf{u}^+ - \alpha^+ \mathbf{u}^-) ds
\end{array}\right\} \text{ in } V. \tag{2.31}$$

$$\left.\begin{aligned} \mathbf{u}^+ \geqslant 0, \quad \mathbf{u}^- \geqslant 0 \\ N\boldsymbol{\rho} = 0 \end{aligned}\right\} \quad \text{on } S_p,$$

$$\mathbf{w} = 0, \quad N\boldsymbol{\rho} - \mathbf{R} = 0 \quad \text{on } S_u.$$

## 3. Comparison of mathematical models of displacement design and body design problems

Comparing the conditions of the generalised Lagrange problem (2.18) and (2.31), we see that firstly they differ in the variables to be found. Besides the fields of residual displacements, deformations and stress velocities (these unknowns being present in both problems) the limit displacements are sought in the displacement design problem and the distribution of $K$ in $V$ in the body design problem.

Secondly, the locking conditions are the constraints on the deformations in the kinematic formulation of both problems. The static formulation provides the constraints on the forces. These conditions for the first problem are expressed by inequalities of the following type:

$$(3.1) \quad \begin{aligned} &\int_V \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \mathbf{u}^+} dv + \int_V \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \mathbf{u}^+} dv \geqslant \mathbf{T}^+, \\ &\int_V \lambda^+ \frac{\partial\varphi(\boldsymbol{\epsilon}^+ + \boldsymbol{\xi})}{\partial \mathbf{u}^+} dv + \lambda^- \frac{\partial\varphi(-\boldsymbol{\epsilon}^- - \boldsymbol{\xi})}{\partial \mathbf{u}^-} dv \geqslant \mathbf{T}^-, \end{aligned}$$

and for the second problem:

$$(3.2) \quad \lambda^+ + \lambda^- \leqslant \Lambda.$$

These conditions define the type of the problem.

It is obvious that the derivation of such conditions by way of physical consideration, regardless of dual pairs of extremum problems, is somewhat difficult, if not impossible. These conditions may be regarded as a fixed power for the first problem and a fixed plastic dissipation for the second problem. This becomes evident if we multiply both sides of the inequalities (3.1) by $\mathbf{u}^+$ and $\mathbf{u}^-$ respectively, and then integrate them on $S_p$, and if the inequality (3.2) is multiplied by $K$ and integrated in $V$. Nevertheless, the conditions (3.1) and (3.2) give tighter constraints than those in an integral form — e.g.

$$\int_{S_p} \mathbf{u}^T \cdot \mathbf{P} ds \geqslant 0, \quad \int_V \lambda K dv \geqslant 0,$$

which is usually applied for dissipative systems.

## 4. Conclusion

The paper concerns the optimisation problem of perfectly elastic-rigid bodies under repeated variable deformation. The mathematical models are derived. These models are common to many problems. Assuming different constraints, various types of optimisation and one-parametrical approaches may arise. Further, they include models for non-repeated deformation, which were dealt with in [5, 6, 7 and 8] for one parameter problems.

The solution of problems on the basic of the mathematical models of repeated-variable deformation is clearly more complicated, than that by applying the mathematical models of non-repeated deformation. In the former case firstly the elastic solution has to be found, and only then is the mathematical programming problem is considered. In the case of non-repeated deformation, the first stage is avoided and only the second one is left.

As already indicated, the present paper concerns the limit analysis — i.e., the limit values of repeated-variable deformation — and the distribution of the body parameters are obtained from the condition of cyclic locking. Thus, the body condition before this state is not dealt with. Such an approach greatly facilitates the analysis when solving various problems, both for physically linear and for nonlinear cases. The computation methods of mathematical programming and electronic computing techniques make practical application of such problems possible.

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