INSTYTUT BADAŃ SYSTEMOWYCH
POLSKIEJ AKADEMII NAUK

TECHNIKI INFORMACYJNE
TEORIA I ZASTOSOWANIA

Wybrane problemy
Tom 1(13)

poprzednio

ANALIZA SYSTEMOWA W FINANSACH I ZARZĄDZANIU

Pod redakcją
Jerzego HOŁUBCA

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AN EFFICIENT ORTHOGONAL ALGORITHMIZATION OF ISOPERIMETRIC TYPE PROBLEM IN THE CLASS OF CLOSED POLYNOMIAL CURVES

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***Abstract**: In this paper we present an efficient algorithm of order $O(n)$, which solves an isoperimetric type problem in the class of closed polynomial curves of degree $n > 2$. This algorithm requires to compute the largest zero of some, recurrently defined, orthogonal polynomial of degree $n-1$ and to solve a sparse system of $n-3$ linear equations. A solution of this system allows to obtain the control points of extremal curve, while maximal zero of this polynomial gives maximal area bounded by this curve.*

1. Introduction

One of the oldest geometrical problem is the isoperimetric problem, which consists in finding among all closed curves of fixed Euclidean length, the one which encloses the maximal area. Since a long time it is known, that the solution of this problem is a circle. However, after imposing some restrictions concerning the set of admissible curves and/or metric, the solution may turn out to be the other curve which may, but not have to, closely approximate the circle. Two examples of such approach may be found in papers [9, 10]. More precisely in the second article there has been presented a numerical algorithm, which solves the isoperimetric type problem in the class $\mathcal{C}_n^0$ of all simple closed and positively oriented Bézier curves $\xi\colon [0,1] \to \mathbb{R}^2$ of degree $n > 2$. This class includes all curves of the form

$$\xi(t) = \sum_{k=1}^{n-1} (x_k, y_k)\, B_k^n(t),$$

where $(x_k, y_k) \in \mathbb{R}^2$, and

$$B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k} \; (0 \leq t \leq 1)$$

are Bernstein polynomials of degree n. Recall, that restricted problem consists in calculating the extremal value

$$\sup_{\xi} \frac{P(\xi)}{L(\xi)} \tag{1}$$

and determining the extremal Bézier curve ξ for which that supremum is attained. Here P(ξ) and L(ξ) mean the area bounded by closed Bézier curve and its linearized length, respectively. Choosing the linearized length instead of the usual length of a curve, turns out to be important from the point of view of simplicity of calculations. However, note that both extremal curves differ only by a scale [10].

If it comes to the algorithm, its form is a consequence of the fact that the area $P(\xi)$ may be represented as a bilinear form with a skew-symmetric matrix A of degree $n-1$, and the linearized length $L(\xi)$ as a sum of two quadratic forms with the same symmetric positive definite matrix B of degree $n-1$. This allows to reduce the isoperimetric problem to the solution of a spectral problem of quadratic matrix $B^{-1}A$ of degree $n-1$, which largest eigenvalue is the requested extremal value, and a corresponding eigenvector gives the control points of requested extremal Bézier curve. For the completeness we present this algorithm [10] as Algorithm 1.

Algorithm 1. Isoperimetric type problem in the class C_n^0 of Bézier curves of degree n with a prescribed linearized length $l > 0$.

Input: A positive integer n and a positive real l.

Output: The control points $(x_k, y_k), k = 0,1,\dots,n$, of extremal Bézier curve ξ, which bounds the maximal area $P(\xi)$.

1. Compute the matrix $B^{-1}A$.
2. Determine the complex eigenvalue $\mu = 4\lambda(n-1)i$ of the matrix $B^{-1}A$ with the largest imaginary part and return $P(\xi) = l^2\lambda$.
3. Compute vector $(x|y)^T = (x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})^T$ satisfying relations $[(B^{-1}A)^2 + |\mu|^2 I]y = 0$ and $x = \frac{B^{-1}A}{|\mu|}y, y \neq 0$.

4. Return the control points of curve ξ equal to $(x_0, y_0) = (x_n, y_n) = (0,0)$ and $(x_k, y_k) = \frac{l}{\sqrt{\frac{n(n-1)}{\binom{2n-1}{n}}(x^T B x + y^T B y)}} (x_k, y_k), k = 1,2, \dots, n-1.$

2. New, more efficient algorithm

It has been observed in [10] that the solution of the isoperimetric type problem in the class of closed Bézier curves of degree n does not depend on the choice of basis in the space $\mathcal{P}_n$ of all polynomials of degree not greater than n and that an appropriate choice of the basis can simplify Algorithm 1 considerably. In the present paper we propose to choose a specific orthogonal basis, which fulfills this requirement. It will allow to reduce the order of cost of solving isoperimetric problem from $O(n^3)$ to $O(n)$. On the other hand, it should be noticed that the new basis does not have the usual geometrical interpretation of Bézier curves with respect to the Bernstein basis.

Let $\xi: [0,1] \to \mathbb{R}^2$ be a simple closed and positively oriented polynomial curve of degree n having the form

$$\xi(t) = \left(\xi_1(t), \xi_2(t)\right) = \sum_{k=1}^{n-1} u_k p_k(t),\ 0 \le t \le 1,$$

where $u_k = (x_k, y_k) \in \mathbb{R}^2$ and

$$\xi_1(t) = \sum_{k=1}^{n-1} x_k p_k(t),\ \xi_2(t) = \sum_{k=1}^{n-1} y_k p_k(t).$$

Then the area $P(\xi)$ bounded by curve ξ may be represented by the formula

$$P(\xi) = \int_0^1 \xi_1'(t)\xi_2(t)dt = x^T A y,$$

where $A = \left[a_{j,k}\right]_{0<j,k<n}$ is the quadratic matrix with elements

$$a_{j,k} = \int_0^1 p_j'(t) p_k(t) dt, \tag{2}$$

and $x = (x_1, \dots, x_{n-1})^T, y = (y_1, \dots, y_{n-1})^T$ are vectors in $\mathbb{R}^{n-1}$. Moreover, we express the linearized length $L(\xi)$ of curve ξ by the formula

$$L(\xi) = \int_0^1 \left[\left(\xi_1'(t)\right)^2 + \left(\xi_2'(t)\right)^2 \right] dt = x^T B x + y^T B y,$$

where $B = \left[b_{j,k}\right]_{0<j,k<n}$ is the quadratic matrix with elements

$$b_{j,k} = \int_0^1 p_j'(t) p_k'(t) dt.$$

It seems that the maximal simplification of the Algorithm 1 can be obtained by requiring matrix B to be a unit matrix, and matrix A to be tridiagonal. In other words, it means that polynomials $p_1(t), \dots, p_{n-1}(t)$ should be chosen in such a way that the derivatives $p_k'(t)$ $(k = 1, \dots, n-1)$ were polynomials of degree k, which are orthonormal with respect to the inner product in the Hilbert space $L^2(0,1)$ and satisfy boundary conditions of the form $p_k(0) = p_k(1) = 0$. For this purpose let us consider the classical Jacobi polynomials

$$J_n^{(\alpha,\beta)}(t) \; (-1 \le t \le 1, \alpha > -1, \beta > -1)$$

orthogonal with respect to the weight $w(t) = (1-t)^\alpha (1+t)^\beta$ [12]. Note that

$$J_k^{(0,0)}(2t-1) = \binom{2k}{k} t^k + \cdots, 0 \le t \le 1,$$

are Legendre polynomials orthogonal with respect to the weight $w(t) = 1$, that are scaled to the interval [0,1]. It is known [2], that they satisfy the following orthogonality relations

$$\int_0^1 J_j^{(0,0)}(2t-1) J_k^{(0,0)}(2t-1)\, dt = \begin{cases} \dfrac{1}{2k+1} & , \text{gdy } j = k, \\ 0 & , \text{gdy } j \ne k, \end{cases}$$

and are expressed by the Rodrigues formula

$$J_k^{(0,0)}(2t-1) = \frac{1}{k!} \frac{d^k}{dt^k} [t^k (t-1)^k].$$

Hence it is clear that $p_k(t)$ $(k = 1, \ldots, n-1)$ have to be defined in such a way that they should satisfy boundary conditions

$$p_k(0) = p_k(1) = 0, \tag{3}$$

and their derivatives should be equal to

$$p_k'(t) = \sqrt{2k+1} J_k^{(0,0)}(2t-1).$$

It is not difficult to notice that they have the form

$$p_k(t) = \frac{\sqrt{2k+1}}{k} t(t-1) J_{k-1}^{(1,1)}(2t-1), k = 1,2, \ldots, n-1. \tag{4}$$

It is not hard to see that these polynomials guarantee that not only the matrix B becomes a unit matrix, but also the matrix A has a simple form. Namely, applying to (2) the integration by parts, conditions (3) and formula (4) we state that

$$a_{j,k} = [p_j(t)p_k(t)]\Big|_0^1 - \int_0^1 p_j(t)p_k'(t)dt = -a_{k,j}$$

and

$$a_{j,j} = \frac{1}{2} p_j^2(t)\Big|_0^1 = 0.$$

Moreover, the orthogonality of polynomials $p_j'(t)$ implies that $a_{j,k} = 0$ for all $k < j-1$ [12, (2.3.1)] and

$$\begin{aligned} a_{j,j-1} &= \sqrt{2j+1} \int_0^1 J_j^{(0,0)}(2t-1) p_{j-1}(t) dt \\ &= \sqrt{2j+1} \frac{\sqrt{2j-1}(2j-2)!}{j!\,(j-1)!} \int_0^1 J_j^{(0,0)}(2t-1) t^j dt \\ &= \sqrt{2j+1} \frac{\sqrt{2j-1}}{2(2j-1)} \int_0^1 \left[J_j^{(0,0)}(2t-1)\right]^2 dt \\ &= \frac{\sqrt{(2j-1)(2j+1)}}{2(2j-1)(2j+1)}. \end{aligned}$$

Hence $A = \left[a_{j,k}\right]_{0<j,k<n}$ is a skew-symmetric and tridiagonal matrix with elements $a_{j,j} = 0, j = 1, \ldots, n-1$, and

$$a_{j,j-1} = -a_{j-1,j} = \frac{\sqrt{(2j-1)(2j+1)}}{2(2j-1)(2j+1)},\ j = 2,3,\dots,n-1.$$

Knowing the forms of matrices A and B we proceed in a similar way as in the paper [10]. Namely, the solution of the isoperimetric type problem is equivalent to finding the largest eigenvalue λ_{2n-2} and the corresponding eigenvector z_{2n-2} of the following pencil of quadratic forms

$$z^T C z - \lambda z^T D z,\ z = (x|y)^T = (x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})^T,$$

where C and D are symmetric block matrices of the form

$$C = \frac{1}{2}\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}, D = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Here I denotes the unit matrix of degree $n-1$.

It is known, that requested eigenvalue λ_{2n-2} is the largest zero of the characteristic polynomial

$$m_{2n-2}(\lambda) = \det(\mathrm{C} - \lambda\mathrm{D}),$$

of degree $2n-2$, which after applying Schur's formula [4] is reduced to the form

$$m_{2n-2}(\lambda) = \det\left(\lambda^2 I + \frac{A^2}{4}\right) = \det M_{n-1}^{(1)} \det M_{n-1}^{(2)},$$

where $M_{n-1}^{(1)} = \frac{A}{2} - i\lambda I$ and $M_{n-1}^{(2)} = \frac{A}{2} + i\lambda I$ with $i^2 = -1$. Note that matrices $M_{n-1}^{(1)}$ and $M_{n-1}^{(2)}$ are tridiagonal and hence their determinants can be expressed by recurrent relations. In particular we have

$$\det M_0^{(1)} = 1, \det M_1^{(1)} = -i\lambda,$$
$$\det M_{n-1}^{(1)} = -i\lambda \det \quad M_{n-2}^{(1)} + \frac{a_{n-1,n-2}^2}{4} \det M_{n-3}^{(1)}, n = 3,4,\dots.$$

One can obtain the similar formula for the determinant of matrix $M_{n-1}^{(2)}$ after replacing above the factor $-i$ by i.

For an efficient algorithm to compute the largest solution λ_{2n-2} of the equation

$$m_{2n-2}(\lambda) = 0,$$

it should be used the fact that characteristic polynomial $m_{2n-2}(\lambda)$ can be represented in the form

$$m_{2n-2}(\lambda) = \bar{m}_{n-1}^2(\lambda), \tag{5}$$

where $\bar{m}_{n-1}(\lambda)$ is the polynomial of degree $n-1$, satisfying the following recurrent relations

$$\begin{aligned} &\bar{m}_0(\lambda) = 1, \bar{m}_1(\lambda) = \lambda, \\ &\bar{m}_{n-1}(\lambda) = \lambda \bar{m}_{n-2}(\lambda) - \tfrac{1}{16(2n-3)(2n-1)} \bar{m}_{n-3}(\lambda), n = 3,4, \dots . \end{aligned} \tag{6}$$

Taking additionally into account the fact, that polynomial $\bar{m}_{n-1}(\lambda)$ is for even n an odd function, and for odd n an even function, it becomes evident that the largest root λ_{n-1}^* of polynomial $\bar{m}_{n-1}(\lambda)$ is also the requested solution λ_{2n-2} of the problem (1). The relation (5) can be easily justified by the mathematical induction. For this purpose, using the recurrent representations of determinants of matrices $M_{n-1}^{(1)}$ and $M_{n-1}^{(2)}$ we check first, that

$$\begin{aligned} m_0(\lambda) &= 1 = \bar{m}_0^2(\lambda), \\ m_2(\lambda) &= \lambda^2 = \bar{m}_1^2(\lambda), \\ m_4(\lambda) &= \left(\lambda^2 - \frac{a_{2,1}^2}{4}\right)^2 = \bar{m}_2^2(\lambda). \end{aligned}$$

Then assuming that the formula

$$m_{2n-4}(\lambda) = \left[\lambda \bar{m}_{n-3}(\lambda) - \frac{a_{n-2,n-3}^2}{4} \bar{m}_{n-4}(\lambda)\right]^2 = \bar{m}_{n-2}^2(\lambda)$$

is true, we again apply the formulae for determinants of matrices $M_{n-1}^{(1)}$ and $M_{n-1}^{(2)}$, and the simple fact that

$$(-I)M_{n-1}^{(1)} = M_{n-1}^{(1)}(-I) = \left[M_{n-1}^{(2)}\right]^T,$$

in order to get

$$
\begin{aligned}
m_{2n-2}(\lambda) = {} & \lambda^2 \det M_{n-2}^{(1)} \det M_{n-2}^{(2)} \\
& + \left(\frac{a_{n-1,n-2}^2}{4}\right)^2 \det M_{n-3}^{(1)} \det M_{n-3}^{(2)} \\
& - i\lambda \frac{a_{n-1,n-2}^2}{4} \left[\det M_{n-2}^{(1)} \det M_{n-3}^{(2)} - \det M_{n-3}^{(1)} \det M_{n-2}^{(2)}\right] \\
& = \lambda^2 \,\overline{m}_{n-2}^2(\lambda) + \left(\frac{a_{n-1,n-2}^2}{4}\right)^2 \overline{m}_{n-3}^2(\lambda) \\
& - 2i\lambda \frac{a_{n-1,n-2}^2}{4} \det M_{n-2}^{(1)} \det M_{n-3}^{(2)}.
\end{aligned}
$$

Since an additional induction leads to

$$\det M_{n-2}^{(1)} \det M_{n-3}^{(2)} = -i\overline{m}_{n-2}(\lambda)\overline{m}_{n-3}(\lambda),$$

we finally obtain

$$m_{2n-2}(\lambda) = \left[\lambda\overline{m}_{n-2}(\lambda) - \frac{a_{n-1,n-2}^2}{4}\overline{m}_{n-3}(\lambda)\right]^2 = \overline{m}_{n-1}^2(\lambda).$$

Therefore it has been proved that the first part of Algorithm 1 is equivalent to the determination of the largest zero λ_{n-1}^* of the orthogonal polynomial $\overline{m}_{n-1}(\lambda)$ of degree $n-1$ defined by recurrent formulae (6).

It is still left to determine control points $u_k = (x_k, y_k), k = 1, \ldots, n-1$, which define curve ξ, extremal to the problem (1). As it has earlier been said they are coordinates of eigenvector

$$z_{2n-2} = (x|y)^T = (x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})^T$$

satisfying equation

$$Cz_{2n-2} = \lambda_{2n-2} D z_{2n-2}.$$

Hence, it follows easily that the requested coordinates satisfy relations

$$(A^2 + 4\lambda_{2n-2}^2 I)y = 0, x = \frac{A}{2\lambda_{2n-2}} y, y \neq 0.$$

Summing up, the previous considerations allow to formulate the following Algorithm 2, which efficiently solves isoperimetric type problem in the class of simple closed polynomial curves ξ of degree n.

Algorithm 2. Isoperimetric type problem in the class of closed polynomial curves of degree n with a prescribed linearized length $l > 0$.

Input: A positive integer n and a positive real l.

Output: The control points $u_k = (x_k, y_k), k = 1, \dots, n-1$, of extremal polynomial curve ξ, which bounds the maximal area $P(\xi)$.

1. Apply formula (6) to compute the largest zero λ_{n-1}^* of polynomial $\bar{m}_{n-1}(\lambda)$ and return $P(\xi) = l^2 \lambda_{n-1}^*$.
2. Compute vector $(x|y)^T = (x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})^T$ satisfying relations $(A^2 + 4\lambda_{2n-2}^2 I)y = 0$ and $x = \frac{A}{2\lambda_{2n-2}} y, y \neq 0$, where $\lambda_{2n-2} = \lambda_{n-1}^*$.
3. Return the control points of curve ξ equal to $u_k = \frac{l}{\sqrt{x^T x + y^T y}} (x_k, y_k), k = 1,2, \dots, n-1$.

3. More details of Algorithm 2

At first, we discuss the numerical computation of the largest zero λ_{n-1}^* of polynomial $\bar{m}_{n-1}(\lambda)$. For this purpose, we note that the cost of computation of the value $\bar{m}_{n-1}(\lambda)$ defined by formula (6) is equal to $O(n)$. Moreover, note that the recurrent relations (6) imply that polynomials $\bar{m}_1(\lambda), \bar{m}_2(\lambda), \dots, \bar{m}_{n-1}(\lambda)$ form a system of monic orthogonal polynomials [6, 12]. It means that their largest zeros $\lambda_1^*, \lambda_2^*, \dots, \lambda_{n-1}^*$ satisfy inequalities

$$\lambda_1^* < \lambda_2^* < \dots < \lambda_{n-1}^*.$$

On the other hand, we have $\lambda_1^* = 0$, $\lambda_2^* = \frac{\sqrt{15}}{60}$ and $\lambda_n^* \uparrow \frac{1}{4\pi}$. Hence it is clear that values λ_k^* $(k = 3,4, \dots)$ should be searched in the interval $\left(\frac{\sqrt{15}}{60}, \frac{1}{4\pi}\right)$. Therefore, in order to compute λ_{n-1}^* one can apply the well-known bisection algorithm for the Sturm's sequences [5, 8, 11]. The cost of this algorithm is proportional to the cost of a single evaluation of polynomial $\overline{m}_{n-1}(\lambda)$. The factor of proportionality depends only on the precision of computations. Since the length of interval $\left(\frac{\sqrt{15}}{60}, \frac{1}{4\pi}\right)$ is less than 2^{-6}, it follows that the factor of proportionality equals 10 in the case when root is computed with precision 2^{-16}. We present the numerical implementation of this algorithm as Algorithm 3.

Algorithm 3. Determination of the largest zero λ_{n-1}^* of polynomial $\overline{m}_{n-1}(\lambda)$ with precision $prec > 0$.

Input: Positive real $prec$ denoting the precision of computing the root λ_{n-1}^* of polynomial $\overline{m}_{n-1}(\lambda)$.

Output: Interval $[\mu, \nu]$ of the length less or equal to $prec$, which contains the largest zero λ_{n-1}^* of polynomial $\overline{m}_{n-1}(\lambda)$.

1. $\mu = \frac{\sqrt{15}}{60}, \nu = \frac{1}{4\pi}$
2. **while** $\nu - \mu > prec$
    2.1. $\kappa = (\mu + \nu)/2$
    2.2. **if** $\overline{m}_{n-1}(\kappa) < 0$
        2.2.1. **then** $\mu = \kappa$
        2.2.2. **else** $\nu = \kappa$

The next step of Algorithm 2 is the computation of non-zero vector $y = (y_1, \dots, y_{n-1})^T$ satisfying the singular system of equations $Hy = 0$, with matrix $H = \left[h_{j,k}\right]_{0<j,k<n} = (A^2 + 4\lambda_{2n-2}^2 I)$. Note that all non-zero elements of the symmetric matrix H are located only on three diagonals:

$$h_{1,1} = -a_{2,1}^2 + 4\lambda_{2n-2}^2, h_{n-1,n-1} = -a_{n-1,n-2}^2 + 4\lambda_{2n-2}^2,$$
$$h_{j,j} = -a_{j,j-1}^2 - a_{j+1,j}^2 + 4\lambda_{2n-2}^2, j = 2, \dots, n-2,$$
$$h_{j+2,j} = h_{j,j+2} = a_{j+1,j}a_{j+2,j+1}, j = 1, \dots, n-3.$$

In order to obtain the non-zero solutions of this system we assume that $y_1 = 0$ and $y_2 = 1$, and calculate the remaining coordinates $\bar{y} = (y_3, \dots, y_{n-1})^T$ by solving the non-singular system of the form

$$H_0\bar{y} = v, \tag{7}$$

where H_0 is the square matrix of degree $n-3$ obtained from matrix H by discarding the first and second row, and the first and second column. Moreover, we denote $v = \left(0, -a_{3,2}a_{4,3}, 0, \dots, 0\right)^T$. Note, that matrix H_0 is a positive definite matrix of the form

$$H_0 = \begin{bmatrix} * & 0 & * & & & 0 \\ 0 & \cdot & \cdot & \cdot & & \\ * & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & * \\ & & \cdot & \cdot & \cdot & 0 \\ 0 & & & * & 0 & * \end{bmatrix}$$

with exactly three non-zero diagonals. Therefore the Cholesky elimination can be applied to solve the equation (7) with the cost $O(n)$.

Next, after obtaining vector y, we apply transformation

$$x = \frac{A}{2\lambda_{2n-2}} y$$

to compute vector x. Note that due to the tridiagonal form of matrix A this transformation can also be performed with only $O(n)$ operations.

The last step of Algorithm 2 is the scaling of obtained control points $u_k = (x_k, y_k), k = 1, \dots, n-1$, in order to get a final curve with the required length l. Since the key operations in this step are $x^T x + y^T y$, $\frac{l}{\sqrt{x^T x + y^T y}} x$ and $\frac{l}{\sqrt{x^T x + y^T y}} y$, it is easy to notice that cost of this step has also the order $O(n)$. Therefore the complexity of Algorithm 2 is equal to $O(n)$.

Finally we present on Figure 1 the plots of extremal polynomial curves of degree $n = 3,4,5,6,7,8$ obtained with the aid of the Algorithm 2 for $l = 2\pi$.

The corresponding areas bounded by these extremal curves are presented in Table 1.

Table 1. Maximal areas bounded by curve $\boldsymbol{\xi}$ of degree $\boldsymbol{n}$ of length $\boldsymbol{l = 2\pi}$.

n	3	4	5	6	7	8
$P(\xi)$	2.54832	3.04583	3.13026	3.14067	3.14154	3.14159

Notice that calculated value of area $P(\xi)$ increases, along with the increase of n, to the area $P(\xi) = \pi \approx 3.14159$ of a circle which is a solution of the unrestricted isoperimetric problem. Moreover, calculated extremal polynomial curve ξ better and better approximates this circle.

Extension of the results of this work to the approximation by (Bézier) curves of other regions in the plane, will be presented elsewhere. There will be also discussed the applications of obtained results to some practical problems of computer aided geometric design considered in the monograph [3], as well as the problems connected with the restoration of maybe blurred images [1].

Note added to proof. A different $O(n^3)$ algorithm based on the Legendre polynomials is presented recently in [7].

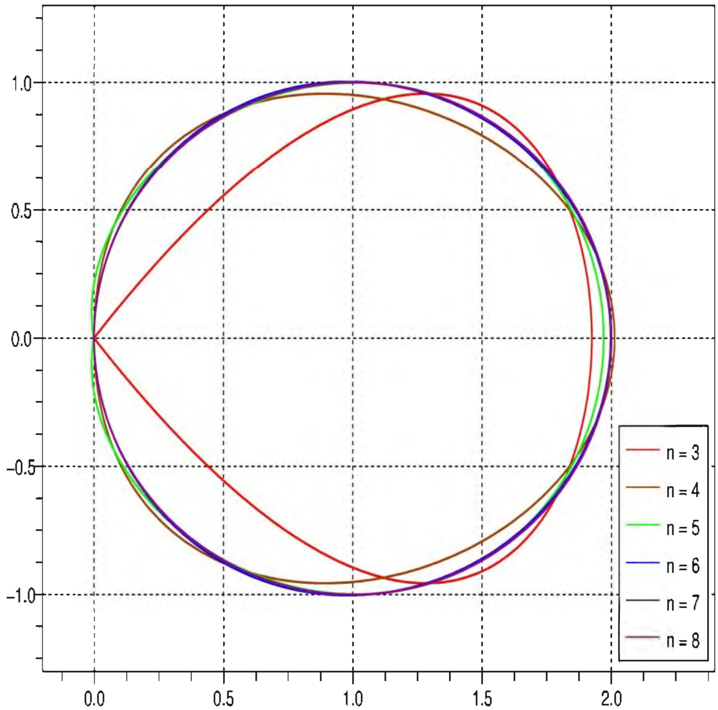

Figure 1. Polynomial curves of degree n= 3, 4, 5, 6, 7, 8
of length l = 2π bounding the maximal area.

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