Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

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Families of fuzzy preferences

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Abstract

We consider fundamental families of relations used in preference modelling. Fuzzification of these models admits rich possibilities of diverse operations on fuzzy preference relations. Some of these operations are internal in given families, other operations lead from one family to another. Such results have important consequences for fuzzy preference theory.
Keywords: preference, strict preference, fuzzy relation, fuzzy preference, triangular norm, T-asymmetric relation, T-transitive relation, T-complete relation.

1 Introduction

Examinations of preferences appear in many domains such as Economics, Psychology, Political Sciences, Operational Research or Artificial Intelligence (cf. [2]). Thus we meet diverse notions and notations connected with considered problems. Mainly we follow after mathematical terminology introduced in [8] and [5] with some complements.

Since relations appearing in real world and in human thinking are usually imprecise, then preference theory naturally includes examinations of fuzzy preferences and their generalizations (e.g. intuitionistic preferences or $L-$preferences). Initial development of fuzzy preference theory was summarized in [1]. Fuzzy preferences were examined in many later publications (cf. e.g. rich references in [3]).

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At first, we reply basic notions of preference theory in the crisp case (Section 2). Next, we consider properties of fuzzy relations connected with preference theory (Sections 3-5). Finally, we discuss fuzzification of preference theory (Sections 6-8).

2 Notion of preference

Let $X \neq \emptyset$. For a given relation $R \subset X \times X$ we use its converse R^{-1}, complement R' and dual $R^* = (R')^{-1} = (R^{-1})'$. We introduce the relation decomposition of R:
• asymmetric part of R, $P = R \setminus R^{-1} = R \cap R^*$,
• symmetric part of R, $I = R \cap R^{-1}$,
• asymmetric part of R', $P^{-1} = R' \setminus R^* = R' \cap R^*$,
• symmetric part of R', $J = R' \cap R^*$.
It can be simply verified, that relations I, J are symmetric and relations P, P^{-1} are asymmetric.

A particular terminology concerning binary relations (independent from set theory) is used in decision making. This is strictly connected with the notion of preference. Let $X \neq \emptyset$ denote a set of considered alternatives. A decision maker has three separate options for arbitrary $x, y \in X$:
a) he prefers x over y or y over x (preference),
b) he cannot distinguish them (indifference),
c) he cannot compare them (incomparability).
This leads to the following definition.

Definition 1 (Structure of preference, cf. [8])**.** Let $X \neq \emptyset$, $R \subset X \times X$, $I_X = \{(x,x) : x \in X\}$.
• R is a preference relation (large preference) if it is reflexive ($I_X \subset R$).
• R is a strict preference relation if it is asymmetric ($R \cap R^{-1} = \emptyset$).
• R is an indifference relation if it is reflexive and symmetric ($I_X \subset R$, $R^{-1} = R$).
By the structure of the preference relation we call a triplet $\{P, I, J\}$, where
• $P = R \cap R^*$ is the asymmetric part of R (strict preference),
• $I = R \cap R^{-1}$ is the symmetric part of R (indifference) and
• $J = R' \cap R^*$ is the symmetric part of R', called incomparability relation.

As a simple observation we get

Lemma 1. *If $\{P, I, J\}$ is the structure of preference relation R, then relations*

P, P^{-1}, I, J are pairwise disjoint. Moreover, $R = P \cup I$, $R' = P^{-1} \cup J$, i.e.

$$P \cup P^{-1} \cup I \cup J = X \times X. \tag{1}$$

Conversely, a given strict preference P can be included in the preference structure of certain preference relation R.

Lemma 2 (Composition of preference)**.** *If P is a strict preference and $I = P' \cap P^*$ (symmetric part of P'), then $R = P \cup I$ is a preference relation with the preference structure $\{P, I, \emptyset\} = \{P, I\}$. In particular, R is the greatest preference relation which asymmetric part is equal P. Moreover, the relation R is complete, i.e. $R \cup R^{-1} = X \times X$.*

Proof. Let $P \subset X \times X$ be asymmetric, i.e. $I_X \subset P'$ and therefore $I_X \subset P^*$. Thus,

$$I_X \subset P' \cap P^* \subset R,$$

which proves that R is reflexive. Since $I = P' \setminus P^{-1}$ and

$$R = P \cup P' \setminus P^{-1} = X \times X \setminus P^{-1} = (P^{-1})' = P^*,$$

then $R \setminus R^{-1} = R \cap R^* = R \cap P = (P \cup I) \cap P = P$, i.e. P is the asymmetric part of R. Thus $I = P' \cap P^* = R^{-1} \cap R$, $J = R' \cap R^* = P^{-1} \cap P = \emptyset$ and we obtain the preference structure $\{P, I, \emptyset\}$.

Finally, let S be an arbitrary relation which has the asymmetric part P, i.e. $P = S \cap S^*$. But $R = P^* = S^* \cup S$ and therefore $S \subset R$. So, R is the greatest preference in X generating the strict preference P. Since $X \times X = J' = R \cup R^{-1}$, then the relation R is complete. □

Now it is clear, that dependence between preference relations and strict preference relations is not bijective, not unique (many preference relations generate the same strict preference). However, the relation R in Lemma 2 is complete, and directly from the above lemmas we get

Theorem 1. *A strict preference P is generated by a complete relation R if and only if the relation R is represented by the formula $R = P \cup (P' \cap P^*)$.*

According to this theorem in consideration of arbitrary strict preferences we can restrict ourselves to these generated by complete relations. Thus, it suffice to consider complete preferences only, with preference structures reduced to the case $\{P, I\}$.

3 Fuzzy relations

We recall basic properties of fuzzy relations on a set $X \neq \emptyset$, which are arbitrary functions $R : X \times X \to [0,1]$. The family of all fuzzy relations on X is denoted by $FR(X)$. The fuzzy relation R on a finite set $X = \{x_1, ..., x_n\}$ can be represented by a square matrix $R = [r_{i,k}]$, where $r_{i,k} = R(x_i, x_k) \in [0,1]$, $i, k = 1, ..., n$. $FR(X)$ is ordered by the point-wise extension of order relation:

$$R \leqslant S \Leftrightarrow (R(x,y) \leqslant S(x,y),\ x,y \in X) \text{ for } R, S \in FR(X).$$

Operations $\vee, \wedge : FR(X)^2 \to FR(X)$ are defined point-wise:

$$(R \vee S)(x,y) = R(x,y) \vee S(x,y),\ (R \wedge S)(x,y) = R(x,y) \wedge S(x,y),\ x,y \in X, \tag{2}$$

which gives a complete and distributive lattice $(FR(X), \vee, \wedge)$.

Useful examples of fuzzy relations are the empty relation $0 = 0_{X \times X}$, the total relation $1 = 1_{X \times X}$ and the identity relation I_X, where

$$0(x,y) = 0,\ 1(x,y) = 1,\ I_X(x,y) = \begin{cases} 1, & if\ x = y \\ 0, & if\ x \neq y \end{cases},\quad x,y \in X.$$

4 Operations on fuzzy relations

We list here commonly used operations on fuzzy relations.

Definition 2 (Unary operations I)**.** Let $R \in FR(X)$. By the complement of R we call fuzzy relation $R' = 1 - R$. The converse relation R^{-1} of R is given by $R^{-1}(x,y) = R(y,x),\ x,y \in X$. Then, dual relation R^* is given by $R^* = (R^{-1})' = 1 - R^{-1}$.

All the presented operations are involutive (e.g. $(R^*)^* = R$). Let us observe that $0^{-1} = 0,\ \ 1^{-1} = 1,\ \ (I_X)^{-1} = I_X,\ \ 0^* = 1,\ \ 1^* = 0, (I_X)^* = (I_X)'$.

Definition 3 (Unary operations II)**.** Let $R \in FR(X)$. The symmetric part $\sqcap R$ of R is given by $\sqcap R = R \wedge R^{-1}$. The asymmetric part $\triangle R$ of R is given by $\triangle R = R \wedge R^*$. The symmetric closure $\sqcup R$ of R is given by $\sqcup R = R \vee R^{-1}$.

All the presented operations are idempotent (e.g. $\triangle(\triangle R) = \triangle R$).

Definition 4 (Relation composition, cf. [10] or [6], Chapter II)**.** The sup–inf composition of fuzzy relations $R, S \in FR(X)$ is the fuzzy relation $R \circ S$ such that

$$(R \circ S)(x,z) = \bigvee_{y \in X} R(x,y) \wedge S(y,z),\ x,z \in X.$$

The powers of R are defined recursively

$$R^1 = R,\ R^{n+1} = R \circ R^n,\ n \in \mathbb{N}.$$

The upper and lower closures of R are defined by

$$R^{\vee} = \bigvee_{n\in\mathbb{N}} R^n,\ \ R^{\wedge} = \bigwedge_{n\in\mathbb{N}} R^n.$$

Remark 1. $(FR(X), \circ)$ is an ordered semigroup with zero element $0_{X\times X}$ and identity element I_X.

5 Fundamental classes of fuzzy relations

Properties of fuzzy relations are based on general relation theory.

Definition 5 ([4]). Fuzzy relation $R \in FR(X)$ is:
• *reflexive*, if $\underset{x\in X}{\forall}\ R(x,x) = 1$,
• *irreflexive*, if $\underset{x\in X}{\forall}\ R(x,x) = 0$,
• *symmetric*, if $\underset{x,y\in X}{\forall}\ R(x,y) = R(y,x)$,
• *asymmetric*, if $\underset{x,y\in X}{\forall}\ R(x,y) \wedge R(y,x) = 0$,
• *antisymmetric*, if $\underset{x,y\in X, x\neq y}{\forall}\ R(x,y) \wedge R(y,x) = 0$,
• *complete*, if $\underset{x,y\in X}{\forall}\ R(x,y) \vee R(y,x) = 1$,
• *weakly complete*, if $\underset{x,y\in X, x\neq y}{\forall}\ R(x,y) \vee R(y,x) = 1$,
• *transitive*, if $\underset{x,y,z\in X}{\forall}\ R(x,y) \wedge R(y,z) \leqslant R(x,z)$.

Let card $X > 1$, $R, S \neq 0$, $R, S \neq 1$. We examine values of operations in particular classes. By a simple verification we obtain Tables 1 and 2 of results, where
• $+$ denotes the positive result for every R,
• $-$ denotes the negative result for every R,
• 0 denotes the empty relation (as the result),
• 1 denotes the total relation (as the result),
• $\#$ denotes mixed results (some positive, some negative),
• (a)-(h) denote additional conditions for positive results (cf. the last column in Table 2).

Class \ Operation	R^*	$\triangle R$	$\sqcap R$	$\sqcup R$	R^n	$R^\wedge$	$R^\vee$
Reflexive	—	—	+	+	+	+	+
Irreflexive	—	+	+	+	(a)	+	(a)
Symmetric	+	0	+	+	+	+	+
Asymmetric	—	+	0	—	(a)	+	(a)
Antisymmetric	—	+	+	—	(a)	+	(a)
Weakly Complete	—	—	—	+	#	#	+
Complete	—	—	—	1	+	+	+
Transitive	#	+	+	#	+	+	+

Table 1. Results for unary operations.

Class\Operation	$R \vee S$	$R \wedge S$	$R \circ S$	Additional Conditions
Reflexive	+	+	+	(a)$R \circ R \leqslant R$
Irreflexive	+	+	(g)	(b)$R \circ S = S \circ R$
Symmetric	+	+	(b)	(c)$R \wedge S^{-1} = 0$
Asymmetric	(c)	+	(g)	(d)$R \wedge S^{-1} \leqslant I_X$
Antisymmetric	(d)	+	(g)	(e)$R \vee S^{-1} = 1$
Weakly Compl.	+	(f)	#	(f)$(I_X)' \leqslant R \vee S^{-1}$
Complete	+	(e)	+	(g)$R \circ S \leqslant R$ or $R \circ S \leqslant S$
Transitive	(h)	+	(b)	(h)$R \circ S \vee S \circ R \leqslant R \vee S$

Table 2. Results for binary operations with additional conditions.

6 Fuzzification of preferences

Now we apply Definition 1 in a fuzzy setting. Let R be a reflexive fuzzy relation and

• $P = R \wedge R^*$, $P(x, y) = \min(R(x, y), 1 - R(y, x))$,
• $I = R \wedge R^{-1}$, $I(x, y) = \min(R(x, y), R(y, x))$,
• $J = R' \wedge R^*$, $J(x, y) = \min(1 - R(x, y), 1 - R(y, x))$.

We ask if $\{P, I, J\}$ forms the preference structure of the fuzzy preference R.

Example 1. Let card $X = 2$. For the fuzzy reflexive relation (preference) $R \in FR(X)$ represented by the matrix

$$R = \begin{bmatrix} 1 & 0.8 \\ 0.4 & 1 \end{bmatrix}$$

we obtain suitable relation matrices R', R^{-1}, R^*,

$$R' = \begin{bmatrix} 0 & 0.2 \\ 0.6 & 0 \end{bmatrix}, R^{-1} = \begin{bmatrix} 1 & 0.4 \\ 0.8 & 1 \end{bmatrix}, R^* = \begin{bmatrix} 0 & 0.6 \\ 0.2 & 0 \end{bmatrix}.$$

Using formulas for P, I, J we get

$$P = \begin{bmatrix} 0 & 0.6 \\ 0.2 & 0 \end{bmatrix}, I = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \end{bmatrix}, J = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix},$$

and in consequence

$$P^{-1} = \begin{bmatrix} 0 & 0.2 \\ 0.6 & 0 \end{bmatrix}, \ P \wedge P^{-1} = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix} \neq 0,$$

$$P \vee I = \begin{bmatrix} 1 & 0.6 \\ 0.4 & 1 \end{bmatrix}, P^{-1} \vee J = \begin{bmatrix} 0 & 0.2 \\ 0.6 & 0 \end{bmatrix}.$$

We see that the fuzzy relation $P = R \wedge R^*$ need not be asymmetric (or strict preference) and the given fuzzy relation R may be not recovered as $P \vee I$. Similarly the fuzzy relation R' may be not recovered as $P^{-1} \vee J$. Moreover, $P \wedge J = P^{-1} \wedge J = I \wedge J = J \neq 0$,

$$P \wedge I = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0 \end{bmatrix} \neq 0, P^{-1} \wedge I = \begin{bmatrix} 0 & 0.2 \\ 0.4 & 0 \end{bmatrix} \neq 0.$$

Thus, fuzzy relations from the fuzzy preference system $\{P, I, J\}$ need not be disjoint.

7 $T-$algebra of fuzzy relations

In the above situation, in the algebra of fuzzy relations another operations should be used, different from $\min$ and $\max$. In particular we use triangular norms and conorms.

Definition 6 ([5], Chapter I)**.** A binary operation $T : [0, 1]^2 \to [0, 1]$ is called a triangular norm if it is increasing, associative, commutative and has the neutral element 1. Similarly, a triangular conorm has the neutral element 0. By the triangular conorm dual to T we call $S : [0, 1]^2 \to [0, 1]$ such that

$$S(x, y) = 1 - T(1 - x, 1 - y) \text{ for } x, y \in [0, 1]. \tag{3}$$

A continuous triangular norm is called strict, if it is strictly increasing in $(0, 1]^2$, and it is called nilpotent, if $([0, 1), T)$ is a nilpotent semigroup.

Example 2 ([7], p. 4)**.** The most important triangular norms and conorms have own names (dual pairs):
• $T_M(x,y) = \min(x,y)$; $S_M(x,y) = \max(x,y)$ (lattice operations);
• $T_P(x,y) = xy$; $S_P(x,y) = x + y - xy$ (product operations);
• $T_L(x,y) = \max(x+y-1,0)$, $S_L(x,y) = \min(x+y,1)$ (Łukasiewicz operations)
where $x, y \in [0,1]$. In particular, T_M is a strict triangular norm, T_L is a nilpotent triangular norm and T_M is an idempotent triangular norm.

Let $P, Q \in FR(X)$. For arbitrary triangular norm T, instead of (2) we can use $T-$algebra of fuzzy relations:

$$T(P,Q)(x,y) = T(P(x,y),Q(x,y)),\ S(P,Q)(x,y) = S(P(x,y),Q(x,y)) \tag{4}$$

for $x, y \in X$, where S is the dual triangular conorm (3). This approach has influence on the classification of fuzzy relations.

Definition 7. Let T, S be dual triangular norm and conorm. Fuzzy relation $R \in FR(X)$ is:
• $T-$*asymmetric*, if $T(R, R^{-1}) = 0$,
• $T-$*antisymmetric*, if $T(R, R^{-1}) \leqslant I_X$,
• $T-$*complete*, if $S(R, R^{-1}) = 1$,
• $T-$*weakly complete*, $(I_X)' \leqslant S(R, R^{-1})$,
• $T-$*transitive*, if $\underset{x,y,z\in X}{\forall} T(R(x,y), R(y,z)) \leqslant R(x,z)$.

8 Preferences with respect to triangular norms

The case of triangular norms and conorms instead of $\min$ and $\max$ for fuzzy preferences was examined in [9]:

Theorem 2. *Let $R \in FR(X)$. If T is a strict triangular norm and*

$$P = T(R, R^*),\ I = T(R, R^{-1}),\ J = T(R', R^*), \tag{5}$$

then P is asymmetric and $S(P, P^{-1}, I, J) = 1$ (cf. (1)*), if and only if the relation R is crisp (the characteristic function of a binary relation), where S is the triangular conorm* (3) *extended to four arguments*

$$S(x,y,z,w) = S(S(x,y), S(z,w)),\ x,y,z,w \in X. \tag{6}$$

Theorem 3. *Let $R \in FR(X)$. If T is a nilpotent triangular norm and we consider fuzzy relations* (5)*, then $\{P, I, J\}$ forms the preference structure of the relation R in $T-$fuzzy algebra, i.e.*
• $T(P, P^{-1}) = 0$ *($T-$asymmetry, $T-$strict preference);*
• $I(x, x) = 1, x \in X$, $I^{-1} = I$ *(indifference);*
• $J(x, x) = 0, x \in X$, $J^{-1} = J$ *(incomparability);*
• $S(P, P^{-1}, I, J) = 1$ *(cf.* (6)*).*

These results show that only particular triangular norms are useful in fuzzification of the preference theory. As a standard case we can consider the Łukasiewicz triangular norm $T = T_L$ and conorm $S = S_L$. Directly from Example 2 and Definition 7 we get

Theorem 4. *Fuzzy relation $R \in FR(X)$ is:*
• T_L-*asymmetric if and only if* $R + R^{-1} \leqslant 1$*;*
• T_L-*antisymmetric if and only if* $R + R^{-1} \leqslant 1 + I_X$*;*
• T_L-*complete if and only if* $R + R^{-1} \geqslant 1$*;*
• T_L-*weakly complete if and only if* $R + R^{-1} \geqslant (I_X)'$*;*
• T_L-*transitive if and only if* $\underset{x,y,z \in X}{\forall} R(x, y) + R(y, z)) \leqslant 1 + R(x, z)$.

9 Concluding remarks

We have discussed here elementary consequences of fuzzification in preference theory. It needs a deeper analysis under additional assumptions about preference relations (such as pre-order, weak order, strict order, semi-order or interval order).

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The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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