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METHODS OF ESTIMATION OF RELATIONS OF: EQUIVALENCE, TOLERANCE AND PREFERENCE IN A FINITE SET

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Warsaw 2011

SYSTEMS RESEARCH INSTITUTE
POLISH ACADEMY OF SCIENCES

Series: SYSTEMS RESEARCH
Volume 69

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Series Editor:
Prof. dr hab. inż. Jakub Gutenbaum

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The work has been supported by the grant No N N111434937 of the Polish Ministry of Science and Higher Education

Printed in Polands
Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
www.ibspan.waw.pl

ISSN 0208-8029
ISBN 9788389475374

Chapter 9

Properties of estimators of the preference relation based on binary and multivalent comparisons – a simulation survey

9.1. Introduction

The analytical properties of the estimators analyzed here do not characterize the precision of estimates for different distributions of comparison errors and the number of comparisons of each pair *N*. These features are examined in this chapter using stochastic simulation approach (Ripley, 2006). The study is based on the set **X,** including nine elements, $\mathbf{X} = \{x_1, ..., x_9\}$; the remaining parameters assume the following values: • three relation form, • five values of the number of comparisons for each pair (i.e. 1, 3, 5, 7, 9) and • three variants of distributions of comparisons errors.

The survey is based on single-dimensional errors, i.e. the sum of absolute errors $\sum_{<i,j> \in R_m} \left| \hat{T}_\upsilon^{(\ell)}(x_i, x_j) - T_\upsilon^{(\ell)}(x_i, x_j) \right|$ and $\sum_{<i,j> \in R_m} \left| \widehat{T}_\upsilon^{(\ell)}(x_i, x_j) - T_\upsilon^{(\ell)}(x_i, x_j) \right|$. It encompasses the frequency of errorless estimates, the average error of estimation and the distributions of errors of multivalent estimators.

The main purpose of the study is to determine the parameters guaranteeing a sufficient precision of estimates.

9.2. Definition of estimation errors

The errors of estimators of the preference relation are in fact multivariate, i.e. of the following form:

$$\hat{T}_{\upsilon}^{(p)}(x_i,x_j)-T_{\upsilon}^{(p)}(x_i,x_j) \quad (<i,j>\in R_m)\,. \tag{9.1}$$

Such form is inconvenient for analytical purposes and it is replaced by single dimensional errors:

$$\hat{\Delta}_l = \sum_{<i,j>\in R_m} \left|\hat{T}_l^{(p)}(x_i,x_j)-T_l^{(p)}(x_i,x_j)\right|, \tag{9.2}$$

$$\widehat{\Delta}_l = \sum_{<i,j>\in R_m} \left|\widehat{T}_l^{(p)}(x_i,x_j)-T_l^{(p)}(x_i,x_j)\right|. \tag{9.3}$$

The values $\hat{\Delta}_l$, $\widehat{\Delta}_l$ equal zero correspond to the errorless estimate; the fraction of such estimates, obtained in an appropriate number of simulations, is a measure of precision of the estimators. A complementary measure of precision is the average value of errors (9.2), (9.3) in simulation runs.

It is obvious that precision of estimates depends not only on distributions of comparison errors and the number of comparisons for each pair, but also on the value of *n* and the number of elements of individual subsets. Therefore, such attributes have been taken into account in the study.

9.3. Parameters of simulation survey

The study has been made for three relation forms, with the use of the following parameters.

- The set **X** comprising nine elements;

- three relation forms:

  - nine subsets relation (linear order): $\{x_1\},\ldots,\{x_9\}$ (*n*=9),
  - six subsets relation: $\{x_1\},\{x_2,x_3\},\{x_4\},\{x_5,x_6\},\{x_7\},\{x_8,x_9\}$ (*n*=6),
  - three subsets relation: $\{x_1,x_2\},\{x_3,x_4,x_5\},\{x_6,x_7,x_8,x_9\}$ (*n*=3);

- binary comparisons with probability functions:

$$P(g_{bk}^{(p)}(x_i,x_j)=T_b^{(p)}(x_i,x_j))=\alpha_{ij}\,,\quad P(g_{bk}^{(p)}(x_i,x_j)\neq T_b^{(p)}(x_i,x_j))=(1-\alpha_{ij})/2\,,$$

with three values of α_{ij}: 0.85, 0.90, 0.95, for all $<i,j>$ (typical levels in statistical tests);

• multivalent comparisons with two variants of the probability functions:
$P(g_{\mu k}^{(p)}(x_i,x_j)=T_{\mu}^{(p)}(x_i,x_j))=\alpha_{ij}$,

variant I

$$P(g_{\mu k}^{(p)}(x_i,x_j)-T_{\mu}^{(p)}(x_i,x_j)=-l)=(1-\alpha_{ij})/2L_{ij}^{(d)}$$
$$(L_{ij}^{(d)}=T_{\mu}^{(p)}(x_i,x_j)+(m-1);\ l=-1,...,-L_{ij}^{(d)}),$$

$$P(g_{\mu k}^{(p)}(x_i,x_j)-T_{\mu}^{(p)}(x_i,x_j)=l)=(1-\alpha_{ij})/2L_{ij}^{(u)}$$
$$(L_{ij}^{(u)}=m-1-T_{\mu}^{(p)}(x_i,x_j);\ l=1,...,L_{ij}^{(u)}),$$

variant II

$$P(g_{\mu k}^{(p)}(x_i,x_j)-T_{\mu}^{(p)}(x_i,x_j)=-l)=\tfrac{1-\alpha_{ij}}{2\Lambda_{ij}^{(d)}}(L_{ij}^{(d)}-l+1)$$

$$(L_{ij}^{(d)}=T_{\mu}^{(p)}(x_i,x_j)+(m-1);\ l=-1,...,-L_{ij}^{(d)};\ \Lambda_{ij}^{(d)}=(L_{ij}^{(d)}(L_{ij}^{(d)}+1)/2),$$

$$P(g_{\mu k}^{(p)}(x_i,x_j)-T_{\mu}^{(p)}(x_i,x_j)=l)=\tfrac{1-\alpha_{ij}}{2\Lambda_{ij}^{(u)}}(L_{ij}^{(u)}-l+1)$$
$$(L_{ij}^{(u)}=m-1-T_{\mu}^{(p)}(x_i,x_j);\ l=1,...,L_{ij}^{(u)};\ \Lambda_{ij}^{(u)}=(L_{ij}^{(u)}(L_{ij}^{(u)}+1)/2)),$$

with three values of α_{ij} : 0.3334; 0.4167; 0.5000 (i.e. approximately $\tfrac{4}{12},\tfrac{5}{12},\tfrac{6}{12}$) for all <*i*, *j*>;

• the number *N* of comparisons of each pair: 1, 3, 5, 7, 9.

The results of simulation comprise – for binary and multivalent comparisons:

• the fraction of errorless estimates (error zero), which are singular solutions of the task (2.19) or (2.20);

• the fraction of errorless estimates, in the case of multiple solutions of the task (2.19) or (2.20);

• the average value of the error (9.2) and (9.3) in 100 simulations;

• the distributions of estimation errors (9.2), (9.3) in the case of multivalent comparisons,
for both types of estimators, i.e. based on sum of differences and sum of differences with medians.

The results of comparisons have been generated in an independent way.

The total number of cases analyzed equals 90 for each type of estimator (3 probability levels, 5 values of *N*, 3 types of relation form, 2 types of comparisons). The number of simulations for each case equals 100 or 200; the double number has been applied for distributions of errors.

The variants of multivalent probability functions reflect two rules: variant I – quasi-uniform probability distribution in each tie, variant II – linear decrease of probabilities for increasing value of error, in each tie. The variant I has, in fact, higher variance and generates lower precision of estimates of the relation. Variant II can be treated as a typical situation, i.e. lower probability of higher errors.

Some parameters used in the case of multivalent comparisons do not satisfy the assumption about the median of comparison errors equal zero; it is so in the case with *n*=9 (linear order) and *N*=1 (singular comparison of each pair), $T_{\mu}^{(p)}(x_i, x_j) = m-1$ and probabilities $\frac{4}{12}, \frac{5}{12}$. Examination of such cases is important from the practical point of view, because sometimes it is not easy to verify precisely the assumptions about zero-value mode and median of distribution.

9.4. Results of simulation survey

The study is based on 100 or 200 repetitions of a set of pairwise comparisons; 200 repetitions have been used for determination of distribution of estimation errors. The optimal solutions of optimization tasks have been determined with the use of partial examination approach - assuming the number of subsets from the set $\{n-1, n, n+1\}$ (for *n*=9 only $n-1$). The algorithms for the approach are presented in Lipski (1982). The results are presented in Tables 1–15, while Tables 7–15 present the distributions of errors.

The efficiency of estimators based on binary comparisons is presented in Tables 1–3. Three parameters are used for characterization of precision of the estimators: the fraction of errorless estimates, which are singular solutions of the tasks (7.2), (7.3), the fraction of errorless estimates, which are non-singular solutions of the tasks (8.1), (8.2), and the average estimation errors (9.2), (9.3) obtained in 100 simulations.

The results concerning binary comparisons (Tables 1–3) can be subsumed in the form of the following observations:

1. The best precision of estimates has been obtained for the relation form with three subsets (Table 3). The precision close to perfect (about 100% of errorless estimates) is obtained for $N \geq 3$ and $\alpha \geq 0{,}90$. The median estimator is slightly worse than those based on total sum of differences. For $N \geq 5$ the fraction of errorless estimates is close to 100% - for all probabilities and for both estimators. Satisfactory results, i.e. fraction of errorless estimates not lower than 50%, for both estimators, and low average errors (about one), have been obtained for $N=1$ and $\alpha \geq 0{,}90$.

2. The lowest precision of estimates has been obtained for the relation form with nine subsets, i.e. linear order (Table 1). For $N=1$ satisfactory results have been obtained only in the case $\alpha = 0{,}95$; for α equal 0.85 and 0.90 three comparisons are indispensable. Five comparisons ($N=5$) and $\alpha \geq 0.90$ or three comparisons and $\alpha = 0.95$ guarantee (nearly) perfect estimates, for both estimators. In the case of moderate precision of the estimators (fraction of errorless estimates 50–75%) the median estimator is slightly worse; it produces more multiple estimates, with higher average errors.

3. The precision of estimates of the relation with $n=6$ (Table 2) is similar to the case $n=3$; however, for $N=1$ and α equal 0.85 or 0.90 the precision is weaker. The precision of the median estimator is similar to precision of the estimator based on the sum of differences for the number of comparisons increased by two, i.e. $N+2$. For $\alpha \geq 0.90$ and $N \geq 3$ both estimators provide precision close to perfect, but the median one is slightly worse.

4. A higher number of comparisons (N) improves significantly the precision of estimates. The estimator based on the sum of differences provides a satisfactory precision of estimates for three comparisons or more; in the case of median estimator it is necessary to have at least five comparisons. Thus, increasing of number N is a “better strategy” than increasing the probability of correct comparison; this results from the exponential evaluation of the probabilities (7.17), (7.18).

5. Average estimation errors assume acceptable values for the number of comparisons N equal at least three. The estimator based on the sum of differences produces the results with considerably higher precision than the median estimator.

6. The difference of precision of both estimators is insignificant for nearly perfect estimates (about 100% of errorless results). In the case of frequency of the estimator $\hat{T}_b^{(p)}(x_i, x_j)$ lower than 75% the difference is significant.

The following general conclusions can be drawn on the basis of the above results:

- both estimators provide precise results for α assuming typical significance levels in statistical tests (at least 0.90) and three or more independent comparisons of each pair;
- increasing number of comparisons N entails a rapid improvement of estimation precision, which reflects the exponential form of the evaluations in the inequalities (7.17), (7.18);
- the median estimator requires more comparisons (at least N+2) than the estimator based on the sum of differences; it is efficient in the case: $\alpha \geq 0{,}95$, $N \geq 3$ or $\alpha \geq 0{,}85$, $N \geq 5$. Moreover, it produces more multiple estimates;
- the average errors of the estimator based on the sum of differences are significantly lower than for the median estimator; low level of errors implies insignificant errors of estimates.

The efficiency of estimators based on multivalent comparisons is presented in Tables 4–9, in similar form as in the case of binary comparisons. Tables 4–6 correspond to variant I of the probability function, while Tables 7–9 correspond to variant II of the probability function. In the case of variant I the distribution functions of errors are additionally analyzed. In the case of variant II the precision of estimates is significantly higher and, therefore, the fraction of errorless estimates can be treated as a sufficient measure of adequacy of estimation.

The following conclusions can be drawn from the results obtained for multivalent estimators – variant I of the distribution functions.

7. The most precise estimates are obtained for the relation form with n=9 (linear order); it is the opposite result to the one for binary comparisons. Satisfactory results are obtained for N=1 and α=0.500. Reliable estimates (about 75% of errorless results) are obtained for N≥3 and α=0.3334. For α=0.500 and N≥3 the fraction of errorless estimates is close to one.

Table 1. The efficiency of estimators based on binary comparisons, n=9 subsets (linear order)

Number of comparisons N	Quantities	Probability of correct comparison:					
		0.85		0.90		0.95	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% CR	20	20	29	29	49	49
	% of CRM	26	26	38	38	60	60
	AE	4.20	4.20	2.78	2.78	1.41	1.41
3	% CR	53	49	77	75	97	96
	% of CRM	56	55	78	80	97	97
	AE	0.88	1.13	0.38	0.45	0.03	0.03
5	% CR	82	82	92	92	99	99
	% of CRM	82	82	92	92	99	99
	AE	0.28	9.30	0.11	0.11	0.01	0.01
7	% CR	91	91	97	97	100	100
	% of CRM	91	91	97	97	100	100
	AE	0.10	0.10	0.03	0.03	0	0
9	% CR	95	95	100	100	100	100
	% of CRM	95	95	100	100	100	100
	AE	0.05	0.05	0	0	0	0

Computations by the author

Symbols: %CR – fraction of errorless singular estimates, % of CRM – fraction of errorless estimates taking into account multiple solutions, AE – average estimation error, taking into account multiple solutions

Table 2. The efficiency of estimators based on binary comparisons, n=6 subsets; number of elements: 1, 2, 1, 2, 1, 2

Number of comparisons, N	Quantities	Probability of correct comparison					
		0.85		0.90		0.95	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% CR	7	7	20	20	48	48
	% of CRM	20	20	33	33	58	58
	AE	5.10	5.10	3.23	3.23	1.39	1.39
3	% of CR	62	46	89	80	98	94
	% of CRM	79	73	96	93	98	95
	AE	0.57	1.11	0.11	0.35	0.02	0.12
5	% CR	88	68	99	92	99	98
	% of CRM	97	92	100	99	99	99
	AE	0.14	0.57	0.10	0.12	0.01	0.03
7	% CR	98	86	99	94	100	100
	% of CRM	99	95	99	98	100	100
	AE	0.02	0.02	0.01	0.09	0	0
9	% CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0

Computations by the author

Symbols: same as in the Table 1

Table 3. The efficiency of estimators based on binary comparisons, *n*=3 subsets; number of elements: 2, 3, 4,

Number of comparisons	Quantities	Probability of correct comparison					
		0.85		0.90		0.95	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% CR	43	43	50	50	76	76
	% of CRM	68	68	84	84	95	95
	AE	1.74	1.74	1.16	1.16	0.38	0.38
3	% CR	94	84	94	93	99	99
	% of CRM	99	98	97	97	99	99
	AE	0.09	0.38	0.06	0.08	0.01	0,01
5	% CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0
7	% CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0
9	% CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0

Computations by the author

Symbols: same as in the Table 1

8. The relations with n=3 and n=6 have similar precision of estimates. Satisfactory precision is obtained for $N \geq 3$ and α equal or greater than 0.3334; results close to errorless - for $N \geq 3$ and α =0.500. Precision close to perfect is obtained for $N \geq 5$ and all values of probabilities. The estimator based on sums produces better results than the median estimator, especially for moderate frequency of errorless estimates (50% - 75% of errorless estimates).

9. An increase in the number of comparisons N improves rapidly the precision of estimation – similarly as in the case of binary comparisons. Three comparisons are sufficient for the estimator based on sums, five comparisons in the case of the median estimator. Nine comparisons guarantee perfect estimates – also in the case of $\alpha = 0{,}3334$.

10. The average errors of estimation are relatively low for the number N greater than one (the average errors corresponding to the estimators $\hat{T}_{\mu}^{(p)}(x_i, x_j)$, $\hat{T}_{\mu}^{(p)}(x_i, x_j)$ assume higher values than those for $\hat{T}_{b}^{(p)}(x_i, x_j)$, $\hat{T}_{b}^{(p)}(x_i, x_j)$ - for binary comparisons). Thus, the estimates different than errorless have insignificant errors in the case of multiple comparisons.

11. The estimator based on the sum of inconsistencies provides better precision than the estimator based on medians. The difference in precision is significant for low and moderate fraction of errorless estimates; for fraction greater than 95% the results are similar. The probability of correct comparison at least 0.3334 and at least five comparisons of each pair produce estimates close to errorless for both estimators.

12. In the cases of $\alpha = 0{,}3334$, $\alpha = 0{,}4167$ and N=1 the multivalent estimators are in fact useless; such situation is marked by a high value of the criterion function (8.1) or (8.2). These cases confirm the necessity of assumption of the median equal zero.

The conclusions obtained on the basis of variant II of distribution function are similar to those concerning variant I. However, precision of estimates is significantly better: the probability of the errorless estimate is higher, the average error – lower. Both distributions can be applied in the case of unknown distributions of comparison errors. However, the uniform distribution – more conservative – is more reliable.

General conclusions concerning multivalent comparisons indicate their higher efficiency relative to the binary case. Especially, it should be emphasized that the probabilities of errorless comparisons can be lower than ½; the remaining probability $1 - \alpha$ is distributed symmetrically about zero. In such case several independent comparisons of each pair guarantee probability of errorless estimate close to one.

The analysis of simulation results based on both types of comparisons indicates some essential conclusions concerning the efficiency of the estimators. Efficiency of the estimators based on multivalent comparisons exceeds, in general, efficiency of the binary estimators. Moreover, these estimators can be applied also in the case of multiple binary comparisons – using the two-step approach. Such approach is applicable, if binary estimates satisfy the requirements of multivalent estimators. The two-step estimator allows also for combining binary and multivalent comparisons, e.g. results of statistical tests, expert opinions, neural networks and other procedures.

The frequencies of estimation errors are presented for variant I of multivalent comparisons only, because efficiency of the binary estimators is adequately characterized by the average error. This is so, because errors of individual estimates assume low values (i.e. the support of distributions includes not more than several points).

The distributions of errors are presented in the interval form – in 10 intervals. The first frequency corresponds to errorless estimate (the error equal zero), the last frequency – to errors greater than 72. The middle intervals have the form $(8\nu, 8(\nu+1)]$ ($\nu=0, ..., 8$); e.g. the second frequency relates to the set of errors $\{1, 2, \ldots, 8\}$. The results take into account multiple solutions, e.g. two solutions with errors – respectively – zero and eight generate the error 3.5. Some of the distributions are presented on graphs 1- 8.

The analysis of results of simulating the distributions of frequencies of estimation errors leads to following conclusions:

13. The estimator based on the sums of differences provides, in the case of multiple comparisons (N>1), better precision than the estimator based on medians. The better precision means higher frequency of errorless estimate and higher concentration of simulated distributions of errors in the neighborhood of zero. The difference in precision is insignificant only in the case of frequency of errorless median estimator higher than 95%.

14. Increase of the number of comparisons N leads to a rapid improvement of precision. For the estimator based on the sum of differences with probability of errorless comparison α=0.5, three comparisons (N=3) guarantee the frequency of errorless estimate close to one. For the median estimator, the number of comparisons has to be increased by two, i.e. equal N+2. The remaining values of α, i.e. 0.3334 and 0.4167, require, respectively, seven and five comparisons. The number of comparisons 7–9 guarantees the frequency of errorless estimate close to 100%.

Table 4. The efficiency of estimators based on multivalent comparisons, variant I of the probability function, n=9 subsets (linear order)

Number of comparisons N	Quantities	Probability of correct comparison:					
		0.3334		0.4167		0.5000	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% of CR	17	17	31	31	60	60
	% of CRM	32	32	51	51	78	78
	AE	43.68	43.68	27.70	27.70	12.50	12.50
3	% of CR	78	58	91	79	97	92
	% of CRM	85	74	95	93	100	98
	AE	6.09	12.35	1.11	3.39	0.26	1.19
5	% of CR	95	71	99	97	100	100
	% of CRM	98	93	100	100	100	100
	AE	0.72	4.0	0.08	0.20	0	0
7	% of CR	98	92	100	100	100	100
	% of CRM	99	99	100	100	100	100
	AE	0.12	0.55	0	0	0	0
9	% of CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0

Computations by the author

Symbols: same as in the Table 1

Table 5. The efficiency of estimators based on multivalent comparisons, *n*=6 subsets, variant I of the probability function; number of elements: 1, 2, 1, 2, 1, 2

Number of comparisons	Quantities	Probability of correct comparison					
		0.3334		0.4167		0.5000	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% of CR	3	3	13	13	15	15
	% of CRM	15	15	39	39	57	57
	AE	38.10	38.10	22.13	22.13	17.00	17.00
3	% of CR	47	21	78	48	89	73
	% of CRM	74	64	89	87	99	96
	AE	6.59	11.53	2.14	5.69	0.56	1.98
5	% of CR	77	45	97	80	100	100
	% of CRM	86	85	98	98	100	100
	AE	1.95	5.52	0.20	1.06	0	0
7	% of CR	99	75	100	100	100	100
	% of CRM	100	96	100	100	100	100
	AE	0.04	1.52	0	0	0	0
9	% of CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0

Computations by the author

Symbols: same as in the Table 1

Table 6. The efficiency of estimators based on multivalent comparisons, *n*=3 subsets, variant I of probability function; number of elements: 2, 3, 4

Number of comparisons	Quantities	Probability of correct comparison					
		0.3334		0.4167		0.5000	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% of CR	4	4	12	12	24	24
	% of CRM	27	27	41	41	58	58
	AE	16.95	16.95	13.66	13.66	9.46	9.46
3	% of CR	57	28	80	54	93	84
	% of CRM	76	69	93	91	100	98
	AE	3.36	7.32	1.18	2.79	0.28	0.89
5	% of CR	90	59	92	87	100	100
	% of CRM	95	95	100	99	100	100
	AE	0.64	2.47	0.32	0.62	0	0
7	% of CR	93	67	100	100	100	100
	% of CRM	97	94	100	100	100	100
	AE	0.43	2.18	0	0	0	0
9	% of CR	100	100	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0	0	0	0	0	0

Computations by the author

Symbols: same as in the Table 1

Table 7. The efficiency of estimators based on multivalent comparisons, variant II of the probability function, n=9 subsets (linear order)

Number of comparisons N	Quantities	Probability of correct comparison					
		0.3334		0.4167		0.5000	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% of CR	19	19	28	28	67	67
	% of CRM	38	38	57	57	88	88
	AE	29.56	29.56	20.8	20.8	7.26	7.26
3	% of CR	70	55	92	81	99	94
	% of CRM	86	88	95	96	100	99
	AE	3.80	6.53	1.03	2.42	0.04	0.36
5	% of CR	96	76	99	92	100	96
	% of CRM	100	97	100	99	99	99
	AE	0.23	2.00	0.04	0.48	0.04	0.24
7	% of CR	99	94	100	100	100	100
	% of CRM	100	100	100	100	100	100
	AE	0.08	0.38	0	0	0	0
9	% of CR	100	96	100	100	100	100
	% of CRM	100	99	100	100	100	100
	AE	0	0.28	0	0	0	0

Computations by the author

Symbols: same as in the Table 1

Table 8. The efficiency of estimators based on multivalent comparisons, n=6 subsets, variant II of the probability function; number of elements: 1, 2, 1, 2, 1, 2

Number of comparisons	Quantities	Probability of correct comparison					
		0.3334		0.4167		0.5000	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% of CR	1	1	8	8	21	21
	% of CRM	14	14	34	34	63	63
	AE	24.51	24.51	18.99	18.99	12.41	12.41
3	% of CR	38	20	72	46	96	78
	% of CRM	63	56	87	83	99	98
	AE	6.91	10.88	2.29	4.90	0.32	1.45
5	% of CR	71	32	97	80	100	91
	% of CRM	83	75	99	97	100	99
	AE	2.48	5.91	0.16	1.38	0	0.40
7	% of CR	93	73	99	95	100	100
	% of CRM	97	94	99	99	100	100
	AE	0.41	1.83	0.08	0.24	0	0
9	% of CR	98	79	100	100	100	100
	% of CRM	100	97	100	100	100	100
	AE	0.08	1.33	0	0	0	0

Computations by the author

Symbols: same as in the Table 1

Table 9. The efficiency of estimators based on multivalent comparisons, *n*=3 subsets, variant II of the probability function; number of elements: 2, 3, 4

Number of comparisons	Quantities	Probability of correct comparison					
		0.3334		0.4167		0.5000	
		Sum.	Median.	Sum.	Median.	Sum.	Median.
1	% of CR	3	3	12	12	22	22
	% of CRM	19	19	44	44	64	64
	AE	18.99	18.99	13.54	13.54	7.9	7.9
3	% of CR	53	26	83	61	93	82
	% of CRM	72	68	95	94	98	96
	AE	18.82	7.70	0.95	2.76	0.42	0.99
5	% of CR	84	54	95	78	100	95
	% of CRM	89	89	99	96	100	100
	AE	1.08	2.88	0.24	1.30	0	0.23
7	% of CR	94	76	100	95	100	98
	% of CRM	99	96	100	98	100	99
	AE	0.28	1.16	0	0.34	0	0.12
9	% of CR	98	88	100	96	100	100
	% of CRM	99	98	100	100	100	100
	AE	0.12	0.69	0	0.16	0	0

Computations by the author

Symbols: same as in the Table 1

Table 10. Frequencies of estimation errors, variant I of the probability function; n=9 subsets, α=0.3334

Value of error	N=1	N=3		N=5		N=7		N=9	
	sum	sum	median	Sum	median	sum	median	Sum	median
0	0.140	0.780	0.545	0.930	0.800	0.995	0.955	0.995	0.980
(0, 8]	0.035	0.075	0.175	0.035	0.100	0	0.020	0	0.010
(8, 16]	0.055	0.045	0.125	0.025	0.050	0	0.005	0.005	0.010
(16, 24]	0.040	0.040	0.050	0	0.030	0	0.015	0	0
(24, 32]	0.080	0.035	0.055	0	0.010	0.005	0.005	0	0
(32, 40]	0.120	0.005	0.045	0.005	0	0	0	0	0
(40, 48]	0.060	0.015	0.020	0	0.005	0	0	0	0
(48, 56]	0.105	0.005	0.020	0	0.005	0	0	0	0
(56, 64]	0.105	0	0.010	0.005	0	0	0	0	0
(64, 72]	0.080	0	0.005	0	0	0	0	0	0
>72	0.180	0	0	0	0	0	0	0	0

Computations by the author

Table 11. Frequencies of estimation errors, variant I of the probability function; n=9 subsets, α=0.4167

Value of error	N=1	N=3		N=5		N=7		N=9	
	sum	sum	median	sum	median	sum	median	sum	median
0	0.350	0.890	0.770	0.990	0.965	1.0	0.985	1.0	1.0
(0,8]	0.090	0.065	0.075	0.005	0.015	0	0.010	0	0
(8,16]	0.085	0.010	0.050	0	0.015	0	0.005	0	0
(16,24]	0.100	0.015	0.040	0.005	0	0	0	0	0
(24,32]	0.065	0.015	0.030	0	0.005	0	0	0	0
(32,40]	0.035	0.005	0.025	0	0	0	0	0	0
(40,48]	0.085	0	0.010	0	0	0	0	0	0
(48,56]	0.050	0	0	0	0	0	0	0	0
(56,64]	0.045	0	0	0	0	0	0	0	0
(64,72]	0.035	0	0	0	0	0	0	0	0
>72	0.060	0	0	0	0	0	0	0	0

Computations by the author

Table 12. Frequencies of estimation errors, variant I of the probability function; n=9 subsets, α=0.5000

Value of error	N=1	N=3		N=5		N=7		N=9	
	sum	sum	median	sum	median	sum	median	sum	median
0	0.600	0.985	0.920	1.0	0.995	1.0	1.0	1.0	1.0
(0,8]	0.095	0.010	0.040	0	0.005	0	0	0	0
(8,16]	0.080	0.005	0.015	0	0	0	0	0	0
(16,24]	0.025	0	0.010	0	0	0	0	0	0
(24,32]	0.050	0	0.010	0	0	0	0	0	0
(32,40]	0.040	0	0.005	0	0	0	0	0	0
(40,48]	0.025	0	0	0	0	0	0	0	0
(48,56]	0.035	0	0	0	0	0	0	0	0
(56,64]	0.015	0	0	0	0	0	0	0	0
(64,72]	0.015	0	0	0	0	0	0	0	0
>72	0.025	0	0	0	0	0	0	0	0

Computations by the author

Table 13. Frequencies of estimation errors, variant I of the probability function; n=6 subsets, α=0.3334

Value of error	N=1	N=3		N=5		N=7		N=9	
	sum	sum	median	sum	median	sum	median	sum	median
0	0.025	0.420	0.155	0.775	0.485	0.915	0.635	0.990	0.825
(0,8]	0.045	0.220	0.240	0.185	0.329	0.075	0.275	0.010	0.165
(8,16]	0.100	0.165	0.245	0.015	0.125	0.005	0.070	0	0.005
(16,24]	0.190	0.110	0.195	0.010	0.060	0.005	0.020	0	0.005
(24,32]	0.195	0.040	0.055	0.015	0.005	0	0	0	0
(32,40]	0.165	0.025	0.075	0	0.01	0	0	0	0
(40,48]	0.090	0.015	0.020	0	0.005	0	0	0	0
(48,56]	0.100	0	0.010	0	0	0	0	0	0
(56,64]	0.025	0	0	0	0	0	0	0	0
(64,72]	0.030	0.005	0	0	0	0	0	0	0
>72	0.035	0	0	0	0	0	0	0	0

Computations by the author

Table 14. Frequencies of estimation errors, variant I of the probability function; n=6 subsets, α=0.4167

Value of error	N=1	N=3		N=5		N=7		N=9	
	sum	sum	median	sum	median	sum	median	sum	median
0	0.115	0.790	0.560	0.970	0.815	0.995	0.945	1.0	0.970
(0,8]	0.090	0.155	0.260	0.025	0.145	0.005	0.055	0	0.030
(8,16]	0.170	0.020	0.120	0	0.030	0	0	0	0
(16,24]	0.190	0.025	0.004	0.005	0.005	0	0	0	0
(24,32]	0.150	0.010	0	0	0.005	0	0	0	0
(32,40]	0.100	0	0.015	0	0.005	0	0	0	0
(40,48]	0.090	0	0.005	0	0	0	0	0	0
(48,56]	0.025	0	0	0	0	0	0	0	0
(56,64]	0.035	0	0	0	0	0	0	0	0
(64,72]	0.020	0	0	0	0	0	0	0	0
>72	0.015	0	0	0	0	0	0	0	0

Computations by the author

Table 15. Frequencies of estimation errors, variant I of the probability function; n=6 subsets, α=0.5000

Value of error	N=1	N=3		N=5		N=7		N=9	
	sum	sum	median	sum	median	sum	median	sum	median
0	0.255	0.880	0.775	1.000	0.950	1.000	1.000	1.000	1.000
(0,8]	0.180	0.100	0.155	0	0.050	0	0	0	0
(8,16]	0.220	0.015	0.045	0	0	0	0	0	0
(16,24]	0.115	0.005	0.015	0	0	0	0	0	0
(24,32]	0.060	0	0.010	0	0	0	0	0	0
(32,40]	0.075	0	0	0	0	0	0	0	0
(40,48]	0.050	0	0	0	0	0	0	0	0
(48,56]	0.020	0	0	0	0	0	0	0	0
(56,64]	0.015	0	0	0	0	0	0	0	0
(64,72]	0.010	0	0	0	0	0	0	0	0
>72	0	0	0	0	0	0	0	0	0

Computations by the author

Table 16. Frequencies of estimation errors, variant I of the probability function; *n*=3 subsets, α=0.3334

Value of error	*N*=1	*N*=3		*N*=5		*N*=7		*N*=9	
	sum	sum	median	sum	median	sum	median	sum	median
0	0	0.530	0.280	0.840	0.540	0.955	0.720	0.980	0.840
(0,8]	0	0.345	0.385	0.125	0.345	0.045	0.275	0.020	0.16
(8,16]	0.015	0.105	0.235	0.035	0.095	0	0.005	0	0
(16,24]	0.120	0.015	0.090	0	0.020	0	0	0	0
(24,32]	0.125	0.005	0.010	0	0	0	0	0	0
(32,40]	0.145	0	0	0	0	0	0	0	0
(40,48]	0.230	0	0	0	0	0	0	0	0
(48,56]	0.225	0	0	0	0	0	0	0	0
(56,64]	0.105	0	0	0	0	0	0	0	0
(64,72]	0.035	0	0	0	0	0	0	0	0
>72	0	0	0	0	0	0	0	0	0

Computations by the author

Table 17. Frequencies of estimation errors, variant I of the probability function; *n*=3 subsets, α=0.4167

Value of error	*N*=1	*N*=3		*N*=5		*N*=7		*N*=9	
	sum	sum	median	sum	median	sum	median	sum	Median
0	0.145	0.785	0.530	0.970	0.820	0.990	0.920	1.000	1.000
(0,8]	0.255	0.190	0.360	0.030	0.165	0.010	0.080	0	0
(8,16]	0.340	0.020	0.100	0	0.010	0	0	0	0
(16,24]	0.170	0.005	0.010	0	0.005	0	0	0	0
(24,32]	0.045	0	0	0	0	0	0	0	0
(32,40]	0.020	0	0	0	0	0	0	0	0
(40,48]	0.015	0	0	0	0	0	0	0	0
(48,56]	0.005	0	0	0	0	0	0	0	0
(56,64]	0.005	0	0	0	0	0	0	0	0
(64,72]	0	0	0	0	0	0	0	0	0
>72	0	0	0	0	0	0	0	0	0

Computations by the author

Table 18. Frequencies of estimation errors, variant I of the probability function; n=3 subsets, α=0.5000

Value of error	N=1	N=3		N=5		N=7		N=9	
	sum	sum	median	sum	median	sum	median	sum	median
0	0.310	0.915	0.795	0.995	0.940	1.000	0.975	1.000	1.000
(0,8]	0.325	0.085	0.190	0.005	0.060	0	0.025	0	0
(8,16]	0.215	0	0.015	0	0	0	0	0	0
(16,24]	0.130	0	0	0	0	0	0	0	0
(24,32]	0.015	0	0	0	0	0	0	0	0
(32,40]	0.005	0	0	0	0	0	0	0	0
(40,48]	0	0	0	0	0	0	0	0	0
(48,56]	0	0	0	0	0	0	0	0	0
(56,64]	0	0	0	0	0	0	0	0	0
(64,72]	0	0	0	0	0	0	0	0	0
>72	0	0	0	0	0	0	0	0	0

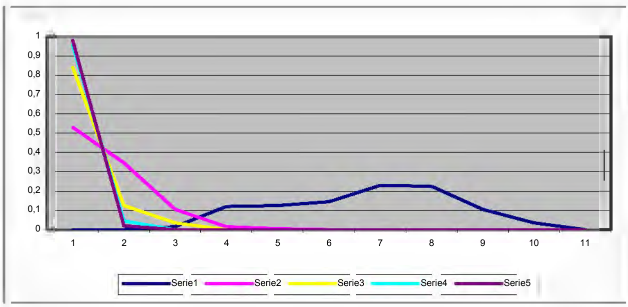
Computations by the author

15. The best precision of estimates has been obtained in the case of linear order, i.e. n=9; remaining cases, i.e. n=3 and n=6, display slightly lower, similar precision. In the case of n=9 and α=1/2, the acceptable precision of estimation (frequency of errorless estimates >50%) is obtained for N=1.

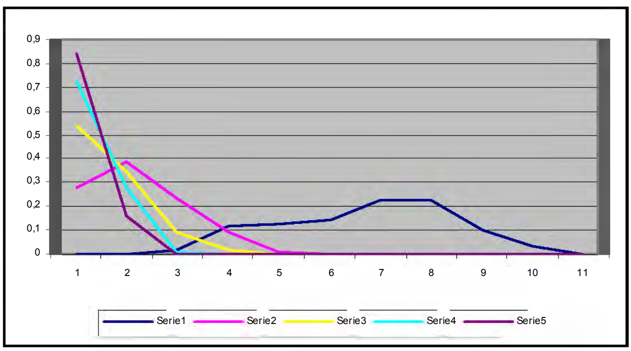
16. The distributions of frequencies of estimation errors are highly concentrated in the case of frequency of errorless estimate higher than 50%. If the frequency of errorless estimate is higher than 75%, the same property is valid for the error equal 8. Insignificant error implies a marginal difference between an estimate and the relation form.

17. When some distributions of comparisons errors do not satisfy the assumptions Z2, i.e. zero is not the median or/and mode of distribution of comparison error, the results of estimation become unacceptable. Such situation takes place for N=1 and α≤0.4167 and for N=3 and α=0.3334.

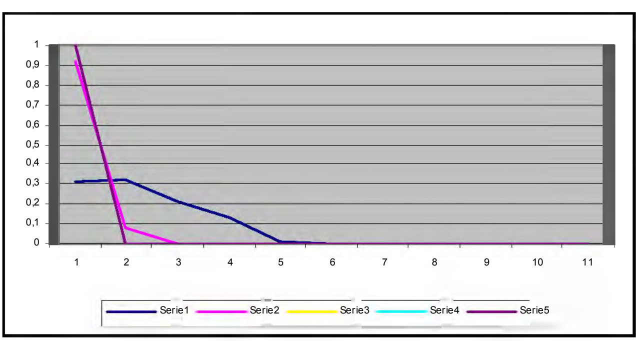
In general, the simulation study confirms the excellent efficiency of the estimator in the form of difference of ranks, based on the sum of differences. It can be postulated that the results of the simulation are valid for any relation form $\chi_1^*, ..., \chi_n^*$, i.e. n>9 and any size of the subsets.



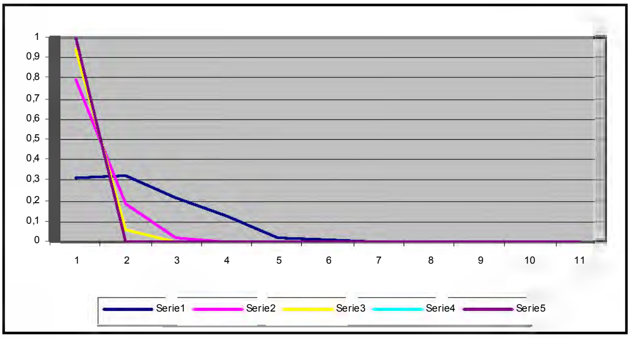
Graph 1. Frequencies of errors of the estimator based on the sums of differences, for N=1, 3, 5, 7, 9 (series 1 – 5); n=3, α=0.3334.



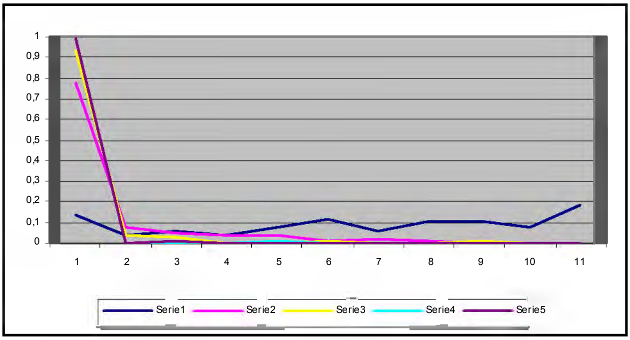
Graph 2. Frequencies of errors of the estimator based on medians, for N=1, 3, 5, 7, 9 (series 1 – 5); n=3, α=0.3334.



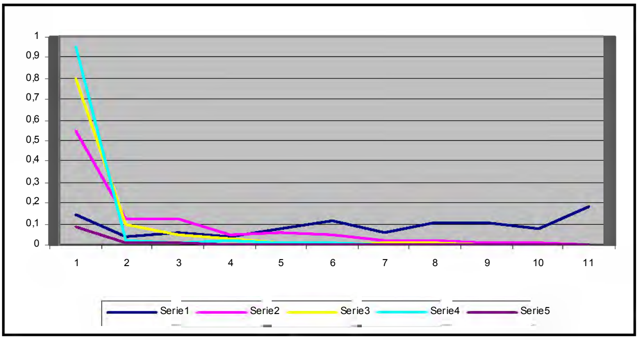
Graph 3. Frequencies of errors of the estimator based on the sums of differences, for N=1, 3, 5, 7, 9 (series 1 – 5); n=3, α=0.5000.



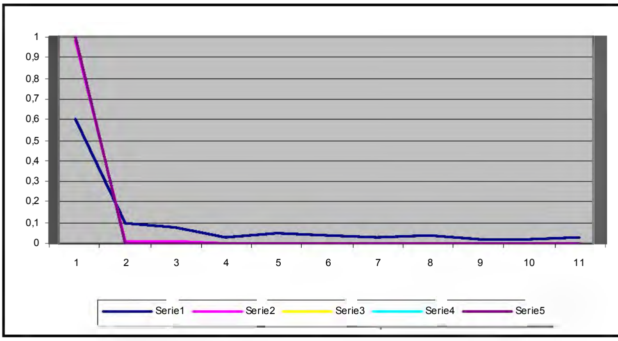
Graph 4. Frequencies of errors of the estimator based on medians, for *N*=1, 3, 5, 7, 9 (series 1 – 5); *n*=3, α=0.5000.



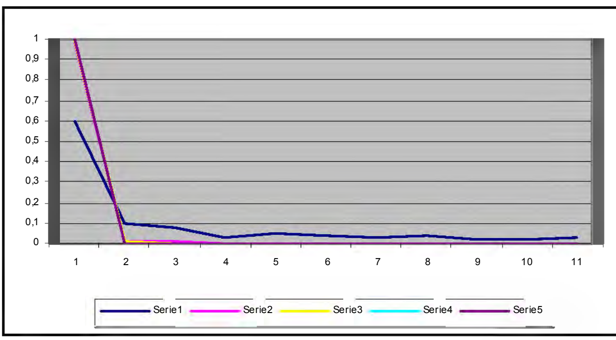
Graph 5. Frequencies of errors of the estimator based on the sums of differences, for *N*=1, 3, 5, 7, 9 (series 1 – 5); *n*=9, α=0.3334.



Graph 6. Frequencies of errors of the estimator based on medians, for *N*=1, 3, 5, 7, 9 (series 1 – 5); *n*=9, α=0.3334.



Graph 7. Frequencies of errors of the estimator based on the sums of differences, for *N*=1, 3, 5, 7, 9 (series 1 – 5); *n*=9, α=0.5000.



Graph 8. Frequencies of errors of the estimator based on medians, for *N*=1, 3, 5, 7, 9 (series 1 – 5); *n*=9, α=0.5000.

9.5. Summary

The simulation study broadens significantly the range of properties of the estimators of the preference relation. It confirms their good statistical properties, especially in the case of multiple comparisons for each pair. One should particularly observe the excellent efficiency of multivalent estimators (comparisons in the form of differences of ranks). It is clear that the results are also valid for remaining relation types, i.e. equivalence and tolerance.

Let us note that the whole estimation process, i.e.: obtaining of pairwise comparisons (using statistical tests), solving of the discrete programming tasks, determining the properties of estimates (using simulation approach) and validation of estimates (see next Chapter) can be computerized; thus the approach is close to data mining techniques.

Further results concerning the empirical distribution functions of errors will be presented in Klukowski (2011a, to appear). In general, these results confirm the conclusions presented in this chapter.

The book presents the estimators of three relations: equivalence, tolerance, and preference in a finite set of data items, based on multiple pairwise comparisons, assumed to be disturbed by random errors. The estimators were developed by the author. They can refer to binary (qualitative), multivalent (quantitative) and combined comparisons. The estimates are obtained on the basis of solutions to the discrete programming problems. The estimators have been developed under weak assumptions on the distributions of comparison errors; in particular, these distributions can have non-zero expected values. The estimators have good statistical properties, including, especially importantly, consistency. Therefore, they produce good results in cases when other methods generate incorrect estimates. The precision of the estimators has been established with the use of simulation methods. The estimates can be validated in a versatile way. The whole estimation process, i.e. comparisons, estimation and validation can be computerized. The approach allows also for inference about the relation type – equivalence or tolerance, on the basis of binary data. Thus, it has features of data mining methods.

The estimators have been applied for ranking and grouping of data from some empirical sets. In particular, estimation of the tolerance relation (overlapping classification) was applied for determination of homogenous shapes of functions expressing profitability of treasury securities and was used for forecasting purposes.

ISSN 0208-8029
ISBN 9788389475374

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