Polska Akademia Nauk • Instytut Badań Systemowych

AUTOMATYKA STEROWANIE ZARZĄDZANIE

Książka jubileuszowa
z okazji
70-lecia urodzin

PROFESORA KAZIMIERZA MAŃCZAKA

pod redakcją
Jakuba Gutenbauma

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AN EXISTENCE RESULT FOR THE BRIDGED CRACK MODEL

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Abstract:*In the paper the existence and uniqueness of solutions for a model of bridged crack is proved by an application of the theory of Hilbert transform and Cauchy - Poisson semi group. Numerical solutions are computed using Matlab.*

Keywords: *bridged crack, singular integral equation, Hilbert transform, Cauchy - Poisson semi group.*

1. Introduction

In the present paper the one-dimensional problem of modeling of a bridged crack is considered. In section 2 the model is derived by standard arguments from linear elasticity. In section 3 a new proof of existence of a unique solution to the model is given. To our knowledge, the proof based on properties of the Cauchy - Poisson semi group is an original contribution of the paper. Finally, in section 4, we provide some numerical results for illustration.

The bridged crack problem, classical in elasticity, is well documented in the literature, we refer the reader to, e.g., the references given in the bibliography for singular integral equation models and asymptotic solutions. However, there is no complete proof of convergence of finite dimensional approximation in the existing literature on the subject. In the present paper we establish the mathematical framework for the analysis of the model, which can be useful for numerical analysis. The convergence of approximation is the next step in the analysis of the model. We provide some preliminary numerical results obtained with Matlab.

2. The bridged crack model

We consider the elastic half plane $S^- = \{(x,y) \in \mathbb{R}^2 | y < 0\}$ with the boundary $\partial S^- = \{(x,0) | x \in \mathbb{R}\}$.
The classical results (Muskhelishvili 1953) on isotropic elasticity in the half plane S^- state that if the forces (X,Y) are applied on ∂S^- then the displacement field (u,v) in S^- is determined from the following relation due to G. V. Kolosov

$$2\mu(u_x + iv_x) = \kappa\Phi(z) + \Phi(\overline{z}) - (z - \overline{z})\overline{\Phi}'(\overline{z}) \tag{1}$$

with the complex potential derived by Muskhelishvili

$$\Phi(z) = -\frac{1}{2\pi z}(X + iY) + O(\frac{1}{z^2})$$

where we denote $z = x + iy \in \mathbb{C}$, $u = u(x,y)$, $v = (x,y)$, $u_x = \dfrac{\partial u}{\partial x}$, μ, κ are given constants.
In the present paper we apply the result for the particular case of the point force $(O, \delta(x))$ at the origin, $\delta(x)$ is the Dirac mass supported at $x = 0$.
In such a case the solution for the unknown displacement v is obtained in the closed form as follows.
The potential $\Phi(z)$ is given by

$$\Phi(z) = -\frac{i}{2\pi z}, \quad \Phi'(z) = \frac{i}{2\pi z^2}$$
$$\Phi(\overline{z}) = -\frac{i}{2\pi \overline{z}}, \quad \Phi'(\overline{z}) = -\frac{i}{2\pi \overline{z}^2}$$

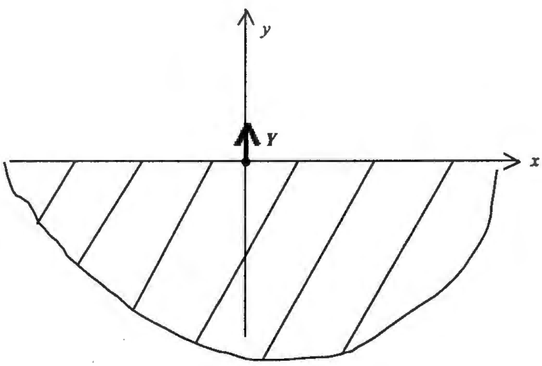

Fig. 1. Elastic half plane

Therefore, the equation (1) takes the form

$$\begin{aligned}
2\mu\left(\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial x}\right) &= \kappa\Phi(z)+\Phi(\overline{z})-(z-\overline{z})\overline{\Phi}'(z)= \\
&= -\frac{i}{2\pi}\left\{\frac{\overline{z}}{|z|^2}\kappa+\frac{z}{|z|^2}-\frac{(z-\overline{z})z^2}{|z|^4}\right\}= \\
&= -\frac{i}{2\pi}\left\{\frac{(\kappa+1)x-i(\kappa-1)y}{|z|^2}+\frac{4xy^2}{|z|^4}-\frac{izy(x^2-y^2)}{|z|^4}\right\}
\end{aligned} \tag{2}$$

and the imaginary part

$$2\mu\frac{\partial v}{\partial x}=-\frac{1}{2\pi}\left\{\frac{(\kappa+1)x}{|z|^2}+\frac{4xy^2}{|z|^4}\right\},\quad |z|^2=x^2+y^2$$

on the boundary S^- for $y=0$,

$$2\mu\frac{\partial v}{\partial x}\bigg|_{y=0}=-\frac{1}{2\pi}\frac{(\kappa+1)x}{x^2}=-\frac{1}{2\pi}(\kappa+1)\frac{1}{x}$$

hence

$$v\bigg|_{y=0}=-\frac{\kappa+1}{4\pi\mu}\ln\frac{|x|}{d}+\text{const}$$

where d (in length dimensions) is a constant. From (Muskhelishvili 1953) we have the values of κ

$$\kappa = \begin{cases} 3-4\nu, & \text{plane deformation} \\ & 1<\kappa<3 \\ \dfrac{3-4\nu}{1+\nu}, & \text{mean tension plane} \\ & \frac{5}{3}\kappa<3 \end{cases}$$

Since we are interested in the modeling of the crack, the next step concerns the superposition of the point force applied at the origin and the symmetric distributed force supported on the intervals $]-\infty,-a]$ and $[a,+\infty[$ for some $a>0$.

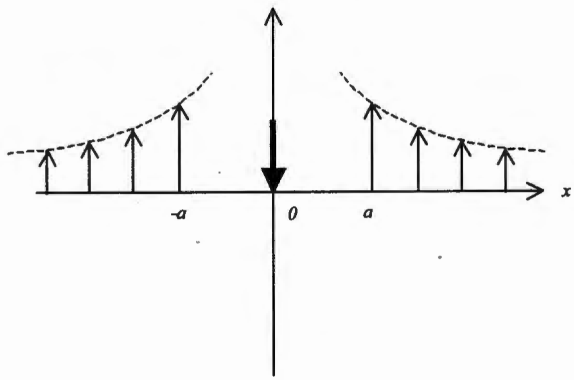

Fig. 2. Elastic half plane loaded on the boundary

In such a case, the following solution is obtained for unknown distribution of the force $p(\zeta), \zeta \in]-\infty,-a[\cup]a,+\infty[$.

$$p^-(-\zeta) = p^+(\zeta),\ \zeta \in [a,+\infty[,$$

$$\int_{\infty}^{-a} p^-(\zeta)\ \ln\frac{|x-\zeta|}{d}d\zeta = \int_a^{+\infty} p^-(-t)\ \ln\frac{|x+t|}{d}\ dt$$

$$\frac{4\pi\mu}{\kappa+1}v\,|_{y=0} = P_0\ln\frac{|x|}{d} - \int_a^{+\infty} p(\zeta)\ln\left(\frac{|x-\zeta|}{d}\right)d\zeta -$$

$$-\int_a^{+\infty} p(\zeta)\ln\left(\frac{|x+\zeta|}{d}\right)d\zeta.$$

Now we are in position to model the problem under consideration in the paper. We have two elastic half planes S^- and S^+ with a thin elastic layer $\{y = 0\}$ between S^- and S^+. Assuming symmetry, we can restrict the analysis to the displacement field in S^-, and we impose the following relation

$$-p(x) = kv^-|_{y=0}$$

between the reaction $p(x)$, $x \in]-\infty, -a] \cup [a, +\infty[$ and the displacement $v^-|_{y=0}$ on ∂S^-, where k is a coefficient which characterize the layer $\{y = 0\}$. In particular, the Poisson coefficient of the layer is assumed to be zero, $\nu = 0$.
This is the limit case, as indicated, for example, in (Solomon 1968, p.73).

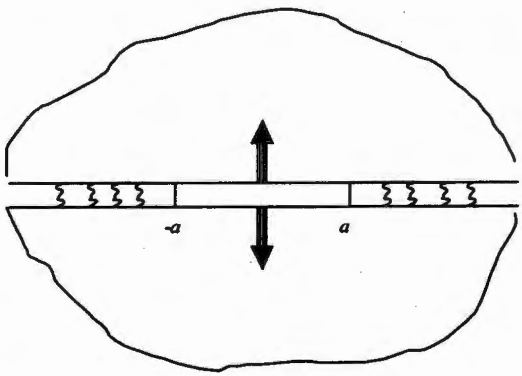

Fig. 3. Bridged crack

So we consider the following crack problem, see Figure 3, described for $|x| > a$ by the equation

$$-\frac{1}{k}\frac{4\pi\mu}{\kappa+1}p(x) = P_0 \ln\frac{|x|}{d} -$$

$$-\int_a^{+\infty} p(\zeta)\left[\ln\frac{|x-\zeta|}{d} + \ln\frac{|x+\zeta|}{d}\right] d\zeta.$$

So by differentiation

$$-A\frac{dp(x)}{dx} = \frac{P_0}{x} - \int_a^{+\infty} p(\zeta)\left[\frac{1}{x-\zeta} + \frac{1}{x+\zeta}\right] d\zeta,$$

where

$$\begin{aligned} P_0 &= 2\int_a^{+\infty} p(\zeta)\, d\zeta \\ A &= \frac{1}{k}\frac{4\pi\mu}{\kappa+1}. \end{aligned}$$

We show that there exists a unique solution p for the above model.

3. The mathematical problem

We consider the following model with unknown function $p(x)$, $x \in \mathbb{R}$.

Find $p(x)$ such that

$$\begin{cases} \int_{-\infty}^{+\infty} \frac{p(\zeta)}{x-\zeta} d\zeta - \frac{P}{x} = A\frac{dp}{dx}, & |x| > a \\ p = 0, & |x| < a \\ \int_{-\infty}^{+\infty} p(\zeta) d\zeta = P \end{cases} \tag{3}$$

Assume P, A and a given. This is an integro - differential equation. The integral is a singular integral - the famous Hilbert transform. The integral itself is defined as a principal value:

$$\begin{aligned} \mathcal{H}f &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(\zeta)}{x-\zeta} d\zeta = \\ &= \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-\zeta|\geq\varepsilon} \frac{f(\zeta)}{x-\zeta}\, d\zeta \end{aligned} \tag{4}$$

It is non - trivial that for each $f \in L^1$ the limit exists almost everywhere. It is a fact that

$$\begin{aligned} &\mathcal{H} : L^p \to L^p, p > 1 \\ &\mathcal{H}^2 = -I \end{aligned}$$

The Hilbert transform is related to the Fourier transform by

$$\widehat{\mathcal{H}f}(\omega) = -i\,(sign\,\omega)\hat{f}(\omega),\ f \in L^2 \ .$$

Also the sine and cosine transforms of a function on $[0, \infty)$ are the Hilbert transforms of each other:

$$\int_0^\infty f(x) \cos \omega x dx = \mathcal{H}\left[\int_0^\infty f(x) \sin \omega x dx\right].$$

Example (appropriate for our case)

If

$$f(x) = \begin{cases} \frac{1}{x}, & |x| > a \\ 0, & |x| < a \end{cases}$$

then

$$\mathcal{H}f(x) = \frac{1}{\pi x} \log\left|\frac{a-x}{a+x}\right|.$$

In our equation both $\mathcal{H}$ and $\frac{d}{dx}$ appear. These are related. There is another method of defining the Hilbert transform:
the function

$$t \longrightarrow \log\left|1 - \frac{x}{t}\right|$$

is in L^q for all $q > 1$, and L^q norm remains bounded for x in a compact set.

For any $f \in L^p$, the integral

$$\int_{-\infty}^{+\infty} f(t)\, log\left|1 - \frac{x}{t}\right| dt$$

is absolutely convergent. It can be shown that the function

$$x \longrightarrow \int_{-\infty}^{+\infty} f(t)\, log\left|1 - \frac{x}{t}\right| dt$$

is absolutely continuous.
One defines

$$\mathcal{H}f(x) = -\frac{d}{dx}\left\{\int_{-\infty}^{+\infty} f(t) \log\left|1 - \frac{x}{t}\right| dt\right\}.$$

Now we can reformulate (3) as:

$$\frac{d}{dx}\{Ap(x) + \int_{-\infty}^{+\infty} p(t) \log\left|1 - \frac{x}{t}\right| dt + \\ + \left(\int_{-\infty}^{+\infty} p(t)dt\right) \log^+ \left|\frac{x}{a}\right|\} = 0, \ |x| > a.$$

so that apparently we need to solve:

$$Ap(x) + \int_{-\infty}^{+\infty} p(t) \log\left|1 - \frac{x}{t}\right| dt + P \ \log^+\left|\frac{x}{a}\right| = \text{const}, \quad |x| > a \ (5)$$

$$\begin{aligned} &p \equiv 0 && \text{in} \quad |x| < a. \\ &\textstyle\int_{-\infty}^{+\infty} p(t)dt = P \end{aligned}$$

This does not involve any singular integral; however, the function $\log^+ |\frac{x}{a}|$ is not in any L^p, and neither is the integral. We do not know any standard procedure for obtaining the solution.
Returning to (3) let us write it as:

$$\begin{aligned} &\mathcal{H}p + f = A\frac{dp}{dx}, && |x| > a \\ &p = 0, && |x| < a. \end{aligned} \tag{6}$$

Note that since p is required to vanish in $(-a, a)$ the values of f cannot be prescribed in this interval.
We noticed that there is yet another connection between the Hilbert transform and $\frac{d}{dx}$.
This is provided by the Cauchy - Poisson semi group:
For any fixed $p > 1$, define $V_t : L^p \to L^p$ by

$$V_t f = \frac{t}{\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2 + t^2} dy. \tag{7}$$

V_t is a C_0 - semi group of contractions on L^p:

$$\begin{aligned} &\| V_t \| \leq 1; \quad V_t V_s = V_{t+s} \\ &\lim_{t\to 0} \| |V_t - I|(f) \| = 0. \end{aligned}$$

The infinitesimal generator $\mathcal{M}$ of the semi group is defined by:

$$\mathcal{M}f = \lim_{t\to 0} \frac{V_t f - f}{t}$$

whenever the limit exists and the linear space where this limit exists is the domain of $\mathcal{M}$. One can show that this domain is the range of the resolvent:

$$D\mathcal{M} = \{R_\lambda f : f \in L^p\} \tag{8}$$

where

$$R_\lambda f = \int_0^\infty e^{-\lambda t} V_t f dt.$$

And we have

$$\lambda R_\lambda f - \mathcal{M}(R_\lambda f) = f, \forall f \in L^p. \tag{9}$$

For the Cauchy - Poisson semigroup the infinitesimal generator has the following characterization (Butzer-Berens 1967, pp.248)

$$D(\mathcal{M}) = \{f \in L^p : (\mathcal{H}f)' \in L^p\} \tag{10}$$

and for $f \in D(\mathcal{M})$

$$\mathcal{M}f = -(\mathcal{H}f)'. \tag{11}$$

Combining (8), (9), (10), (11) we see that for each $f \in L^p$ and each $\lambda > 0$,

$$\lambda R_\lambda f + \frac{d}{dx}(\mathcal{H}(R^\lambda f)) = f. \tag{12}$$

Recall that $\mathcal{H}^2 = -I$, and so, by writing

$$v = \mathcal{H}(R_\lambda f)$$

we get from (12):
for each $f \in L^p, v = \mathcal{H}(R_\lambda f)$ solves:

$$-\lambda \mathcal{H} v + \frac{d}{dx} v = f$$

i.e.,

$$\frac{\lambda}{\pi} \int_{-\infty}^{+\infty} \frac{v(y)dy}{x-y} + f = \frac{dv}{dx}. \tag{13}$$

With a little computation, if

$$f(x) = \begin{cases} \dfrac{1}{x} & |x| > a \\ 0 & |x| < a. \end{cases}$$

the solution of (13) is given by

$$v(x) = \int_{-\infty}^{+\infty} K(x-y)\frac{1}{\pi y}\log\left|\frac{y-a}{y+a}\right| dy$$

where

$$K(\lambda) = \frac{1}{\pi}\int_0^{\infty} e^{-\lambda t}\frac{t}{\lambda^2+t^2}dt = \frac{1}{\pi}\int_0^{\infty}\frac{cos\lambda\zeta}{\lambda+\zeta}d\zeta.$$

It looks as though we have solved our problem. Unfortunately not, because we require v to be zero in$(-a,a)$ so that f cannot be prescribed inside this interval.

In other words we have to determine f in $(-a,a)$ and the solution v. We formulate the problem:

Let $f_0 \in L^2$. Find f and v so that $f(x) = f_0(x), |x| > a, v(x) =$ constant in $|x| < a$ and such that

$$\frac{\lambda}{\pi}\int_{-\infty}^{+\infty}\frac{v(y)}{x-y}dy + f(x) = \frac{dv}{dx}(x). \tag{14}$$

To solve this we proceed as follows:

Let v_0 solve (14) with $f = f_0$.

Then $v_0 = \mathcal{H}(R_\lambda f_0) = R_\lambda(\mathcal{H}f_0)$.

Let

$$f_1 = \begin{cases} f_0 & |x| > a \\ -\lambda\mathcal{H}v_0 & |x| < a \end{cases}$$

and $v_1 = \mathcal{H}(R_\lambda f_1)$.

Then v_1 solves (14) with $f = f_1$.

If $f_1, \ldots, f_n,\ v_1, \ldots, v_n$ have been defined, we define

$$\begin{cases} f_{n+1} & = & \begin{cases} f_0, & |x| > a \\ -\lambda\mathcal{H}v_n, & |x| < a \end{cases} \\ v_{n+1} & = & \mathcal{H}(R_\lambda f_{n+1}). \end{cases} \tag{15}$$

We have

$$v_{n+1} - v_n = \mathcal{H}[R_\lambda(f_{n+1} - f_n)]$$

so that, $\mathcal{H}$ being an isometry

$$\begin{aligned} \| v_{n+1} - v_n \|_2 &= \| R_\lambda(f_{n+1} - f_n) \|_2 \\ &= \| \lambda R_\lambda \{[\mathcal{H}(v_n - v_{n-1})] \mathbf{1}_{\{|x|<a\}} \|_2 \end{aligned} \tag{16}$$

because by definition (15) of f_n

$$f_{n+1} - fn = \begin{cases} 0 & , \quad |x| > a \\ \lambda \mathcal{H}(v_n - v_{n-1}) & , \quad |x| < a. \end{cases}$$

To estimate the norm in (16) we use the following result.

Proposition

Let $0 < \varphi \in L^1$. Let $f \in L^p$ with support in $(-a, a)$.
Then

$\varphi \star f \in L^p$ and

$$\| \varphi \star f \|_p \leq \alpha \| f \|_p$$

where $\alpha^q = \sup_I \mu(I), I$ any interval of length $2a$ and μ has density φ.

Proof:

The easy proof is:

$$\begin{aligned} &\int \varphi(x-y) f(y) dy = \int \varphi(x-y) f(y) \mathbf{1}_{\{|y| \leq a\}} dy \\ &\leq \left(\int \varphi(x-y) f^p(y) dy\right)^{\frac{1}{p}} \left(\int \varphi(x-y) \mathbf{1}_{\{|y| \leq a\}}\right)^{\frac{1}{q}} \\ &= \alpha \left(\int \varphi(x-y) f^p(y) dy\right)^{\frac{1}{p}}. \end{aligned}$$

Now raise both sides to power p, integrate and use Fubini. □

Since λR_λ has strictly positive density, the Proposition can be used to estimate (16):

There is $\alpha < 1$ such that

$$\begin{aligned} \| v_{n+1} - v_n \|_2 &\leq \alpha \| (\mathcal{H}(v_n - v_{n-1})) \mathbf{1}_{\{|x|<a\}} \|_2 \\ &\leq \alpha \| \mathcal{H}(v_n - v_{n-1} \|_2 \\ &= \alpha \| v_n - v_{n-1}) \|_2 . \end{aligned}$$

This implies that the sequence v_n converges to v in L^2, $\mathcal{H} v_n$ converges to $\mathcal{H} v$ in L^2. Then f_n converges to f in L^2, where f is defined by

$$f = \begin{cases} f_0 & |x| > a \\ -\lambda \mathcal{H} v & |x| < a. \end{cases}$$

Since

$$v_n = \mathcal{H}R_\lambda f_n$$

we see that

$$v = \mathcal{H}R_\lambda f = R_\lambda \mathcal{H} f$$

i.e., that v satisfies (14).
Using (17) in (14) we see that $\frac{dv}{dx} = 0$ for $|x| < a$ or v is a constant. Thus (v, f) is a solution of our problem. Note that since $v = R_\lambda \mathcal{H} f$, v is *absolutely continuous.*
Finally, we prove the uniqueness of the solution. Suppose u and v are solutions of the above problem. In other words

$$\lambda \mathcal{H}_u + f_1 = u_x \,, \quad \begin{array}{ll} u_x = 0 \,, & |x| < a \\ f_1 = f_0 \,, & |x| > a \end{array}$$

$$\lambda \mathcal{H}_v + f_2 = v_x \,, \quad \begin{array}{ll} v_x = 0 \,, & |x| < a \\ f_2 = f_0 \,, & |x| > a \end{array}$$

where we denote $u_x = \dfrac{du}{dx}$, $v_x = \dfrac{dv}{dx}$.
Then

$$w = u - v$$

satisfies

$$\begin{cases} \lambda \mathcal{H} w + f & = \frac{d}{dx} w \\ w' = 0 & |x| < a \\ f = 0 & |x| > a. \end{cases}$$

Since $\frac{d}{dx}\, w = 0$, $|x| < a$, we must have $f = -\lambda \mathcal{H} w$, $|x| < a$.
From what we have said, the unique solution w is given by:

$$w = \mathcal{H} R_\lambda f.$$

Hence

$$\begin{aligned} &\| w \|_2 = \| R_\lambda f \|_2 = \| R_\lambda [(-\lambda \mathcal{H}_w) \, \mathbf{1}_{\{|x|<a\}} \|_2 \\ &= \| \lambda R_\lambda (\mathbf{1}_{\{|x|<a\}} \mathcal{H}_w) \|_2 \leq \alpha \| \mathcal{H} w \|_2 \\ &= \alpha \| w \|_2 \end{aligned}$$

where $\alpha < 1$, so we must have $w = 0$.
This shows that the solution is unique.

Now suppose the given function f_0 is odd in $(|x| > a)$ i.e. $f_0(x) = -f_0(-x)$ for $|x| > a$. Extend f_0 to be odd. Now R_λ is a function of $|x|$ so R_λ takes odd functions into odd functions. The same is true of the Hilbert transform. It follows that v_0 is odd. The definition of f_1 shows that f_1 is also odd, etc. All v_n and hence the limit v are odd. But v is constant in $(-a, a)$. So it must be zero.
Thus we have:

Let $f_0 \in L^2(x : |x| > a)$ and be odd. Then there exists the unique $v, f \in L^2$ such that f is odd, $f = f_0$ in $(|x| > a)$, $v = 0$ in $|x| < a$ and

$$\frac{\lambda}{\pi} \int_{-\infty}^{+\infty} \frac{v - (y)}{x - y} dy + f = \frac{dy}{dx}. \tag{17}$$

4. Numerical example

Numerical solutions to the model can be computed using Matlab. To this end, the equation

$$-A\frac{dp}{dx}(x) = \frac{P_0}{x} - \int_a^\infty p(\zeta)\left(\frac{1}{x-\zeta} + \frac{1}{x+\zeta}\right) d\zeta,\ x > a$$

is transported to the interval $s \in (0, 1]$ using the change of variable $s = \frac{a}{x}, x > a$,

$$\frac{dp}{ds} = \frac{dp}{dx}\left(-\frac{a}{s^2}\right) \Rightarrow \frac{dp}{dx} = -\frac{s^2}{a}\frac{dp}{dx}.$$

So, the equation becomes

$$\begin{aligned} A\frac{s^2}{a}\frac{dp}{ds} &= \tfrac{sP_0}{a} - \textstyle\int_1^0 p(t)\left(\frac{1}{\frac{a}{s} - \frac{a}{t}} + \frac{1}{\frac{a}{s} + \frac{a}{t}}\right)\left(-\frac{a}{t^2}\right) dt \\ &= \tfrac{sP_0}{a} - \textstyle\int_0^1 \frac{2p(t)}{t^2 - s^2} dt \end{aligned}$$

i.e.,

$$As\frac{dp}{ds} + 2a\int_0^1 \frac{p(t)}{t^2 - s^2} dt = P_0.$$

This equation is now discretized. We use the standard two point discretization for the derivative, and a somewhat modified mid-point rule for the singular integral. The sampling points for the integration variable t are chosen to be the midpoints between the sampling points for s. In this way, the singularity does not introduce an added difficulty. However, we cannot evaluate p at these points and replace $p(t_j)$ by the average of the two closest points $(p(s_j) + p(s_{j+1}))/2$. A Matlab M-file program for this algorithm is listed in Figure 4.

Numerical results are shown in Figures 5-8. Figure 5 and Figure 6 contain the solution to the integral equation when $a = 3$ and $P_0 = 4$. The upper graph in both figures shows the solution in the re-scaled finite domain, the lower graph shows the solution in the domain $[a, \infty)$. In Figure 5 we used a discretization with 500 sub-intervals, Figure 6 uses a discretization with 1000 sub-intervals. One can see that the value of $p(x)$ at $x = a$ is independent of the discretization.

Figures 7 and 8 show the dependence of $p(x)$ at $x = a$ on the value of the parameter a (Figure 7) and the value of the parameter P_0, the L^1-norm of the solution (Figure 8). Figure 8 suggests that the dependence on the norm is linear. All values were computed using discretization with 500 sub-intervals. The dependence on a was computed using a norm of 4, the norm dependence was computed with $a = 3$.

Concluding remarks

In the paper the existence of a unique solution to a bridged crack model is proved using some properties of the Cauchy - Poisson semi group and the Hilbert transform. The method of proof is to our knowledge original. To complete the analysis one should determine the singularity of the solution at $x = a^+$ and prove the convergence of the finite dimensional approximation to the model. On the other hand the dependence of the solution on the tip position $\pm a$ is important for applications to the crack propagation modeling.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%
%%% Matlab M-file for the Numerical Solution of a Singular Integral Equation
%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear;
N=input('number of subintervals?');
length=input('length of the crack?' );
norm=input('Norm of Solution? ');
h=1/N
t=linspace(0,1,N+1);
s=zeros(1,N);
for j=1:N
   s(j)=t(j)+h/2;
end
A=zeros(N+1,N+1);
B=zeros(N+1,N+1);
C=zeros(N+1,N+1);
for j=1:N+1
    for k=1:N
       A(j,k)=1/(t(j)^2-s(k)^2);
       B(j,k+1)=1/(t(j)^2-s(k)^2);
   end
   end
for j=2:N
   C(j,j-1)=-N/2*t(j);
   C(j,j+1)=N/2*t(j);
end
C(N+1,N)=-N*t(N+1);
C(N+1,N+1)=N*t(N+1);
C(1,2)=N*t(1);
C(1,1)=-N*t(1);
D=-h*(A+B)*length+C;
Right=norm*ones(N+1,1);
w=D\Right;
for j=1:200
   x(j)=length/t(N+2-j);
   v(j)=w(N+2-j);
end
figure(1)
subplot(2,1,1)
h=plot(t,w);
xlabel('s=a/x')
ylabel('p(s)')
title('\bf{Solution in the rescaled domain}')
subplot(2,1,2)
h=plot(x,v);
xlabel('x')
ylabel('p(x)')
title('\bf{Solution in the domain x > a}')
w(N)
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%
% The integro-differential
%
% equation is transformed into
%
% a linear system
%
% Dw=right
%
% The matrix D is not sparse
% and consist of a discretized
% Integral A+B and the discrete
% Derivative C.
%
% The linear system is solved
% the standard Matlab procedure
%
%
% The solution is transformed into
% the original domain [a,\infinity)
%
% Both representations of the
% solution are plotted:
%  1. over the interval [0,1]
%
%
%
%
%  2. over [a,\infinity)
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Fig. 4. Matlab Program

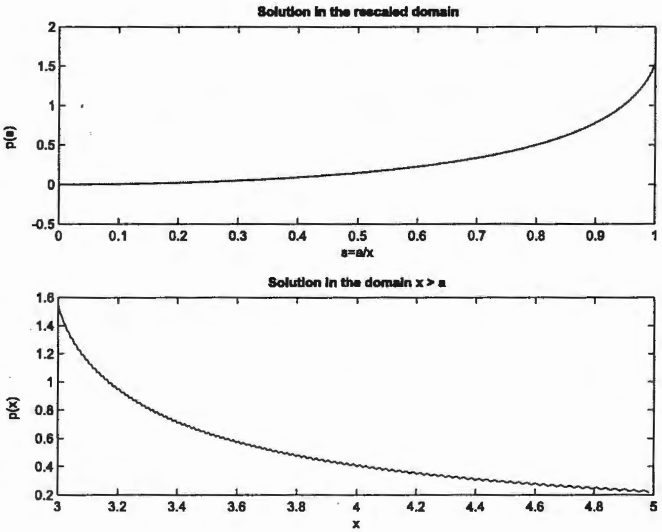

Fig. 5. Solution to the integral equation with $a = 3$ and $P_0 = 4$ and 500 sub-intervals.

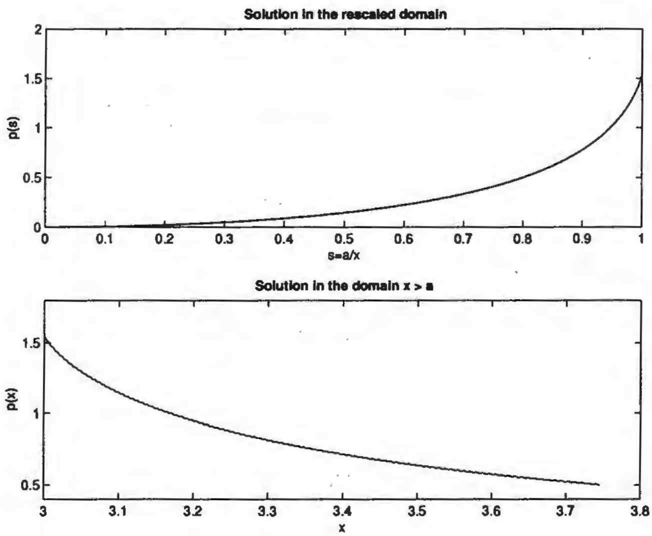

Fig. 6. Solution to the integral equation with $a = 3$ and $P_0 = 4$ and 1000 sub-intervals.

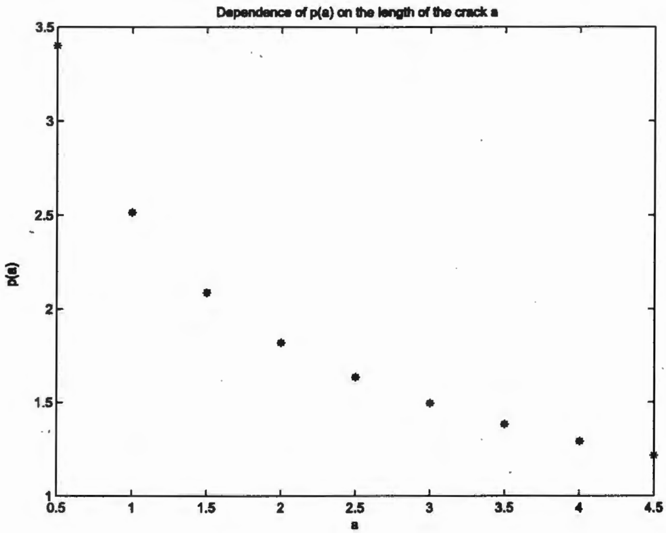

Fig. 7. Dependence of the solution on a.

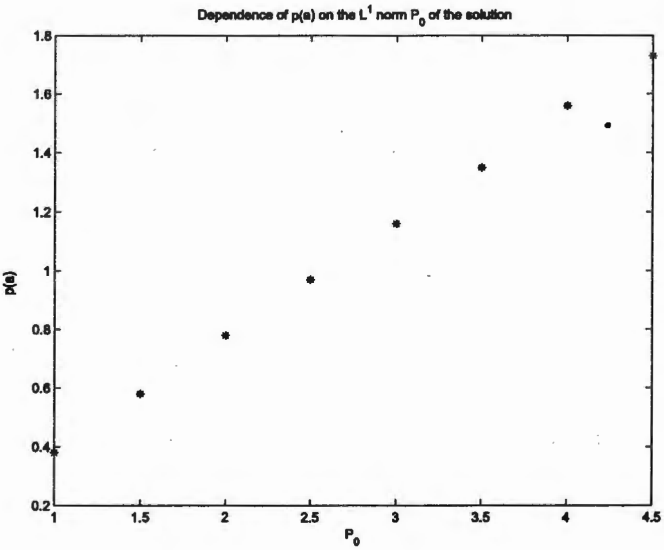

Fig. 8. Dependence of the solution on P_0.

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